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Research Article

Fixed and Best Proximity Points of Cyclic Jointly Accretive and Contractive Self-Mappings

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$p(\geq 2)$-cyclic and contractive self-mappings on a set of subsets of a metric space which are simultaneously accretive on the whole metric space are investigated. The joint fulfilment of the p-cyclic contractiveness and accretive properties is formulated as well as potential relationships with cyclic self-mappings in order to be Kannan self-mappings. The existence and uniqueness of best proximity points and fixed points is also investigated as well as some related properties of composed self-mappings from the union of any two adjacent subsets, belonging to the initial set of subsets, to themselves.

1. Introduction

In the last years, important attention is being devoted to extend the fixed point theory by weakening the conditions on both the mappings and the sets where those mappings operate [1, 2]. For instance, every nonexpansive self-mappings on weakly compact subsets of a metric space have fixed points if the weak fixed point property holds [1]. Another increasing research interest field relies on the generalization of fixed point theory to more general spaces than the usual metric spaces, for instance, ordered or partially ordered spaces (see, e.g., [3–5]). It has also to be pointed out the relevance of fixed point theory in the stability of complex continuous-time and discrete-time dynamic systems [6–8]. On the other hand, Meir-Keeler self-mappings have received important attention in the context of fixed point theory perhaps due to the associated relaxing in the required conditions for the existence of fixed points compared with the usual contractive mappings [9–12]. Another interest of such mappings is their usefulness as formal tool for the study p-cyclic contractions even if the involved subsets of the metric space under study of do not intersect [10]. The underlying idea is that the best proximity points are fixed points if such subsets intersect while they play a close role to fixed points, otherwise. On the other hand, there are close links between contractive self-mappings and Kannan self-mappings [2, 13–16]. In fact, Kannan self-mappings are contractive for values of the contraction constant being less than $1/3$, [15, 16] and can be simultaneously p-cyclic Meir-Keeler contractive self-mappings. The objective of this paper is the investigation of relevant properties of contractive $p(\geq 2)$-cyclic self-mappings of the union of set of subsets of a Banach space $(X, \|\ \|)$ which are simultaneously λ^*-accretive on the whole X, while investigating the existence and uniqueness of potential fixed points on the subsets of X if they intersect and best proximity points. For such a purpose, the concept of λ^*-accretive self-mapping is established in terms of distances as a, in general, partial requirement of that of an accretive self-mapping. Roughly speaking, the self-mapping T from X to X under study can be locally increasing on X but it is still p-cyclic contractive on the relevant subsets A_i $(i \in \overline{p})$ of X. For the obtained results of boundedness of distances between the sequences of iterates through T, it is not required for the set of subsets of X to be either closed or convex. For the obtained results concerning fixed points and best proximity points, the sets A_i $(i \in \overline{p})$ are required to be convex but they are not necessarily closed if the self-mapping T can be defined on the union of the closures of the sets A_i $(i \in \overline{p})$. Consider a metric space (X, d) associated to the Banach space $(X, \|\ \|)$ and a self-mapping $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$, where A and B are nonempty subsets of X. Then, $T : A \cup B \to A \cup B$ is a 2-cyclic self-mapping. It is said to be a 2-cyclic k-contraction self-mapping if it satisfies, furthermore,

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)\,\mathrm{dist}(A, B); \quad \forall x \in A,\ \forall y \in B, \tag{1.1}$$

for some real $k \in [0, 1)$. A best proximity point of convex subsets A or B of X is some $z \in \mathrm{cl}(A \cup B)$ such that $d(z, Tz) = \mathrm{dist}(A, B)$. If A and B are closed then either z (resp., Tz) or Tz (resp., z) is in A (resp., in B). The distance between subsets A and B of the metric space $\mathrm{dist}(A, B) = 0$ if either $A \cap B \neq \emptyset$ or if either A or B is open with $\mathrm{Fr}(A) \cap \mathrm{Fr}(B) \neq \emptyset$. In this case, if z is a best proximity point either z or Tz is not in $A \cup B$ (in particular, neither z nor Tz is in $A \cup B$ if both of them are open). It turns out that if $A \cap B \neq \emptyset$ then $z \in \mathrm{Fix}(T) \subset A \cup B$; that is, z is a fixed point of T since $\mathrm{dist}(A, B) = 0$, [9–11]. If $k = 1$ then $d(Tx, Ty) \leq d(x, y)$; for all $x \in A$, for all $y \in B$ and $T : A \cup B \to A \cup B$ is a 2-cyclic nonexpansive self-mapping, [10].

1.1. Notation

$$\mathbf{R}_{0+} := \mathbf{R}_{+} \cup \{0\}; \qquad \mathbf{Z}_{0+} := \mathbf{Z}_{+} \cup \{0\}; \qquad \overline{p} := \{1, 2, \ldots, p\} \subset \mathbf{Z}_{+}, \tag{1.2}$$

superscript T denotes vector or matrix transpose, $\mathrm{Fix}(T)$ is the set of fixed points of a self-mapping T on some nonempty convex subset A of a metric space $(X, d)\mathrm{cl}\,A$ and $\overline{A}$ denote, respectively, the closure and the complement in X of a subset A of X, $\mathrm{Dom}(T)$ and $\mathrm{Im}(T)$ denote, respectively, the domain and image of the self-mapping T and 2^X is the family of subsets of X, $\mathrm{dist}(A, B) = d_{AB}$ denotes the distance between the sets A and B for a 2-cyclic self-mapping $T : A \cup B \to A \cup B$ which is simplified as $\mathrm{dist}(A_i, A_{i+1}) = d_{A_i A_{i+1}} = d_i$; for all $i \in \overline{p}$ for distances between adjacent subsets of p-cyclic self-mappings T on $\bigcup_{i=1}^{p} A_i$.

$BP_i(T)$ which is the set of best proximity points on a subset A_i of a metric space (X, d) of a p-cyclic self-mapping T on $\bigcup_{i=1}^{p} A_i$, the union of a collection of nonempty subsets of (X, d) which do not intersect.

2. Some Definitions and Basic Results about 2-Cyclic Contractive and Accretive Mappings

Let $(X, \| \|)$ be a normed vector space and (X, d) be an associate metric space endowed with a metric (or distance function or simply "distance") $d : X \times X \to \mathbf{R}_{0+}$. For instance, the distance function may be induced by the norm $\| \|$ on X. If the metric is homogeneous and translation-invariant, then it is possible conversely to define the norm from the metric. Consider a self-mapping $T : X \to X$ which is a 2-cyclic self-mapping restricted as $T : \mathrm{Dom}(T) \subseteq X \mid A \cup B \to \mathrm{Im}(T) \subseteq X \mid A \cup B$, where A and B are nonempty subsets of X. Such a restricted self-mapping is sometimes simply denoted as $T : A \cup B \to A \cup B$. Self-mappings which can be extended by continuity to the boundary of its initial domain as well as compact self-mappings, for instance, satisfy such an extendibility assumption. In the cases that the sets A and B are not closed, it is assumed that $\mathrm{Dom}(T) \supset \mathrm{cl}(A \cup B)$ and $\mathrm{Im}(T) \supset \mathrm{cl}(A \cup B)$ in order to obtain a direct extension of existence of fixed points and best proximity points. This allows, together with the convexity of A and B, to discuss the existence and uniqueness of fixed points or best proximity points reached asymptotically through the sequences of iterates of the self-mapping T. In some results concerning the accretive property, it is needed to extend the self-mapping $T : \mathrm{Dom}(T) \subseteq X \to \mathrm{Im}(T) \subseteq X$ in order to define successive iterate points through the self-mapping which do not necessarily belong to $A \cup B$. The following definitions are then used to state the main results.

Definition 2.1. $T : \mathrm{Dom}(T) \subset X \to X$ is an accretive mapping if

$$d(x, y) \le d(x + \lambda T x, y + \lambda T y); \quad \forall x, y \in \mathrm{Dom}(T), \tag{2.1}$$

for any $\lambda \in \mathbf{R}_{0+}$.

Note that, since X is also a vector space, $x + \lambda T x$ is in X for all x in X and all real λ. This fact facilitates also the motivation of the subsequent definitions as well as the presentation and the various proofs of the mathematical results in this paper. A strong convergence theorem for resolvent accretive operators in Banach spaces has been proved in [17]. Two more restrictive (and also of more general applicability) definitions than Definition 2.1 to be then used are now introduced as follows.

Definition 2.2. $T : \mathrm{Dom}(T) \subset X \to X$ is a λ^*-accretive mapping, some $\lambda^* \in \mathbf{R}_{0+}$ if

$$d(x, y) \le d(x + \lambda T x, y + \lambda T y); \quad \forall x, y \in \mathrm{Dom}(T);\ \forall \lambda \in [0, \lambda^*], \tag{2.2}$$

for some $\lambda^* \in \mathbf{R}_{0+}$. A generalization is as follows: $T : \mathrm{Dom}(T) \subset X \to X$ is $[\lambda_1^*, \lambda_2^*]$-accretive for some $\lambda_1^*, \lambda_2^* (\ge \lambda_1^*) \in \mathbf{R}_{0+}$ if

$$d(x, y) \le d(x + \lambda T x, y + \lambda T y); \quad \forall x, y \in \mathrm{Dom}(T);\ \forall \lambda \in \left[\lambda_1^*, \lambda^*\right]. \tag{2.3}$$

Definition 2.3. $T : \mathrm{Dom}(T) \subset X \to X$ is a weighted λ-accretive mapping, for some function $\lambda : X \times X \to \mathbf{R}_{0+}$, if

$$d(x, y) \le d(x + \lambda(x, y) T x, y + \lambda(x, y) T y); \quad \forall x, y \in \mathrm{Dom}(T). \tag{2.4}$$

The above concepts of accretive mapping generalize that of a nondecreasing function. Contractive and nonexpansive 2-cyclic self-mappings are defined as follows on unions of subsets of X.

Definition 2.4. $T : A \cup B \to A \cup B$ is a 2-cyclic k-contractive (resp., nonexpansive) self-mapping if

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)\,\mathrm{dist}(A, B); \quad \forall x \in A,\ \forall y \in B, \tag{2.5}$$

for some real $k \in [0, 1)$ (resp., $k = 1$), [12, 13].

The concepts of Kannan-self mapping and 2-cyclic (α, β)-Kannan self-mapping which can be also a contractive mapping, and conversely if $k < 1/3$, [16], are defined below.

Definition 2.5. $T : X \to X$ is a α-Kannan self-mapping if

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)); \quad \forall x, y \in X, \tag{2.6}$$

for some real $\alpha \in [0, 1/2)$, [12, 13].

Definition 2.6. $T : A \cup B \to A \cup B$ is an 2-cyclic (α, β)-Kannan self-mapping for some real $\alpha \in [0, 1/2)$ if it satisfies, for some $\beta \in \mathbf{R}_+$.

$$\begin{aligned} d(Tx, Ty) &\leq \alpha(d(x, Tx) + d(y, Ty)) \\ &\quad + \beta(1 - \alpha)\,\mathrm{dist}(A, B); \quad \forall x \in A,\ \forall y \in B. \end{aligned} \tag{2.7}$$

The relevant concepts concerning 2-cyclic self-mappings are extended to $p(\geq 2)$-cyclic self-mappings in Section 3. Some simple explanation examples follow.

Example 2.7. Consider the scalar linear mapping from $X \equiv A \equiv \mathbf{R}$ to X as $Tx = \gamma x + \gamma_0$ with $\gamma, \gamma_0 \in \mathbf{R}$ endowed with the Euclidean distance $d(x, y) = |x - y|$; for all $x, y \in X$. Then,

$$\begin{aligned} d(x + \lambda Tx, y + \lambda Ty) &= |x - y + \lambda\gamma(x - y)| = |1 + \lambda\gamma||x - y| \\ &= |1 + \lambda\gamma| d(x, y) \geq d(x, y), \end{aligned} \tag{2.8}$$

for all $x, y \in \mathbf{R}$ for any $\lambda \in \mathbf{R}_{0+}$ provided that $\gamma \in \mathbf{R}_{0+}$. In this case, $T : A \cup B \to X$ is accretive. It is also k-contractive if since $d(Tx, Ty) = |Tx - Ty| = \gamma,\ d(z, y) \leq kd(x, y)$; for all $x, y \in \mathbf{R}$. Also, if $\gamma \in \mathbf{R}_-$, then $d(x + \lambda Tx, y + \lambda Ty) \geq |\lambda|\gamma| - 1| d(x, y) \geq d(x, y)$; for all $x, y \in \mathbf{R}$ if $\lambda|\gamma| \geq 2$, that is, if $\lambda \geq \lambda_1^* := 2|\gamma|^{-1}$. Then, $T : \mathbf{R} \to \mathbf{R}$ is $[\lambda_1^*, \infty)$-accretive and k-contractive if $|\gamma| \leq k < 1$.

Example 2.8. Consider the metric space $(\mathbf{R}, d)$ with the distance being homogeneous and translation-invariant and a self-mapping $T : \mathbf{R} \to \mathbf{R}$ defined by $Tx = -t|x|^p \mathrm{sgn}_e\, x = -t|x|^{p-1} x$ with $t \in \mathbf{R}_{0+}$, $p \in \mathbf{R}_{0+}$, and $\mathrm{sgn}_e\, x = \mathrm{sgn}\, x$ if $x \neq 0$ and $\mathrm{sgn}_e 0 = 0$. If $pt = 0$, then $T : \mathbf{R} \to \mathbf{R}$ is accretive since

$$d(x + \lambda Tx, y + \lambda Ty) = d(x, y); \quad \forall x, y \in X;\ \forall \lambda \in \mathbf{R}_{0+}. \tag{2.9}$$

Furthermore, if $t = 0$, then $0 \in \mathbf{R}$ is the unique fixed point with $T^j x = 0$; for all $j \in \mathbf{Z}_+$. If $p = 0$ then, $T^j x = t^j \to z = 0$ as $j \to \infty$ if $|t| < 1$ and then $z = 0$ is again the unique fixed point of T. In the general case, $Tx = t|x|^p \mathrm{sgn}_e x$ implies

$$\begin{aligned} T^2 x = T(Tx) &= -t\left(-t|x|^p \mathrm{sgn}_e x\right)^p \mathrm{sgn}_e Tx = t^{p+1}|x|^{2p}\left(\mathrm{sgn}_e x\right)^{p+1}, \\ d(x + \lambda Tx, y + \lambda Ty) &= d\left(\left(1 + \lambda t x^{p-1}\right)x, \left(1 + \lambda t y^{p-1}\right)y\right) \\ &\geq \min\left(\left|1 - \lambda t |x|^{p-1}\right|, \left|1 - \lambda t |y|^{p-1}\right|\right) d(x, y), \\ &\geq d(x,\ y); \quad \forall x, y \in X,\ \forall \lambda \in [0, \lambda^*],\ \text{some } \lambda^* \in \mathbf{R}_+, \end{aligned} \tag{2.10}$$

holds if $\lambda^* |t|\ |x|^{p-1} \leq 1$ that is, $T : \mathbf{R} \to \mathbf{R}$ is weighted $\lambda^*(x, y)$-accretive with $\lambda^*(x, y) := t^{-1} \min(|x|^{1-p}, |y|^{1-p})$. The restricted self-mapping $T : [-1, 1] \subset X \to [-1, 1]$ is $\lambda^*(\equiv t^{-1})$-accretive. Furthermore, if $p \geq 1$, then $T : [-1, 1] \subset X \to [-1, 1]$ is $|t|$-contractive if $|t| < 1$ and the iteration $T^j x \to 0$ as $j \to \infty$ with $z = 0$ being the unique fixed point since

$$d(Tx, Ty) \leq |t| \min\left(|x|^{p-1}, |y|^{p-1}\right) d(x, y) \leq |t| d(x, y); \quad \forall x, y \in [-1, 1]. \tag{2.11}$$

Note from the definition of the self-mapping $Tx = -t|x|^{p-1}x$ on $[-1, 1]$ that it is also a 2-cyclic self-mapping from $[-1, 0] \cup [0, 1]$ to itself with the property $T([-1, 0]) = [0, 1]$ and $T([0, 1]) = [-1, 0]$.

All the given definitions can also be established mutatis-mutandis if X is a normed vector space. A direct result from inspection of Definitions 2.1 and 2.2 is the following.

Assertions 1. (1) If $T : D(T) \subset X \to X$ is an accretive mapping, then it is λ^*-accretive, for all $\lambda^* \in \mathbf{R}_{0+}$. (2) If $T : D(T) \subset X \to X$ is λ^*-accretive, then it is λ_1^*-accretive; for all $\lambda_1^* \in [0, \lambda^*]$. (3) Any nonexpansive self-mapping $T : D(T) \subset X \to X$ is 0^*-accretive and conversely.

Theorem 2.9. *Let $(X, \|\ \|)$ be a Banach vector space with (X, d) being the associated complete metric space endowed with a norm-induced translation-invariant and homogeneous metric $d : X \times X \to \mathbf{R}_{0+}$. Consider a self-mapping $T : X \to X$ which restricted to $T : A \cup B \to A \cup B$ is a 2-cyclic k-contractive self-mapping where A and B are nonempty subsets of X. Then, the following properties hold.*

(i) *Assume that the self-mapping $T : X \to X$ satisfies the constraint:*

$$\begin{aligned} d(Tx, Ty) &\leq kd(x, y) + (1 - k)d_{AB} \\ &\leq kd(x + \lambda Tx, y + \lambda Ty) + (1 - k)d_{AB}; \quad \forall x \in A,\ \forall y \in B \end{aligned} \tag{2.12}$$

with $k, \lambda \in \mathbf{R}_{0+}$ satisfying the constraint $k(1 + k\lambda) < 1$. Then, the restricted self-mapping $T : A \cup B \to A \cup B$ satisfies

$$\limsup_{j \to \infty} d\left(T^j x, T^j y\right) < \infty; \quad \forall x \in A,\ \forall y \in B \tag{2.13}$$

irrespective of A and B being bounded or not.

If, furthermore, A and B are closed and convex and $A \cap B \neq \emptyset$, then there exists a unique fixed point $\omega \in A \cap B$ of $T : A \cup B \to A \cup B$ such that there exists $\lim_{j\to\infty} d(T^j x, T^j y) = 0$; for all $x \in A$, for all $y \in B$, implying that $\lim_{j\to\infty} T^j x = \lim_{j\to\infty} T^j y = \omega$. If, in addition, $\operatorname{dist}(A, B) > 0$ so that $A \cap B = \emptyset$, then there exists $\lim_{j\to\infty} d(T^j x, T^j y) = d\,(z, Tz)$; for all $x \in A$, for all $y \in B$ for some best proximity points $z \in A$, $Tz \in B$ which depend in general on x and y. Furthermore, if $(X, \|\,\|)$ is a uniformly convex Banach space, then $T^{2j}x, T^{2j+1}y \to z_1 \in A$ and $T^{2j}y$, $T^{2j+1}x \to Tz_1 \in B$ as $\to \infty$; for all $(x, y) \in A \times B$, where z_1 and z_2 are unique best proximity points in A and B of $T : A \cup B \to A \cup B$.

(ii) *Assume that A and B are nondisjoint. Then, $T : A \cup B \to X$ is also k_c contractive and λ^*-accretive for any nonnegative $\lambda^* \leq k^{-2}(k_c - k)$ and any $k_c \in [k, 1)$. It is also nonexpansive and λ^*-accretive for any nonnegative $\lambda^* \leq k^{-2}(1 - k)$.*

(iii) *If $k = 0$ then $T : A \cup B \to X$ is weighted λ-accretive for $\lambda : X \times X \to \mathbf{R}_{0+}$ for any $\lambda^* \in \mathbf{R}_+$ and its restriction $T : A \cup B \to A \cup B$ is 2-cyclic 0-contractive.*

(iv) *$T : A \cup B \to X$ is weighted λ-accretive for $\lambda : X \times X \to \mathbf{R}_{0+}$ satisfying $\lambda(x, y) \leq k^{-2}(k_c(x, y) - k)(d(x, y) - d_{AB})$ for some $k_c : X \times X \to [k, \infty)$. The restricted self-mapping $T : A \cup B \to A \cup B$ is also $\overline{k}_c$-contractive with $\overline{k}_c \in [k, \overline{k}_c) \subseteq [k, 1)$ if $k_c : X \times X \to [k, \overline{k}_c)$ with $\overline{k}_c < 1$. Also, $T : A \cup B \to X$ is nonexpansive and weighted λ-accretive for $\lambda : X \times X \to \mathbf{R}_{0+}$ satisfying $\lambda(x, y) \leq k^{-2}(k_c(x, y) - k)(d(x, y) - d_{AB})$ if $k_c : X \times X \to [k, 1]$ which implies, furthermore, that $\lambda : X \times X \to \mathbf{R}_{0+}$ is bounded.*

Proof. Let us denote $d_{AB} := \operatorname{dist}(A, B)$. Consider that the two following relations are verified simultaneously:

$$
\begin{gathered}
d(x, y) \leq d(x + \lambda Tx, y + \lambda Ty) \quad \text{for some } \lambda \in \mathbf{R}_{0+};\ \forall x \in A,\ \forall y \in B, \\
d(x, y) \leq kd(x, y) + (1 - k)d_{AB} \quad \text{for some } k \in [0, 1) \in R,\ \lambda \in \mathbf{R}_{0+};\ \forall x \in A,\ \forall y \in B.
\end{gathered} \tag{2.14}
$$

Since the distance $d : X \times X \to \mathbf{R}_{0+}$ is translation-invariant and homogeneous, then the substitution of (2.14) yields if A and B are disjoint sets, after using the subadditive property of distances, the following chained relationships since $0 \in X$:

$$
\begin{aligned}
d(Tx, Ty) &\leq kd(x, y) + (1 - k)d_{AB} \leq kd(x + \lambda Tx, y + \lambda Ty) + (1 - k)d_{AB} \\
&\leq kd(x + \lambda Tx, y + \lambda Tx + \lambda Ty - \lambda Tx) + (1 - k)d_{AB} \\
&\leq kd(x + \lambda Tx, y + \lambda Tx) + kd(y + \lambda Tx, y + \lambda Tx + \lambda Ty - \lambda Tx) + (1 - k)d_{AB} \\
&= kd(x, y) + k\lambda d(0, \lambda Ty - \lambda Tx) + (1 - k)d_{AB} \\
&\leq kd(x, y) + k^2\lambda d(0, y - x) + (1 - k)d_{AB} \\
&\leq kd(x, y) + k^2\lambda d(x, y) + (1 - k)d_{AB} \leq k(1 + k\lambda)d(x, y) + (1 - k)d_{AB} \\
&\leq k_c d(x, y) + (1 - k)d_{AB};\quad \forall x \in A,\ \forall y \in B;\ \forall \lambda \in [0, \lambda^*],\ \text{for } \lambda^* \leq k^{-2}(1 - k),
\end{aligned} \tag{2.15}
$$

with $k_c := k(1+k\lambda^*) \geq k$. Note from (2.15) that

$$d_{AB} \leq d\left(T^j x, T^j y\right) \leq k_c^j d(x,y) + (1-k) d_{AB} \left(\sum_{i=0}^{j-1} k_c^i\right) = k_c^j d(x,y) + (1-k) d_{AB} \left(\sum_{i=0}^{\infty} k_c^i - \sum_{i=j}^{\infty} k_c^i\right) \leq k_c^j d(x,y) + \frac{(1-k)\left(1-k_c^j\right)}{1-k_c} d_{AB}; \quad \forall x \in A,\ \forall y \in B, \tag{2.16}$$

and, if $k_c < 1$, then

$$d_{AB} \leq \limsup_{j\to\infty}\ d\left(T^j x, T^j y\right) \leq \frac{1-k}{1-k_c} d_{AB} = \frac{1-k}{1-k(1+k\lambda)} d_{AB} < \infty; \quad \forall x \in A, \forall y \in B. \tag{2.17}$$

If $d_{AB} = 0$ then $\lim_{j\to\infty} d(T^j x, T^j y) = 0$. It is first proven that the existence of the limit of the distance implies that of the limit $\lim_{j\to\infty} T^j z$; for all $z \in A \cup B$. Let be $x_j = T^j x$, $y_j = T^j y$ with $x_j, y_j \in A \cup B$. Then,

$$\lim_{j\to\infty} d\left(T^j x, T^j y\right) = \lim_{j\to\infty} d(x_j, y_j) = \lim_{j\to\infty} d\left(T^\ell x_j, T^\ell y_j\right) = 0; \quad \forall \ell \in \mathbf{Z}_{0+} \Longrightarrow \left(x_j\left(= T^j x\right) - y_j\left(= T^j y\right)\right) \longrightarrow T^\ell (x_j - y_j) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty;\ \forall \ell \in \mathbf{Z}_{0+} \tag{2.18}$$

since $T : A \cup B \to A \cup B$ being contractive is globally Lipschitz continuous. Then, $\lim_{j\to\infty} d(T^j x, T^j y) = d(\lim_{j\to\infty} T^j x, \lim_{j\to\infty} T^j y) = 0$ since, because the fact that the metric is translation-invariant, one gets

$$\lim_{j\to\infty} d\left(T^j x, T^j y\right) = d\left(\lim_{j\to\infty} T^j x, \lim_{j\to\infty} T^j y\right) = \lim_{j\to\infty} d\left(0, T^j y - T^j x\right), = d\left(0, \lim_{j\to\infty}\left(T^j y - T^j x\right)\right) = 0. \tag{2.19}$$

As a result, $\lim_{j\to\infty} d(T^j x, T^j y) = 0$ if $d_{AB} = 0$ what implies which $\lim_{j\to\infty}(T^j x - T^j y) = 0$; for all $x \in A$, for all $y \in B$, since $T : A \cup B \to A \cup B$ is globally Lipschitz continuous since it is contractive.

In addition, there exists $\lim_{j\to\infty} T^j x = \lim_{j\to\infty} T^j y = \omega \in A \cup B$; for all $x \in A$, for all $y \in B$. Assume not so that there exists $x \in A$ such that $\neg\exists \lim_{j\to\infty} T^j x$ and there exists a subsequence on nonnegative integers $\{j_k\}_{k\in \mathbf{Z}_{0+}}$ such that $T^{j_k+1} x \neq T^{j_k} x$. If so, one gets by taking $y = Tx \in B$ that $d(T^{j_k}(Tx), T^{j_k} x) > 0$ which contradicts $\lim_{j\to\infty} d(T^j(Tx), T^j x) = 0$. Then $\{T^j x\}_{j\in\mathbf{Z}_{0+}}$ is a Cauchy sequence for any $x \in A \cup B$ and then converges to a limit. Furthermore, $\omega \in A \cup B$ since $T^j(A\cup B) \subseteq A \cup B$ for any $j \in \mathbf{Z}_{0+}$ and as $j \to \infty$ since A and B are nonempty and closed. It has been proven that $\lim_{j\to\infty} T^j x = \lim_{j\to\infty} T^j y = \omega \in A \cup B$; for all $x \in A$, for all $y \in B$.

It is now proven that $\omega = T\omega \in \mathrm{Fix}(T)$. Assume not, then, from triangle inequality,

$$0 < d(T\omega,\omega) \leq d\left(\omega,T^j\omega\right) + d\left(T\omega,T^j\omega\right); \quad \forall j \in \mathbf{Z}_{0+} \Longrightarrow \liminf_{j\to\infty} d\left(\omega,T^j\omega\right) > 0, \tag{2.20}$$

which contradicts $\lim_{j\to\infty} T^j\omega = \omega$ so that $\omega = T\omega \in \mathrm{Fix}(T)$. It is now proven that $\omega \in \mathrm{Fix}(T) \cap (A \cap B)$. Assume not, such that, for instance, $T^jx \in A$ and $T^{j+1}x \in \overline{A} \cap B$. If so, since $T(A) \subseteq B$; $T(B) \subseteq A$, then the existing limit fulfils $\lim_{j\to\infty}T^jx \in A \cap \overline{A}(=\emptyset)$ which is impossible so that there would be no existing limit $\lim_{j\to\infty}T^jx$ in $A\cup B$, contradicting the former result of its existence. Then, $\omega \in \mathrm{Fix}(T) \cap (A \cap B)$ implying that $\mathrm{Fix}(T) \subset A \cap B$.

It is now proven by contradiction that $\omega = \lim_{j\to\infty}T^jx$; for all $x \in A \cup B$ is the unique fixed point of $T : A \cup B \to A \cup B$. Assume that $\exists\omega_1(\neq\omega) \in \mathrm{Fix}(T)$, then $\lim_{j\to\infty}T^jy_1 = \omega_1$ for some $y_1(\neq y) \in B$ with no loss in generality and all $x \in A$. Thus, $\lim_{j\to\infty}d(T^jx,\ T^jy_1) = d(\omega,\ \omega_1) = 0 \Longrightarrow \omega = \omega_1$ which contradicts $\omega \neq \omega_1$ so that $\mathrm{Fix}(T) = \{\omega\}$.

Now, assume that A and B do not intersect so that dist $(A,B) = d_{AB} > 0$. Then, one gets from the first inequality in (2.15) that for all $x \in A$, $y \in B$, one gets

$$\begin{gathered} d\left(T^jx,T^jy\right) \leq k^jd(x,y) + (1-k)d_{AB}\left(\sum_{i=0}^{\infty}k^j\right) = k^jd(x,y) + d_{AB}; \quad \forall j \in \mathbf{Z}, \\ \limsup_{j\to\infty} d\left(T^jx,T^jy\right) \leq d_{AB}. \end{gathered} \tag{2.21}$$

Note that since $T(A) \subseteq B$, $T(B) \subseteq A$ and $\mathrm{dist}(A,B) = d_{AB} > 0$, then $x \in A \Rightarrow T^jx \in A$ and $T^jx \notin B$ if j is even and $T^jx \in B$ and $T^jx \notin A$ if j is odd $y \in B \Rightarrow T^jy \in B$ and $T^jy \notin A$ if j is even and $T^jy \in A$ and $T^jy \notin B$ if j is odd.

Then, T^jx and T^jy are not both in either A or B if x and y are not both in either A or B for any $j \in \mathbf{Z}_{0+}$. As a result, $\lim_{j\to\infty}\sup d(T^jx,T^jy) < d_{AB}$ is impossible so that

$$\exists \lim_{j\to\infty} d\left(T^jx,T^jy\right) = \limsup_{j\to\infty} d\left(T^jx,T^jy\right) = d_{AB} = d(z,Tz), \tag{2.22}$$

for some best proximity points $z \in A$ and $Tz \in B$ or conversely. Then,

$$\begin{aligned} \lim_{j\to\infty} d\left(T^{j+1}x,T^{j+1}y\right) &= \lim_{j\to\infty} d\left(Tz_j,T^2z_j\right) = d_{AB} \leq k\lim_{j\to\infty} d\left(T^jx,T^jy\right) + (1-k)d(z,Tz) \\ &= k\lim_{j\to\infty} d(z_j,Tz_j) + (1-k)d(z,Tz) = k\lim_{j\to\infty} d(z_j,Tz_j) + (1-k)d_{AB}, \end{aligned} \tag{2.23}$$

where $z_j = T^jx$. Thus, $\lim_{j\to\infty}d\ (z_j,Tz_j) = d_{AB} = d(z,Tz)$. It turns out that $\mathrm{dist}(z_j,\mathrm{Fr}(A\cup B)) \to 0$ and $\mathrm{dist}(Tz_j,\mathrm{Fr}(A \cup B)) \to 0$ as $j \to \infty$. Otherwise, it would exist an infinite subsequence $\{d(z_j,Tz_j)\}_{j\,\in\,\widehat{Z}_{0+}}$ of $\{d\ (z_j,Tz_j)\}_{j\,\in\,Z_{0+}}$ with $\widehat{\mathbf{Z}}_{0+}$ being an infinite subset of $\mathbf{Z}_{0+}$ such that $d(z_j,Tz_j) > d_{AB}$ for $j \in \widehat{\mathbf{Z}}_{0+}$. On the other hand, since $(X,\|\ \|)$ is a normed space, then by taking the norm-translation invariant and homogeneous induced metric and since there exists $\lim_{j\to\infty} d\ (T^{j+1}x, T^{j+1}y) = d_{AB}$, it follows that there exist $j_1 \in \mathbf{Z}_{0+}$ and $\delta = \delta(\varepsilon, j_1) \in \mathbf{R}_+$ such that

$$\begin{aligned} 2d_{AB} + \delta < d\left(T^j x + T^{j+1}y, 0\right) \le d\left(T^j x, 0\right) + d\left(T^{j+1}y, 0\right), \\ \le 2(d_{AB} + \delta) \Longrightarrow d\left(T^j x, T^{j+1}y\right) < \varepsilon, \end{aligned} \tag{2.24}$$

for any given $\varepsilon \in \mathbf{R}_+$; for all $x \in A$, for all $y \in B$ with $T^j x \in A$, $T^{j+1}y \in A$ for any even $j(\ge j_1) \in \mathbf{Z}_{0+}$ and $T^j x \in B$, $T^{j+1}y \in B$, for any odd $j(\ge j_1) \in \mathbf{Z}_{0+}$. As a result, by choosing the positive real constant arbitrarily small, one gets that $T^{2j}x \to T^{2j+1}y \to z = z(x,y) \in A$ (a best proximity point of A) and $T^{2j+1}x \to T^{2j}y \to Tz \in B$ (a best proximity point of B), or vice-versa, as $j \to \infty$ for any given $x \in A$ and $y \in B$. A best proximity point $z \in A \cup B$ fulfils $z = T^2 z$. Best proximity points are unique in A and B as it is now proven by contradiction. Assume not, for instance, and with no loss in generality, assume that there exist two distinct best proximity points z_1 and z_2 in A. Then $T^2 z_1 = z_1$ and $T^2 z_1 = z_2$ contradict $z_1 \ne z_2$ so that necessarily $z_1 = T^2 z_1 \ne z_2 = T^2 z_2$. Since $(X, \|\ \|)$ is a uniformly convex Banach space, we take the norm-induced metric to consider such a space as the complete metric space (X, d) to obtain the following contradiction:

$$d_{AB} = d(z_1, Tz_1) = d(z_1, Tz_2) = \left\| \frac{Tz_2 - z_1}{2} + \frac{Tz_2 - z_1}{2} \right\| < 2\left\| \frac{Tz_2 - z_1}{2} \right\| = d_{AB}, \tag{2.25}$$

since $(X, \|\ \|)$ is also a strictly convex Banach space and A and B are nonempty closed and convex sets. Then, $z = T^2 z \in A$ is the unique best proximity point of $T : A \cup B \to A \cup B$ in A and Tz is its unique best proximity point in B. Then, Property (i) has been fully proven. Since A and B are not disjoint, then $d_{AB} = 0$, and $T : A \cup B \to A \cup B$ is k_c-contractive and λ^*-accretive if $\lambda^* = k^{-2}(k_c - k)$ with $k_c \in [k, 1)$. By taking $k_c = 1$, note that $T : A \cup B \to X$ is nonexpansive and $k^{-2}(1-k)$-accretive. Property (ii) has been proven.

To prove Property (iii), we now discuss if

$$\begin{aligned} d(Tx, Ty) \le k(1 + k\lambda)d(x,y) + (1-k)d_{AB} \le k_c d(x,y) + (1 - k_c)d_{AB}; \\ \forall x \in A,\ \forall y \in B. \end{aligned} \tag{2.26}$$

is possible with $1 \ge k_c$ and $d_{AB} > 0$. Note that $d_{AB} = \operatorname{dist}(A, B) = d(z, Tz)$ for some $z \in A$. Define $\overline{d} := \max(d_A, d_B) = k_D d_{AB}$, if $d_{AB} \ne 0$ for some $k_{DA}, k_{DB}, k_D \in \mathbf{R}_+$, where $d_A :=$ diam $A = k_{DA} d_{AB}$ and $d_B :=$ diam $B = k_{DB} d_{AB}$. Three cases can occur in (2.26), namely,

(a) If $k = k_c$ then $k^2 \lambda d(x,y) \le 0 \Leftrightarrow [k\lambda = 0 \vee d(x,y) \le 0]$ which is untrue if $x \ne y$ and $k\lambda > 0$ and it holds for either $k = 0$ or $\lambda = 0$,

(b) $k_c > k$, then (2.26) is equivalent to

$$d(x,y) \ge \frac{k_c - k}{k_c - k(1 + k\lambda)} d_{AB}; \quad \forall x \in A,\ \forall y \in B. \tag{2.27}$$

Take $x \in A$ to be a best proximity point with so that $d(x, Tx) = d_{AB} \ge (k_c - k)/(k_c - k(1 + k\lambda))d_{AB} > d_{AB}$ which is untrue if $k\lambda > 0$ and true for $k\lambda = 0$,

(c) $1 \geq k(1+k\lambda) \geq k_c < k$, then (2.16) is equivalent to $(k-k_c)d_{AB} \geq [k(1+k\lambda)-k_c]d(x,y)$; for all $x \in A$, for all $y \in B$, but $d(x,y) \leq 2\overline{d} + d_{AB} = (2k_D + 1)d_{AB}$. Thus, the above constraint is guaranteed to hold in the worst case if $k - k_c \geq (k + k^2\lambda - k_c)(2k_D + 1) > k - k_c$ which is a contradiction.

Property (iii) follows from the above three cases (a)–(c).

To prove Property (iv), consider again (2.26) by replacing the real constants λ and k_c with the real functions $\lambda : X \times X \to \mathbf{R}_{0+}$ and $k_c : X \times X \to [k,1)$. Note that (2.26) holds through direct calculation if $\lambda(x,y) \leq k^{-2}(k_c(x,y) - k)(d(x,y) - d_{AB})$; for all $x \in A$, for all $y \in B$ for some $k_c : X \times X \to [k,\infty)$. Thus, the self-mapping $T : A \cup B \to X$ is weighted λ-accretive for $\lambda : X \times X \to \mathbf{R}_{0+}$ satisfying $\lambda(x,y) \leq k^{-2}(k_c(x,y)-k)(d(x,y)-d_{AB})$ for some $k_c : X \times X \to [k,\infty)$; and it is also $\overline{k}_c$-contractive with $\overline{k}_c \in [k,\overline{k}_c) \subseteq [k,1)$ if $k_c : X \times X \to [k,\overline{k}_c)$ with $\overline{k}_c < 1$ and nonexpansive if $k_c : X \times X \to [k,1]$. On the other hand, note that $d(x,y) - d_{AB} \leq k_{DA}d_A + k_{DB}d_B \leq 2k_D\overline{d}$. If A and B are bounded and $k_c : X \times X \to [k,1]$, then

$$\begin{aligned} \lambda(x,y) &\leq k^{-2}(k_c(x,y) - k)(d(x,y) - d_{AB}) \leq k^{-2}(k_c(x,y) - k)(k_{DA}d_A + k_{DB}d_B) \\ &\leq 2k^{-2}(k_c(x,y) - k)k_D\overline{d} \leq \infty; \quad \forall x \in A,\ \forall y \in B. \end{aligned} \tag{2.28}$$

Property (iv) has been proven. □

Remark 2.10. Note that Theorem 2.9 (iii) allows to overcome the weakness of Theorem 2.9 (ii) when A and B are disjoint by introducing the concept of weighted accretive mapping since for best proximity points $z \in A \cup B$, $\lambda(z,Tz) = 0$.

Remark 2.11. Note that the assumption that $(X, \| \|)$ is a uniformly convex Banach space could be replaced by a condition of strictly convex Banach space since uniformly convex Banach spaces are reflexive and strictly convex, [18]. In both cases, the existence and uniqueness of best proximity points of the 2-cyclic $T : A \cup B \to A \cup B$ in A and B are obtained provided that both sets are nonempty, convex, and closed.

Remark 2.12. Note that if either A or B is not closed, then its best proximity point of $T : A \cup B \to A \cup B$ is in its closure since $T(A) \subseteq B \subseteq \mathrm{cl}\, B$, $T(B) \subseteq A \subseteq \mathrm{cl}\, A$ leads to $T(A \cup B) \subseteq A \cup B \subseteq \mathrm{cl}(A \cup B)$ and $T^k(A \cup B) \subseteq \mathrm{cl}(A \cup B)$ for finitely many and for infinitely many iterations through the self-mapping $T : A \cup B \to A \cup B$ and Theorem 2.9 is still valid under this extension.

Note that the relevance of iterative processes either in contractive, nonexpansive and pseudocontractive mappings is crucial towards proving convergence of distances and also in the iterative calculations of fixed points of a mapping or common fixed points of several mappings. See, for instance, [19–25] and references therein. Some results on recursive multi-estimation schemes have been obtained in [26]. On the other hand, some recent results on Krasnoselskii-type theorems and related to the statement of general rational cyclic contractive conditions for cyclic self-maps in metric spaces have been obtained in [27] and [28], respectively. Finally, the relevance of certain convergence properties of iterative schemes for accretive mappings in Banach spaces has been discussed in [29] and references therein. The following result is concerned with norm constraints related to 2-cyclic accretive self-mappings which can eventually be also contractive or nonexpansive.

Theorem 2.13. *The following properties hold.*

(i) *Let (X, d) be a metric space endowed with a norm-induced translation-invariant and homogeneous metric $d : X \times X \to \mathbf{R}_{0+}$. Consider the λ^*-accretive mapping $T : A \cup B \to X$ for some $\lambda^* \in \mathbf{R}_{0+}$ which restricted as $T : A \cup B \to A \cup B$ is 2-cyclic, where A and B are nonempty subsets of X subject to $0 \in A \cup B$. Then,*

$$d\Big((I + \lambda T)^j x, 0\Big) \geq 1; \quad \forall j \in \mathbf{Z}_{0+},\ \forall x(\neq 0) \in A \cup B,\ \forall \lambda \in [0, \lambda^*]. \tag{2.29}$$

If, furthermore, $T : A \cup B \to A \cup B$ is k-contractive, then

$$1 \leq d\Big((I + \lambda T)^j x, 0\Big) < k^{-1}; \quad \forall j \in \mathbf{Z}_{+},\ \forall x(\neq 0) \in A \cup B,\ \forall \lambda \in [0, \lambda^*]. \tag{2.30}$$

$T : A \cup B \to A \cup B$ is guaranteed to be nonexpansive (resp., asymptotically nonexpansive) if

$$d\Big((I + \lambda T)^j x, 0\Big) = 1; \quad \forall j \in \mathbf{Z}_{+},\ \forall x(\neq 0) \in A \cup B,\ \forall \lambda \in [0, \lambda^*], \tag{2.31}$$

respectively,

$$\limsup_{j \to \infty} d\Big((I + \lambda T)^j x, 0\Big) = 1; \quad \forall x(\neq 0) \in A \cup B,\ \forall \lambda \in [0, \lambda^*]. \tag{2.32}$$

(ii) *Let $(X, \| \|)$ be a normed vector space. Consider a λ^*-accretive mapping $T : A \cup B \longrightarrow X$ for some $\lambda^* \in \mathbf{R}_{0+}$ which restricted to $T : A \cup B \longrightarrow A \cup B$ is 2-cyclic, where A and B are nonempty subsets of X subject to $0 \in A \cup B$ then*

$$\Big\|(I + \lambda T)^j\Big\| \geq 1; \quad \forall j \in \mathbf{Z}_{0+},\ \forall \lambda \in [0, \lambda^*]. \tag{2.33}$$

If, furthermore, $T : A \cup B \to A \cup B$ is k-contractive, then

$$1 \leq \Big\|(I + \lambda T)^j\Big\| < k^{-1}; \quad \forall j \in \mathbf{Z}_{+},\ \forall \lambda \in [0, \lambda^*]. \tag{2.34}$$

$T : A \cup B \to A \cup B$ is nonexpansive (resp., asymptotically nonexpansive, [30]) if

$$\Big\|(I + \lambda T)^j\Big\| = 1; \quad \forall j \in \mathbf{Z}_{+},\ \forall \lambda \in [0, \lambda^*], \tag{2.35}$$

respectively,

$$\limsup_{j \to \infty} \Big\|(I + \lambda T)^j\Big\| = 1; \quad \forall \lambda \in [0, \lambda^*]. \tag{2.36}$$

Proof. To prove Property (i), define an induced by the metric norm as follows $\|x\| = d(x,0)$ since the metric is homogeneous and translation-invariant. Define the norm of $T : A \cup B \to A \cup B$, that is, the norm of T on X restricted to $A \cup B$ as follows:

$$\begin{aligned} \|T\| : &= \min\{c \in \mathbf{R}_{0+} : \|Tx\| \le c\|x\|;\ \ \forall x \in A \cup B\} \\ &\equiv \min\{c \in \mathbf{R}_{0+} : d(Tx,0) \le cd(x,0);\ \ \forall x \in A \cup B\}, \end{aligned} \tag{2.37}$$

with the above set being closed, nonempty, and bounded from below. Since $T : A \cup B \to A \cup B$ is 2-cyclic and $T : A \cup B \to X$ is λ^*-accretive (Definition 2.2), one gets by proceeding recursively

$$\begin{aligned} d(x,y) &\le d(x + \lambda Tx, y + \lambda Ty) \le d(x + \lambda Tx + \lambda T(x + \lambda Tx), y + \lambda Ty) \\ &= d\Big((I + \lambda T)^2 x, (I + \lambda T)^2 y\Big) \\ &\le \cdots \le d\Big((I + \lambda T)^j x, (I + \lambda T)^j y\Big) \le \Big\|(I + \lambda T)^j\Big\| d(x,y); \\ &\qquad\qquad\qquad \forall x \in A,\ \forall y \in B,\ \forall j \in \mathbf{Z}_{0+},\ \forall \lambda \in [0,\lambda^*], \end{aligned} \tag{2.38}$$

since the metric is homogeneous and $0 \in A \cup B$, and I is the identity operator on X, where

$$\begin{aligned} \Big\|(I + \lambda T)^j\Big\| &:= \min\Big\{c \in \mathbf{R}_{0+} : \Big\|(I + \lambda T)^j x\Big\| \le c\|x\|;\ \forall x \in A \cup B\Big\}, \\ &\equiv \min\Big\{c \in \mathbf{R}_{0+} : d\Big((I + \lambda T)^j x, 0\Big) \le cd(x,0);\ \forall x \in A \cup B\Big\}, \end{aligned} \tag{2.39}$$

with the above set being closed, nonempty, and bounded from below. If $\|(I + \lambda T)^j\| < 1$ for some $\|(I + \lambda T)^j\| < 1$, then we get the contradiction $d(x,y) < d(x,y)$; for all $x \in A$, for all $y \in B$ in (2.38). Thus, $\|(I + \lambda T)^j\| = d((I + \lambda\ T)^j x, 0) \ge 1$; for all $j \in \mathbf{Z}_{0+}$, for all $x(\neq 0) \in A \cup B$, for all $\lambda \in [0,\lambda^*]$. If now x and y are replaced with $T^i x$ and $T^i y$ for any $i \in \mathbf{Z}_{0+}$ in (2.30), one gets if $T : A \cup B \to A \cup B$ is a 2-cyclic k-contractive for some real $k \in [0,1)$ and λ^*-accretive mapping:

$$\begin{aligned} d\Big(T^i x, T^i y\Big) &\le d\Big((I + \lambda T)^j T^i x, (I + \lambda T)^j T^i y\Big) \le \Big\|(I + \lambda T)^j\Big\| d\Big(T^i x, T^i y\Big) \\ &\le k^i \Big\|(I + \lambda T)^j\Big\| d(x,y) + (1 - k)d_{AB} < d(x,y) + (1 - k)d_{AB}, \end{aligned} \tag{2.40}$$

for all $x \in A$, for all $y(\neq x) \in B$, for all $j(\ge i) \in \mathbf{Z}_+$, for all $i \in \mathbf{Z}_+$, for all $\lambda \in [0,\lambda^*]$. Then, $1 \le \|(I + \lambda T)^j\| = d((I + \lambda T)^j x, 0) < k^{-1}$; for all $j \in \mathbf{Z}_+$, for all $x(\neq 0) \in A \cup B$, for all $\lambda \in [0,\lambda^*]$. If $\|(I + \lambda T)^j\| = d((I + \lambda T)^j x, 0) = 1$; for all $j \in \mathbf{Z}_+$, for all $x(\neq 0) \in A \cup B$, for all $\lambda \in [0,\lambda^*]$, it turns out that $T : A \cup B \to X$ is λ^*-accretive and $T : A \cup B \to A \cup B$ is a 2-cyclic nonexpansive self-mapping. It is asymptotically nonexpansive if $\lim_{j\to\infty} \sup\ d((I + \lambda T)^j x, 0) = 1$; for all $x(\neq 0) \in A \cup B$, for all $\lambda \in [0,\lambda^*]$. Property (i) has been proven. The proof of Property (ii) for $(X, \|\ \|)$ being a normed vector space is identical to that of Property (i) without associating the norms to a metric. □

Example 2.14. Assume that $X = \mathbf{R}$, $A = \mathbf{R}_-$, $B = \mathbf{R}_+$ and the 2-cyclic self-mapping $T(\equiv)\ t : (A \cup B) \times \mathbf{Z}_{0+} \to (A \cup B) \times \mathbf{Z}_{0+}$ defined by the iteration rule $A \cup B \ni x_{j+1} = k_j\ x_j \in A \cup B$ with $\mathbf{R} \ni k_j(\in [-k,k]) \leq k \leq 1$, $\operatorname{sgn} k_{j+1} = -\operatorname{sgn} k_j = \operatorname{sgn} x_j$; for all $j \in \mathbf{Z}_{0+}$, and $x_0 \in A \cup B$. Let $d : \mathbf{R}_{0+} \to \mathbf{R}_{0+}$ be the Euclidean metric.

(a) If $k < 1$, then $\lim_{j\to\infty} \prod_{i=0}^{j}[k_j] = 0$ so that for any $x_0 \in A \cup B$, $x_j \in A \cup B$; for all $j \in \mathbf{Z}_{0+} x_j \to z = 0 \notin A \cup B$ as $j \to \infty$ with $0 \in \operatorname{cl}(A \cup B)$, $\operatorname{Fix}(t) = \{0\} \subset \operatorname{cl}(A \cap B)$ but it is not in $A \cap B$ which is empty. If $k = 1$ and $\lim_{j\to\infty} \prod_{i=0}^{j}[k_j] = 0$ (i.e., there are infinitely many values $|k_i|$ being less than unity), then the conclusion is identical. If A and B are redefined as $A = \mathbf{R}_{0-}$, $B = \mathbf{R}_{0+}$, then $\operatorname{Fix}(t) = \{0\} \subset A \cap B \neq \emptyset$.

(b) If $k_j = k = 1$; for all $j \in \mathbf{Z}_{0+}$ the self-mapping $t : (A \cup B) \times \mathbf{Z}_{0+} \to (A \cup B) \times \mathbf{Z}_{0+}$ is not expansive and there is no fixed point.

(c) If $k = 1 - \sigma$ for some $\sigma(< 1) \in \mathbf{R}_+$, then for $\mathbf{R}_0 \ni \lambda \in [0, \lambda^*]$,

$$\begin{gathered} d(tx, ty) \leq K d(x,y) \leq k(1+\lambda)|x-y| \leq k|x-y|; \\ d(x,y) \leq d(x+\lambda x, y+\lambda y) \leq (1+\lambda)|x-y|, \end{gathered} \tag{2.41}$$

so that $t : (A \cup B) \times \mathbf{Z}_{0+} \to (A \cup B) \times \mathbf{Z}_{0+}$ is also λ^*-accretive and $k_1 \in [k, 1)$-contractive with $\lambda^* = k_1 k^{-1} - 1$.

(d) Now, define closed sets $\mathbf{R}_{\varepsilon+} := \{r(\geq \varepsilon) \in \mathbf{R}_+\}$ and $\mathbf{R}_{\varepsilon-} := \{r(\leq -\varepsilon) \in \mathbf{R}_+\}$ for any given $\varepsilon \in \mathbf{R}_{0+}$ so that $d_{AB} = \varepsilon$. The 2-cyclic self-mapping $T(\equiv)\ t : (A \cup B) \times \mathbf{Z}_{0+} \to (A \cup B) \times \mathbf{Z}_{0+}$ is re-defined by the iteration $x_{j+1} = \overline{x}_{j+1}$ if $|x_{j+1}| \geq \varepsilon$ and $x_{j+1} = -\varepsilon \operatorname{sgn} x_j$, for $i = 1, 2$, otherwise, where $\overline{x}_{j+1} = k_j x_j$ for $i = 1, 2$ with the real sequence $\{k_j\}_{j \in \mathbf{Z}_{0+}}$ being subject to $k_j(\in [-k,k]) \leq k \leq 1$, $\operatorname{sgn} k_{j+1} = -\operatorname{sgn} k_j = \operatorname{sgn} x_j^{(i)}$; $i = 1, 2$, for all $j \in \mathbf{Z}_{0+}$ and $x_0 \in A \cup B$.

Then, for any $\varepsilon \in \mathbf{R}_+$ and any $x_0 \in A \cup B$, there are two best proximity points $z = -\varepsilon \in A$ and $z_1 = \varepsilon \in B$ fulfilling $-\varepsilon = t\varepsilon = -t^2\varepsilon$ and $d_{AB} = d(z, z_1) = d(z, tz) = d(tz_1, z_1)$.

(e) Redefine $X = \mathbf{R}^2$ so that $\mathbf{R}^2 \ni x = (x^{(1)}, x^{(2)})^T$ with $x^{(1)}, x^{(2)} \in \mathbf{R}$; $A = \mathbf{R}^2_{\varepsilon-}$, $B = \mathbf{R}^2_{\varepsilon+}$. In the case that $\varepsilon = 0$, then A and B are open disjoint subsets (resp., $A = \mathbf{R}^2_{0-}$, $B = \mathbf{R}^2_{0+}$ are closed nondisjoint subsets with $A \cap B = \{(0,\ x)^T : x \in \mathbf{R}\}$).

The 2-cyclic self-mapping $T(\equiv)\ t : (A \cup B) \times \mathbf{Z}_{0+} \to (A \cup B) \times \mathbf{Z}_{0+}$ is re-defined by the iteration rule:

$$\begin{gathered} x_{j+1}^{(i)} = \overline{x}_{j+1}^{(i)}, \quad \text{if } \left|x_{j+1}^{(i)}\right| \geq \varepsilon, \\ x_{j+1}^{(i)} = -\varepsilon \operatorname{sgn} x_j^{(i)}, \quad \text{for } i = 1, 2, \end{gathered} \tag{2.42}$$

otherwise, where

$$\overline{x}_{j+1}^{(i)} = k_j x_j^{(i)}, \quad \text{for } i = 1, 2 \tag{2.43}$$

with the real sequence $\{k_j\}_{j \in \mathbf{Z}_{0+}}$ being subject to $k_j(\in [-k,k]) \leq k \leq 1$, $\operatorname{sgn} k_{j+1} = -\operatorname{sgn} k_j = \operatorname{sgn} x_j^{(i)}$; for $i = 1, 2$; for all $j \in \mathbf{Z}_{0+}$ and $x_0 \in A \cup B$.

The same parallel conclusions to the above ones (a)–(c) follow related to the existence of the unique fixed point $z = 0$ in the closure of A and B but not in its empty intersection if either A or B is open, respectively, in the intersection of A and B (the vertical real line of zero abscissa) if they are closed. The same conclusion of (d) is valid for the best proximity points if $\varepsilon > 0$.

The following result which leads to elementary tests is immediate from Theorem 2.13.

Corollary 2.15. *The following properties hold.*

(i) *Let $(X, \|\ \|)$ be a normed vector space with (X, d) being the associate metric space endowed with a norm-induced translation-invariant and homogeneous metric $d : X \times X \to \mathbf{R}_{0+}$ and consider the self-mapping $T : X \to X$ so that the restricted $T : A \cup B \to X$ is λ^*-accretive for some $\lambda^* \in \mathbf{R}_{0+}$, where A and B are nonempty subsets of X subject to $0 \in A \cup B$, and the restricted $T : A \cup B \to A \cup B$ is 2-cyclic. Then,*

$$d((I + \lambda T)x, 0) \geq 1; \quad \forall x(\neq 0) \in A \cup B,\ \forall \lambda \in [0, \lambda^*]. \tag{2.44}$$

If, furthermore, $T : A \cup B \to A \cup B$ is k-contractive, then

$$1 \leq d((I + \lambda T)x, 0) < k^{-1}; \quad \forall x(\neq 0) \in A \cup B,\ \forall \lambda \in [0, \lambda^*]. \tag{2.45}$$

$T : A \cup B \to A \cup B$ is guaranteed to be nonexpansive (resp., asymptotically nonexpansive) if

(ii) *Let $(X, \|\ \|)$ be a normed vector space. Then if $T : A \cup B \to X$ is a λ^*-accretive mapping and $T : A \cup B \to A \cup B$ is 2-cyclic for some $\lambda^* \in \mathbf{R}_{0+}$ where A and B are nonempty subsets of X subject to $0 \in A \cup B$, then*

$$\|I + \lambda T\| \geq 1; \quad \forall \lambda \in [0, \lambda^*]. \tag{2.46}$$

If, furthermore, $T : A \cup B \to A \cup B$ is 2-cyclic k ($\in [0, 1)$)-contractive, then

$$1 \leq \|I + \lambda T\| < k^{-1}; \quad \forall \lambda \in [0, \lambda^*]. \tag{2.47}$$

Outline of Proof

It follows since the basic constraint of $T : A \cup B \to X$ being λ^*-accretive holds if

$$\|I + \lambda T\| \geq 1 \Longrightarrow \|I + \lambda T\|^j \geq \left\|(I + \lambda T)^j\right\| \geq 1; \quad \forall j \in \mathbf{Z}_+,\ \forall \lambda \in [0, \lambda^*], \tag{2.48}$$

while it fails if

$$\|I + \lambda T\| < 1 \Longrightarrow \left\|(I + \lambda T)^j\right\| \leq \|I + \lambda T\|^j < 1; \quad \forall j \in \mathbf{Z}_+,\ \forall \lambda \in [0, \lambda^*]. \tag{2.49}$$

Remark 2.16. Theorem 2.13 and Corollary 2.15 are easily linked to Theorem 2.9 as follows. Assume that $T : A \cup B \rightarrow A \cup B$ is 2-cyclic k-contractive and $T : A \cup B \rightarrow X$ is a λ^*-accretive mapping. Assume that there exists $x \in A \cup B$ such that $\|x\| = d(x, 0) \leq 1$. Then, $1 \leq \|I + \lambda T\| < k^{-1}$; for all $\lambda \in [0, \lambda^*]$ from (2.47). This is guaranteed under sufficiency-type conditions with

$$\begin{aligned}
&\|T\| = \max_{\|x\|\leq 1} d(Tx, 0) = \max_{d(x,0)\leq 1} d(Tx, 0) \leq k \\
&\text{if } \ 1 \leq \|I + \lambda T\| \leq 1 + \lambda \|T\| \leq 1 + \lambda k < k^{-1}; \quad \forall \lambda \in [0, \lambda^*], \\
&\quad \Longleftrightarrow (1 + \lambda k) k \leq k_c < 1; \quad \forall \lambda \in [0, \lambda^*],
\end{aligned} \tag{2.50}$$

with $\lambda^* = k^{-2}(k_c - k)$ for some real constants $k_c \in [k, 1)$, $k \in [0, 1)$. It is direct to see that $\mathrm{Fix}(T) = \{0 \in \mathbf{R}^n\}$ if $0 \in A \cap B$.

Example 2.17. Constraint (2.50) linking Theorem 2.13 and Corollary 2.15 to Theorem 2.9 is tested in a simple case as follows. Let $A \equiv \mathrm{Dom}(T) = B \equiv \mathrm{Im}(T) \subset X \equiv \mathbf{R}^n$. $\mathbf{R}^n$ is a vector space endowed with the Euclidean norm induced by the homogeneous and translation-invariant Euclidean metric $\mathrm{d} : X \times X \rightarrow \mathbf{R}_{0+}$. T is a linear self-mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ represented by a nonsingular constant matrix $\mathbf{T}$ in $\mathbf{R}^{n\times n}$. Then, $\|\mathrm{T}\|$ is the spectral (or ℓ_2-) norm of the k-contractive self-mapping $T : X \rightarrow X$ which is the matrix norm induced by the corresponding vector norm (the vector Euclidean norm being identical to the ℓ_2 vector norm as it is well-known) fulfilling

$$\begin{aligned}
\|T\| &= \max_{\mathrm{Dom}(T)\ni\|x\|_2\leq 1} \|Tx\|_2 = \max_{\mathrm{Dom}(T)\ni\|x\|_2 = 1} \|Tx\|_2, \\
&= d(Tx, 0) = \lambda_{\max}^{1/2}\left(\mathbf{T}^T\mathbf{T}\right) \leq k < 1, \\
d\left(T^j x, T^j y\right) &= \left\|T^j (x - y)\right\|_2 \leq \left[\lambda_{\max}^{1/2}\left(T^T\mathbf{T}\right)\right]^j \|x - y\|_2, \\
&= \left[\lambda_{\max}^{1/2}(\mathbf{T}^T\mathbf{T})\right]^j d(x, y) \quad \forall x, y \in \mathrm{Dom}(T) \subset X,
\end{aligned} \tag{2.51}$$

with the symmetric matrix $\mathbf{T}^T\mathbf{T}$ being a matrix having all its eigenvalues positive and less than one, since $\mathbf{T}$ is nonsingular, upper-bounded by a real constant k which is less than one. Thus, $T : A \cup B \rightarrow X$ is also λ^*-accretive for any real constant $\lambda^* < k^{-2}(1-k)$ and k_c-contractive for any real $k_c \in [k, 1)$. Assume now that

$$\mathbf{T} = \mathrm{diag}\left(k_1 k_2 \cdots k_p \ \overbrace{0 \cdots \ 0}^{n-p} \right) \tag{2.52}$$

for some integer $0 < p \leq n$ with

$$A = \mathrm{Dom}(T) = X = \mathbf{R}^n,$$
$$B = \mathrm{Im}(T) = \left\{ x \in X : x = \left(x_1 x_2 \cdots x_p \overbrace{0 \cdots 0}^{n-p} \right)^T \right\} \subset X = \mathbf{R}^n, \tag{2.53}$$

$-k \leq k_i(\neq 0) \leq k < 1$; for all $i \in \overline{p}$. If $p = n$, then $\mathrm{Fix}(T) = \{0 \in \mathbf{R}^n\}$. Also, $\mathrm{Fix}(T) = \{0 \in \mathbf{R}^n\}$for any integer $0 < p < n$ (then $\mathbf{T}$ is singular) but the last $(n-p)$-components of any $x \in A = X = \mathbf{R}^n$ are zeroed at the first iteration via $\mathbf{T}$ so that if e_i is the ith unit vector in $\mathbf{R}^n$ with its ith component being one, then

$$\begin{gathered} e_i^T T^j x \neq 0; \quad \forall i \in \overline{p},\ \forall x(\neq 0) \in \mathbf{R}^n;\ \forall j \in \mathbf{Z}_{0+}, \\ e_i^T T^j x = 0; \quad \forall i(> p) \in \overline{n},\ \forall x \in \mathbf{R}^n;\ \forall j \in \mathbf{Z}_{0+}, \\ T^j x \longrightarrow 0; \quad \forall x \in \mathbf{R}^n \text{ as } j \longrightarrow \infty. \end{gathered} \tag{2.54}$$

Now, assume that the matrix $\mathbf{T}$ is of rank one with its first column being of the form

$$t_1 = \left(k_1 k_2 \cdots k_p \ \overbrace{0 \cdots\ 0}^{n-p} \right)^T \tag{2.55}$$

with $0 < p < n$, $-k \leq k_i(\neq 0) \leq k < 1$; for all $i \in \overline{p}$. Then, (2.54) still holds by changing $x \neq 0$ in the first equation to $x_1 \neq 0$. Finally, assume that

$$\mathbf{T} \ = \ \mathrm{diag}\left(k_1 k_2 \cdots k_p \ \overbrace{1 \cdots\ 1}^{n-p} \right) \tag{2.56}$$

with $0 < p < n$. Then, the self-mapping $T : X \ \rightarrow \ X$ is nonexpansive also noncontractive and $\mathrm{Fix}(T) = \{0 \in \mathbf{R}^p\} \oplus \mathbf{R}^{n-p}$ which is a vector subspace of $\mathbf{R}^n$, that is, there exist infinitely many fixed points each one being reached depending on the initial x in X with the property $\exists \lim_{j \to \infty} T^j x = (0^T, y^T) \in \mathrm{Fix}(T)$ for any given $x = (z^T, y^T)^T \in \mathbf{R}^n$ with $x \in \mathbf{R}^p$, $y \in \mathbf{R}^{n-p}$.

The following result is concerned with the distance boundedness between iterates through the self-mapping $T : A \cup B \ \rightarrow \ A \cup B$.

Theorem 2.18. *Let $(X, \|\ \|)$ be a normed vector space with (X, d) being the associated metric space endowed with a norm-induced translation-invariant and homogeneous metric $d : X \times X \ \rightarrow \ \mathbf{R}_{\,0+}$. Let $T : X \mid A{\cup}B \ \rightarrow \ X \mid A{\cup}B$ be a 2-cyclic k-contractive self-mapping so that $T : A \cup B \rightarrow X$ is λ^*-accretive for some $\lambda^* \in \ \mathbf{R}_{0+}$ where A and B are nonempty subsets of X. Then,*

$$d\left(T^j x, T^{j+1} x\right) \leq m_1 d(x, 0) + m_2; \quad \forall x \in A \cup B;\ \forall j \in \mathbf{Z}_+, \tag{2.57}$$

for some finite real constants $m_1 \in \mathbf{R}_+$, *and* $m_2 \in \mathbf{R}_{0+}$, *which are independent of* x *and the jth power, and* m_2 *is zero if* A *and* B *intersect. Furthermore,* $\lim_{j\to\infty}\sup\, d(T^j x, T^{j+1}x)$ *is finite irrespective of* $x \in A \cup B$.

Proof. One gets for $\lambda \in [0,\lambda^*]$, some $\lambda^* \in \mathbf{R}_{0+}$ and $x \in A \cup B$ that

$$\begin{aligned}
d(x,Tx) &\le d\left(x+\lambda Tx, Tx+\lambda T^2x\right) = d\left(\lambda Tx, Tx+\lambda T^2x - x\right) \\
&= d\left(Tx+(\lambda-1)Tx, T^2x+(\lambda-1)T^2x+Tx-x\right) \\
&= d\left(Tx, T^2x+(1-\lambda)Tx+(\lambda-1)T^2x+Tx-x\right) \\
&\le d\left(Tx,T^2x\right) + d\left(T^2x, T^2x+(1-\lambda)Tx+(\lambda-1)T^2x+Tx-x\right) \\
&= d\left(Tx,T^2x\right) + d\left((\lambda-1)Tx, (\lambda-1)T^2x+Tx-x\right) \\
&= d\left(Tx,T^2x\right) + d\left(\lambda Tx, (\lambda-1)T^2x-x\right) \\
&\le d\left(Tx,T^2x\right) + d\left(\lambda Tx, \lambda T^2x\right) + d\left(\lambda T^2x, (\lambda-1)T^2x-x\right) \\
&= d\left(Tx,T^2x\right) + d\left(\lambda Tx, \lambda T^2x\right) + d\left(T^2x, -x\right) \\
&= d\left(Tx,T^2x\right) + d\left(\lambda Tx, \lambda T^2x\right) + d\left(T^2x, 0\right) + d(x,0) \\
&\le kd(x,Tx) + (1-k)d_{AB} + k\lambda d(x,Tx) + \lambda(1-k)d_{AB} + kd(Tx,0) \\
&\quad + (1-k)d_{AB} + d(x,0) \le kd(x,Tx) + (1-k)d_{AB} + k\lambda d(x,Tx) \\
&\quad +\lambda(1-k)d_{AB} + k^2 d(x,0) + k(1-k)d_{AB} + (1-k)d_{AB} + d(x,0) \\
&\le k(1+\lambda)d(x,Tx) + (1-k)(2+\lambda+k)d_{AB} + \left(k^2+1\right)d(x,0); \quad \forall \lambda \in [0,\lambda^*],
\end{aligned} \tag{2.58}$$

so that one has for $\lambda^* := 1 - k^{-1}\varepsilon$ with $\varepsilon \in [\varepsilon_0, 1)$ for some real constant $\varepsilon_0 \in [0,1)$ provided that $k \in (0,1)$:

$$\begin{aligned}
d(x,Tx) &\le \frac{(2+\lambda+k)(1-k)}{1-k(1-\lambda)} d_{AB} + \frac{k^2+1}{1-k(1-\lambda)} d(x,0) \\
&\le \frac{1}{\varepsilon_0}\left((2+\lambda+k)(1-k)d_{AB} + \left(k^2+1\right)d(x,0)\right),
\end{aligned} \tag{2.59}$$

and if $k = 0$ then

$$d(x,Tx) \le (2+\lambda)d_{AB} + d(x,0); \quad \forall \lambda \in \mathbf{R}_{0+}. \tag{2.60}$$

Also,

$$\begin{aligned} d\left(T^j x, T^{j+1} x\right) &\le \frac{(2+\lambda+k)(1-k)}{1-k(1-\lambda)} d_{AB} + \frac{k^2+1}{1-k(1-\lambda)} d\left(T^j x, 0\right) \\ &\le \left[2+\lambda+k+\left(k^2+1\right)\left(\sum_{i=0}^{j-1} k^i\right)\right] \frac{1-k}{1-k(1-\lambda)} d_{AB} + \frac{\left(k^2+1\right)k^j}{1-k(1-\lambda)} d(x,0) \\ &\le \left[2+\lambda+k+\left(k^2+1\right)\left(\sum_{i=0}^{j-1} k^i\right)\right] \frac{1-k}{\varepsilon_0} d_{AB} + \frac{\left(k^2+1\right)k^j}{\varepsilon_0} d(x,0) \\ &= \left[2+\lambda+k+\frac{\left(k^2+1\right)\left(1-k^j\right)}{1-k}\right] \frac{1-k}{\varepsilon_0} d_{AB} + \frac{\left(k^2+1\right)k^j}{\varepsilon_0} d(x,0) \\ &\le \frac{1}{\varepsilon_0}\left[\left(3+\lambda+\frac{2}{1-k}\right) d_{AB} + 2d(x,0)\right]; \quad \forall \lambda \in [0,\lambda^*],\ \forall j \in \mathbf{Z}_+ \end{aligned} \tag{2.61}$$

$$\limsup_{j\to\infty} d\left(T^j x, T^{j+1} x\right) \le \frac{\left(3+\lambda+k+k^2\right)(1-k)}{\varepsilon_0} d_{AB} \le \frac{5+\lambda}{\varepsilon_0} d_{AB}; \quad \forall \lambda \in [0,\lambda^*] \tag{2.62}$$

if $k \in (0,1)$, and

$$d\left(T^j x, T^{j+1} x\right) \le \frac{3+\lambda}{\varepsilon_0} d_{AB}; \quad \forall \lambda \in \mathbf{R}_{0+},\ \forall j \in \mathbf{Z}_+ \tag{2.63}$$

$$\limsup_{j\to\infty} d\left(T^j x, T^{j+1} x\right) \le \frac{3+\lambda}{\varepsilon_0} d_{AB}; \quad \forall \lambda \in [0,\lambda^*] \tag{2.64}$$

if $k = 0$. □

The subsequent result has a close technique for proof to that of Theorem 2.18.

Theorem 2.19. *Let $(X, \|\ \|)$ be a normed space with an associate metric space (X, d) endowed with a norm-induced translation-invariant and homogeneous metric $d : X \times X \to \mathbf{R}_{0+}$ and let $T : X \to X$ be a self-mapping on X which is k-contractive with $k \in [0, 1/3)$ and 2-cyclic on $A \cup B$, where A and B are nonnecessarily disjoint nonempty subsets of X. If such sets A and B intersect then $T : X \to X$ is also k_c-contractive with $k_c := k/(1-2k) = k/(1-(2+\lambda^*)k) \in [0,1)$ and λ^*-accretive with $\lambda^* = \infty$ if $k_c = k = 0$ and with $\lambda^* = k^{-1} - k_c^{-1} - 2$ if $k \in (0, 1/3)$. Irrespective of A and B being disjoint or not, $T : A \cup B \to X$ is still λ^*-accretive and the following inequalities hold:*

$$d(Tx, Ty) \le k_c d(x,y) + m d_{AB}; \quad \forall (x,y) \in A \times B,\ \forall \lambda \in [0,\lambda^*], \tag{2.65}$$

$$d\left(T^j x, T^j y\right) \le k_c^j d(x,y) + m d_{AB}\left(\sum_{i=0}^{j-1} k_c^i\right) = k_c^j d(x,y) + \frac{1-k_c^j}{1-k_c} m d_{AB}; \quad \forall j \in \mathbf{Z}_+, \quad \forall (x,y) \in A \times B,\ \forall \lambda \in [0,\lambda^*], \tag{2.66}$$

$$\limsup_{j\to\infty} d\left(T^j x, T^j y\right) \le \frac{m d_{AB}}{1-k_c} < \infty; \quad \forall j \in \mathbf{Z}_+,\ \forall (x,y) \in A\times B,\ \forall \lambda \in [0,\lambda^*]. \tag{2.67}$$

Proof. Direct calculations yield

$$\begin{aligned}
d(Tx,Ty) &\le d\left(Tx+\lambda T^2x, Ty+\lambda T^2y\right)\\
&= d\left(T^2x+(\lambda-1)T^2x, T^2y+(\lambda-1)T^2y+Ty-Tx\right)\\
&= d\left(T^2x, T^2y+(\lambda-1)T^2y+(1-\lambda)T^2x+Ty-Tx\right)\\
&\le d\left(T^2x,T^2y\right)+d\left(T^2y, T^2y+(\lambda-1)T^2y+(1-\lambda)T^2x+Ty-Tx\right)\\
&= d\left(T^2x,T^2y\right)+d\left(\lambda T^2x, \lambda T^2y-T^2y+T^2x+Ty-Tx\right)\\
&\le d\left(T^2x,T^2y\right)+\lambda d\left(T^2x,T^2y\right)+d\left(\lambda T^2y, \lambda T^2y-T^2y+T^2x+Ty-Tx\right)\\
&= d\left(T^2x,T^2y\right)+\lambda d\left(T^2x,T^2y\right)+d\left(T^2y-Ty, T^2x-Tx\right)\\
&\le d\left(T^2x,T^2y\right)+\lambda d\left(T^2x,T^2y\right)+d\left(T^2y,T^2x\right)+d(Tx,Ty)\\
&\le (2+\lambda)\left(kd(Tx,Ty)+(1-k)d_{AB}\right)+kd(x,y)+(1-k)d_{AB};\\
&\qquad\qquad \forall (x,y)\in A\times B, \quad \forall \lambda\in[0,\lambda^*],
\end{aligned} \tag{2.68}$$

which leads to the inequalities (2.65)–(2.67) with $k_c := k/(1-(2+\lambda^*)k) \in [0,1)$ and $m := (3+\lambda)(1-k)/kk_c$ where $k_c \in [0,1)$ with $\lambda^* := \infty$ if $k = k_c = 0$ and $\lambda^* := k^{-1}-k_c^{-1}-2 \ge 0$ if $k_c := k/(1-2k) \in (0,1)$ which holds if and only if $k \in (0,1/3)$. The proof is complete. □

Remark 2.20. Compared to Theorem 2.9, Theorem 2.19 guarantees the simultaneous maintenance of the λ^*-accretive and contractive properties if the subsets of X intersect. Otherwise, the contractive property is not guaranteed if $k > 0$ to be λ^*-accretive for the nontrivial case of $\lambda^* > 0$ since md_{AB} is larger than $(1-k_c)d_{AB}$ in general. However, the guaranteed value of λ^* is larger than that guaranteed in Theorem 2.9 to make compatible the accretive and contractive properties of the self-mapping. Also, the relevant properties (2.65)–(2.67) hold irrespective of the sets A and B being bounded or not. Note, in particular, that the uniformly bounded limit superior distance (2.67) is also independent of the boundedness or not of such subsets of X.

The following result follows directly from Theorem 2.9 concerning 2-cyclic Kannan self-mappings which are also contractive (see [16]) which are proven to be accretive.

Theorem 2.21. *Let $(X, \|\,\|)$ be a normed vector space with A and B being bounded nonempty subsets of X and $0 \in A\cup B$. Consider a 2-cyclic $(k<1/3)$-contractive self-mapping $T : A\cup B \to A\cup B$ with $k\in[0,1/3)$. Then, $T : A\cup B \to A\cup B$ is also a $(k_c/(1-k_c),\beta)$-Kannan self-mapping and $T : A\cup B \to X$ is $(1/3-k)k^{-2}$-accretive for $k_c(\in \mathbf{R}_+) = k(1+k\,\lambda^*)$, for all $\beta(\in \mathbf{R}_+) \ge \beta_0 := (2\lambda^*+1-\sqrt{1+4\lambda^* k_c})/2\lambda^*(1-\alpha)$.*

Proof. Since $T : A \cup B \to A \cup B$ is a 2-cyclic $k(<1/3)$-contractive self-mapping, then one gets for that the following relationships hold from the distance sub-additive property from the proof of Theorem 2.9(i), (2.15):

$$\begin{aligned} d(Tx,Ty) &\le kd(x,y) + (1-k)d_{AB} \le kd(x+\lambda Tx, y+\lambda Ty) + (1-k)d_{AB} \\ &\le k_c d(x,y) + (1-k)d_{AB} \\ &\le k_c(d(x,Tx) + d(y,Ty)) + k_c d(Tx,Ty) + (1-k)d_{AB}; \quad \forall x \in A,\ \forall y \in B \\ &\Longrightarrow d(Tx,Ty) \le \alpha(d(x,Tx) + d(y,Ty)) + (1-k)d_{AB} \\ &\le \alpha(d(x,Tx) + d(y,Ty)) + \beta(1-\alpha)d_{AB}; \quad \forall x \in A,\ \forall y \in B, \end{aligned} \tag{2.69}$$

provided that

$$\begin{gathered} \alpha := \frac{k_c}{1-k_c} < \frac{1}{2}\left(\Longrightarrow k_c < \frac{1}{3}\right), \\ k_c := k(1+k\lambda^*), \\ \beta \ge \beta_0 := \frac{2\lambda^* + 1 - \sqrt{1+4\lambda^*}k_c}{2\lambda^*(1-\alpha)}, \end{gathered} \tag{2.70}$$

since $1/3 > k_c := k(1+k\ \lambda^*) \ge k$ if $\lambda^* := (1/3-k)k^{-2}$ so that $T : A \cup B \to X$ is $(1/3-k)k^{-2}$-accretive. Note that the function $k = k(k_c)$ for a contractive self-mapping is the positive solution of $\lambda^* k^2 + k - k_c = 0$, that is, $k = (1+2\lambda^* - \sqrt{1+4\lambda^*}k_c)\ /2\lambda^*$, which is wellposed since $0 \le k < 1$ for $0 \le k_c < 1$. Thus, $T : A \cup B \to A \cup B$ is also a 2-cyclic $(k_c/(1-k_c), \beta)$-Kannan self-mapping from Definition 2.6 since $0 \le k_c < 1/3$ implies $\alpha := k_c/(1-k_c\ ) < 1/2$ with

$$\beta \ge \beta_0 := \frac{2\lambda^* + 1 - \sqrt{1+4\lambda^*}k_c}{2\lambda^*(1-\alpha)}, \quad \lambda^* \le K^{-2}(1-K). \tag{2.71}$$

□

3. Extended Results for p-Cyclic Nonexpansive, Contractive, and Accretive Mappings

This section generalizes the main results of Section 2 to p-cyclic self-mappings with $p \ge 2$. Now, it is assumed that there are p nonempty subsets A_i of X; for all $i \in \overline{p}$ which can be disjoint or not and a so-called p-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$such that $T\ (A_i) \subseteq T(A_{i+1})$ with $A_{p+1} \equiv A_1$. Inspired in the considerations of Remark 2.12 claiming that Theorem 2.9 can be directly extended to the case that the subsets A and B are not necessarily closed, it is not assumed in the sequel that the subsets A_i of X; for all $i \in \overline{p}$ are necessarily closed. A simple notation for distances between adjacent sets is $\operatorname{dist}(A_i, A_{i+1}) = d_{A_iA_{i+1}} := d_i$. Definition 2.4 is generalized as follows.

Definition 3.1. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic weakly k-contractive (resp., weakly nonexpansive) self-mapping if

$$d(Tx,Ty) \le k_i d(x,y) + (1-k_i)d_i; \quad \forall x \in A_i,\ \forall y \in A_{i+1};\ \forall i \in \overline{p}, \tag{3.1}$$

for some real constants $k_i \in \mathbf{R}_{0+}$ (resp., $k_i \in \mathbf{R}_+$); for all $i \in \overline{p}$ [12, 13] such that $k := \prod_{i \in \overline{p}} [k_i] < 1$ (resp., $k = 1$).

Definition 3.2. $T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is a p-cyclic k-contractive (resp., nonexpansive) p-cyclic self-mapping if

$$d(Tx, Ty) \leq k_i d(x, y) + (1 - k_i) d_i; \quad \forall x \in A_i,\ \forall y \in A_{i+1};\ \forall i \in \overline{p}, \tag{3.2}$$

for some real constants $k_i \in [0, 1)$ (resp., $k_i = 1$); for all $i \in \overline{p}$ [12, 13].

Assertion 1. A p-cyclic weakly nonexpansive self-mapping $T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ may be locally expansive for some $(x, y) \in A_i \times A_{i+1}$; for all $i \in \overline{p}$ which cannot be best proximity points.

Proof. Assume that $k_i > 1$. Then, the following inequalities can occur for given $x \in A_i$, $y \in A_{i+1}$:

(1)

$$d(Tx, Ty) \leq k_i d(x, y) + (1 - k_i) d_i \leq d(x, y) \Longrightarrow d(x, y) \leq d_i \Longrightarrow d(Tx, Ty) \leq d_i, \tag{3.3}$$

In this case, and since $d(x, y) < d_i$ is impossible, one concludes that

$$d(Tx, Ty) \leq k_i d(x, y) + (1 - k_i) d_i \leq d(x, y) \Longrightarrow d(x, y) = d_i \Longrightarrow d(Tx, Ty) \leq d_i, \tag{3.4}$$

so that (3.3) can only hold for best proximity points $x \in A_i$, $y = Tx \in A_{i+1}$ for which $T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is nonexpansive. If $d_i = d_{i+1}$, then the last inequality of (3.4) becomes $d(Tx, Ty) = d_i = d_{i+1}$ so that $Ty \in A_{i+2}$ is also a best proximity point if A_i are convex, for all $i \in \overline{p}$,

(2)

$$d(Tx, Ty) \leq d(x, y) \leq k_i d(x, y) + (1 - k_i) d_i \Longrightarrow d(x, y) \geq d_i, \tag{3.5}$$

and then $T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$ is nonexpansive for $(x, y) \in A_i \times A_{i+1}$;

(3)

$$\begin{aligned} d(x, y) < d(Tx, Ty) \leq k_i d(x, y) + (1 - k_i) d_i \Longrightarrow d(x, y) > d_i \Longrightarrow d(Tx, Ty) \\ > d(x, y) > d_i, \end{aligned} \tag{3.6}$$

and then $T : \bigcup_{i \in \overline{p}} A_i \to \bigcup_{i \in \overline{p}} A_i$, is expansive for $(x, y) \in A_i \times A_{i+1}$ which cannot be best proximity points since $d\ (x,\ y) > d_i$. □

Remark 3.3. Note from Definitions 3.1 and 3.2 that a p-cyclic weakly contractive (resp., contractive) self-mapping is also weakly nonexpansive (resp., weakly contractive). Also, a nonexpansive (resp., contractive) self-mapping is also weakly nonexpansive (resp., weakly contractive). Note that if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$, is p-cyclic weakly nonexpansive and $d_i = d_1$; for all $i \in \overline{p}$, then

$$\begin{aligned} d\left(T^j x, T^j y\right) &\leq k_{i+j-1} d\left(T^{j-1}x, T^{j-1}y\right) + (1 - k_{i+j-1}) d_1, \\ &\leq \left(\prod_{i=1}^{j} [k_i]\right) d(x,y) + \left(1 - \prod_{i=1}^{j} [k_i]\right) d_1; \\ &\forall x \in A_i,\ \forall y \in A_{i+1}, \end{aligned} \tag{3.7}$$

where $k_{\ell p+i} = k_i$; for all $\ell \in \mathbf{Z}_{0+}$, for all $i \in \overline{p-1} \cup \{0\}$. Note that if $d(\ T^{j-1}x,\ T^{j-1}y) > d_1$; that is, $T^{j-1}x$, $T^{j-1}y$ are not best proximity points, then if $k_{i+j-1} > 1$, then $d(T^j x, T^j y) > d(T^{j-1}x, T^{j-1}y)$ since $k_{i+j-1} d(T^{j-1}x, T^{j-1}y) + (1 - k_{i+j-1}) d_1 > d(T^{j-1}x, T^{j-1}y)$. Thus, a weakly nonexpansive self-mapping is not necessarily nonexpansive for each iteration. However, the composed self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ defined as $\mathbf{T}x = T(T^{p-1}x) = T^p x$; for all $x \in \bigcup_{i\in\overline{p}} A_i$ is nonexpansive in the usual sense since if $j = p$, then $k = \prod_{i\in\overline{p}} [k_i] = 1$, implies

$$d\left(T^j x, T^j y\right) \leq d\left(T^{j-1}x, T^{j-1}y\right) \leq d(x,y); \quad \forall x \in A_i,\ \forall y \in A_{i+1}. \tag{3.8}$$

It has been commented in Remark 2.12 for the case of 2-cyclic self-mappings that results about best proximity and fixed points are extendable to the case that some of the subsets are not closed by using their closures. We use this idea to formulate the main results for p-cyclic self-mappings with $p \geq 2$. The following technical result stands related to the fact that nonexpansive p-cyclic self-mappings have identical distances between all the adjacent subsets in the set $\{A_i \subset X : i \in \overline{p}\}$.

Lemma 3.4. *Assume that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is p-cyclic and nonexpansive. Then, $d_i = d_1$; for all $i \in \overline{p}$.*

Proof. If $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i \neq \emptyset$ (i.e., the closures of the subsets intersect), then the proof is direct since $d_i = 0$; for all $i \in \overline{p}$. Now, assume that $0 \leq d_j < d_1 \neq 0$ for some $j \in \overline{p}$. Let $z \in A_1$ and $z_1 = Tz \in A_2$ best proximity points such that

$$\begin{aligned} d\left(T^{j+1}z, T^{j+1}z_1\right) = d_j \leq d\left(T^{i+1}z, T^{i+1}z_1\right) = d_i \leq d(Tz, Tz_1) \leq d(z, z_1) = d_1; \\ \forall j, i(<j) \in \overline{p}, \end{aligned} \tag{3.9}$$

since $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic nonexpansive self-mapping. Thus, any iterates $T^j z$ and $T^j z_1$ are also best proximity points of some subset in $\{A_i \subset X : i \in \overline{p}\}$; for all $j \in \mathbf{Z}_+$. If $d_j = d_1$; for all $j \in\ \overline{p}$, does not hold, then from (3.9):

$$d\left(T^{j+1}z, T^{j+1}z_1\right) < d(Tz, Tz_1); \quad \forall j \in \overline{p} \Longrightarrow \exists \lim_{j\to\infty} d\left(Tz, T^j z_1\right) = 0. \tag{3.10}$$

Then $d_i = 0$; for all $i \in \overline{p}$ which contradicts $0 \leq d_j < d_1 \neq 0$ for some $j \in \overline{p}$ what is a contradiction or $d_i = 0$; for all $i \in \overline{p}$, and $\bigcap_{i\in\overline{p}} A_i \neq \emptyset$. □

Note that Lemma 3.4 applies even if the subsets are neither bounded or closed. In this way, note that the contradiction to $0 \le d_j < d_1 \ne 0$ for some $j \in \overline{p}$ established in the second part of the proof does not necessarily imply that $\bigcap_{i\in\overline{p}} \operatorname{cl} A_i \ne \emptyset$ which would require for the subsets A_i, for all $i \in \overline{p}$, to be bounded and, in particular, $\bigcap_{i\in\overline{p}} A_i \ne \emptyset$ if such subsets are bounded and closed. The following result stands concerning the limit iterates of p-cyclic nonexpansive self-mappings:

Lemma 3.5. *The following properties hold.*

(i) *If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic weakly nonexpansive self-mapping, then*

$$\begin{aligned}
\limsup_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) &\le \left(\prod_{\mu=i}^{i+\ell}[k_\mu]\right)\limsup_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) \\
&\quad + \sum_{\mu=i}^{i+\ell}\left(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma]\right)(1-k_\mu)d_\mu \\
&\le \lim_{j\to\infty}\left[\left(\prod_{\mu=i}^{i+\ell}[k_\mu]\right)k^j\right]d(x,y) + \sum_{\mu=i}^{i+\ell}\left(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma]\right)(1-k_\mu)d_\mu \\
&= \sum_{\mu=i}^{i+\ell}\left(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma]\right)(1-k_\mu)d_\mu; \quad \forall x \in A_i,\ \forall y \in A_{i+1}
\end{aligned} \tag{3.11}$$

if $i, \ell \in \overline{p}$ satisfy

$$\begin{gathered}
\sum_{\mu=i}^{i+\ell}\left(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma]\right)(1-k_\mu)d_\mu \ge d_i \\
\limsup_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) \le d\left(T^{\ell p}x, T^{\ell p}y\right) \le d(x,y); \quad \forall x \in A_i,\ \forall y \in A_{i+1}, \\
\forall i \in \overline{p},\ \forall \ell \in \mathbf{Z}_+.
\end{gathered} \tag{3.12}$$

(ii) *If $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic nonexpansive self-mapping, then*

$$\begin{gathered}
\limsup_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) \le d\left(T^{\ell p}x, T^{\ell p}y\right) \le d(x,y); \\
\forall x \in A_i,\ \forall y \in A_{i+1},\ \forall i \in \overline{p},\ \forall \ell \in \mathbf{Z}_+
\end{gathered} \tag{3.13}$$

(iii) *If* $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ *is a p-cyclic weakly contractive self-mapping, then*

$$\limsup_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) \le \sum_{\mu=i}^{i+\ell}\left(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma](1-k_\mu)\right)d_\mu;$$
$$\exists \lim_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) = d_i, \tag{3.14}$$

for all $x \in A_i$, *for all* $y \in A_{i+1}$ *if* $i, \ell \in \overline{p}$ *satisfy the feasibility constraints* $\sum_{\mu=i}^{i+\ell}(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma])(1-k_\mu)d_\mu \ge d_i$ *and* $\sum_{\mu=i}^{i+p}(\prod_{\sigma=\mu+1}^{i+p}[k_\sigma])(1-k_\mu)d_\mu = d_i$. *If* $d_i = d_1$; *for all* $i \in \overline{p}$, *then*

$$\limsup_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) \le d_1\left(\sum_{\mu=i}^{i+\ell}\left(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma](1-k_\mu)\right)\right);$$
$$\exists \lim_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) = d_1 \tag{3.15}$$

for all $x \in A_i$, *for all* $y \in A_{i+1}$, *for all* $\ell, i \in \overline{p}$ *if* $i, \ell \in \overline{p}$ *satisfy the feasibility constraints* $\sum_{\mu=i}^{i+\ell}(\prod_{\sigma=\mu+1}^{i+\ell}[k_\sigma])(1-k_\mu) \ge 1$ *and* $\sum_{\mu=i}^{i+p-1}(\prod_{\sigma=\mu+1}^{i+p-1}[k_\sigma])(1-k_\mu) = 1$

(iv) *If* $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ *is a p-cyclic contractive self-mapping, then*

$$\limsup_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) \le d_1 \Longrightarrow \exists \lim_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) = d_1;$$
$$\forall x \in A_i,\ \forall y \in A_{i+1},\ \forall \ell, i \in \overline{p}. \tag{3.16}$$

(v) *If* $\bigcap_{i\in\overline{p}} A_i \ne \emptyset$ *and* $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ *is a p-cyclic weakly contractive self-mapping, then*

$$\exists \lim_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) = \lim_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) = 0; \quad \forall x \in A_i,\ \forall y \in A_{i+1},\ \forall i, \ell \in \overline{p}. \tag{3.17}$$

Proof. Property (i) follows from (3.7) for $k = \prod_{i\in\overline{p}}[k_i] = 1$. Property (iii) follows from Property (i) since $k < 1$implies $(\prod_{\mu=i}^{p+i}[k_\mu])^j = k^j \to 0$ as $j \to \infty$. Property (ii) Follows from Property (i) for $k_i = 1$; for all $i \in \overline{p}$ since $d_i = d_1$; for all $i \in \overline{p}$ from Lemma 3.4. Property (iv) follows from Property (ii) for $k_i < 1$; for all $i \in \overline{p}$ since $d_i = d_1$; for all $i \in \overline{p}$ from Lemma 3.4.

Property (v) follows from Property (iii) since if all the subsets A_i; $i \in \overline{p}$ intersect, then it follows necessarily $d_i = d_1 = 0$; for all $i \in \overline{p}$ so that

$$\lim_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) = 0; \quad \forall x \in A_i,\ \forall y \in A_{i+1},\ \forall i \in \overline{p}$$
$$\lim_{j\to\infty} d\left(T^{jp+\ell}x, T^{jp+\ell}y\right) \le \left(\prod_{\mu=i}^{\ell}[k_\mu]\right)\lim_{j\to\infty} d\left(T^{jp}x, T^{jp}y\right) = 0; \tag{3.18}$$
$$\forall x \in A_i,\ \forall y \in A_{i+1},\ \forall i, \ell \in \overline{p}.$$

□

Remark 3.6. Note that Lemma 3.5(v) also applies to contractive self-mappings since contractive self-mappings are weakly contractive.

The following result is concerned to the identical distance between adjacent subsets for p-cyclic contractive self-mappings. A parallel result is discussed in [10] for Meir-Keeler contractions.

Theorem 3.7. *Assume that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic weakly k-contractive self-mapping and the closures of the p subsets A_i; $i \in \overline{p}$ of X intersect. Then, it exists a unique fixed point in $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i$ which is also in $\bigcap_{i\in\overline{p}} A_i$ if all such subsets A_i; for all $i \in \overline{p}$ of X, are closed.*

Proof. The existence of a fixed point follows from Lemma 3.5(v). Its uniqueness follows by contradiction. Assume that there exist $z_1, z_2(\ne z_1) \in \mathrm{Fix}(T) \subset \bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i$. Then, for some $i \in \overline{p}$, $\exists x \in A_i$, $y \in A_{i+1}$ such that $T^j x \to z_1$ and $T^j y \to z_2$ as $j \to \infty$. Then, by using triangle inequality for distances,

$$d(z_1, z_2) \le d\left(z_1, T^j x\right) + d\left(T^j x, T^j y\right) + d\left(z_2, T^j y\right); \quad \forall j \in \mathbf{Z}_{0+} \tag{3.19}$$

which implies by using Lemma 3.5(v)

$$d(z_1, z_2) \le \limsup_{j\to\infty}\left(d\left(z_1, T^j x\right) + d\left(T^j x, T^j y\right) + d\left(z_2, T^j y\right)\right) = 0$$
$$\Longrightarrow d(z_1, z_2) \Longleftrightarrow z_1 = z_2, \tag{3.20}$$

what contradicts $z_1 \ne z_2$. Therefore, $\mathrm{Fix}(T)$ consists of a unique point in $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i$ which is also in $\bigcap_{i\in\overline{p}} A_i$ if the sets A_i; $i \in \overline{p}$ are all closed. □

Theorem 3.7 also applies to p-cyclic contractive self-mappings since they are weakly contractive. The following result follows from Theorem 2.9, Lemma 3.5 and some parallel result provided in [12].

Theorem 3.8. *Let $(X, \|\ \|)$ be a uniformly convex Banach space endowed with the translation-invariant and homogeneous metric $d : X \times X \to \mathbf{R}_{0+}$ with nonempty convex subsets $A_i \subset X$, for all $i \in \overline{p}$ of pair-wise disjoint closures. Let $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ be a p-cyclic weakly k-contractive* *self-mapping so that the composed 2-cyclic self-mappings.* $T_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$, *for all* $i \in \overline{p}$ *are defined as* $T_i x = T(T^{p-1}x)$; *for all* $x \in A_i \cup A_{i+1}$; *for all* $i \in \overline{p}$. *Then, the following properties hold:*

(i) *Any composed 2-cyclic self-mapping* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$, $i \in \overline{p}$ *is* k*-contractive provided that the constraint* $\sum_{\mu=i}^{i+p} (\prod_{\sigma=\mu+1}^{i+p} [k_\sigma])(1-k_\mu)d_\mu = d_i$ *holds. If, furthermore, it is assumed that* A_i *and* A_{i+1} *are convex, then the 2-cyclic self-mapping* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ *self-mapping is extendable to* $\mathbf{T}_i : \mathrm{cl}(A_i \cup A_{i+1}) \to \mathrm{cl}(A_i \cup A_{i+1})$, *and that* $T(\mathrm{cl}\, A_i) \subseteq \mathrm{cl}\, A_{i+1}$; *for all* $i \in \overline{p}$. *Thus, the iterates* $\mathbf{T}_i^{\,j} x = T\,(T^{jp-1}x)$ *and* $\mathbf{T}_i^{\,j} y = T(T^{jp-1}y)$; *for all* $x \in A_i$, *for all* $y \in A_{i+1}$ *converge as* $j \to \infty$ *to best proximity points in* $\mathrm{cl}(A_i)$ *and* $\mathrm{cl}(A_{i+1})$ *which are also in* A_i *if* A_i *is closed, respectively, in* A_{i+1} *if* A_{i+1} *is closed.*

(ii) *If for some given* $i \in \overline{p}$, *the sets* A_i *and* A_{i+1} *are convex and closed, if any, then both best proximity points of* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ *of Property (i) are unique and belong, respectively, to* A_i *and* A_{i+1}.

(iii) *Assume that the subsets* A_i *of* X *are convex, for all* $i \in \overline{p}$. *If* $\bigcap_{i \in \overline{p}} \mathrm{cl}\, A_i \neq \emptyset$, *then the best proximity points of Property (i) become a unique fixed point for all the composed 2-cyclic self-mappings* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ *which are* k*-contractive, for all* $i \in \overline{p}$. *Such a fixed point is in* $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i$ *(and also in* $\bigcap_{i\in\overline{p}} A_i$ *if all the subsets* A_i, $i \in \overline{p}$, *are closed).*

Proof. Since $T(A_i) \subseteq A_{i+1}$; for all $i \in \overline{p}$, then for any $i \in \overline{p}$, $x \in A_i \Rightarrow \mathbf{T}_i x \in A_i$ and $x \in A_{i+1} \Rightarrow T_i x \in A_{i+1}$ if p is even and $x \in A_i \Rightarrow T_i x \in A_{i+1}$ and $x \in A_{i+1} \Rightarrow T_i x \in A_i$ if p is odd. Since $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is p-cyclic weakly k-contractive then $k = \prod_{i=1}^{p}[k_i] < 1$, then $T_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ is 2-cyclic contractive provided that $\sum_{\mu=i}^{i+p} (\prod_{\sigma=\mu+1}^{i+p} [k_\sigma])(1-k_\mu)d_\mu = d_i$; $i \in \overline{p}$. One has from Lemma 3.5(iv) that $\exists \lim_{j\to\infty} d\,(T_i^{2j}x,\ T_i^{2j}y) = d_i = d(z, z_1)$; for all $x \in A_i$, for all $y \in A_{i+1}$ for the given $i \in \overline{p}$, where $z = z\ (i) \in \mathrm{cl}(A_i)(z \in A_i$ if A_i is closed), $z_1 = z_1(i) \in \mathrm{cl}(A_{i+1})(z \in \mathrm{cl}(A_i)(z \in A_{i+1}$ if A_{i+1} is closed)) are best proximity points. Using Theorem 2.9(i) for 2-cyclic self-mappings in uniformly convex Banach spaces endowed with translation-invariant and homogeneous metric, one gets $\mathbf{T}_i^{2j} x \to$ z and $\mathbf{T}_i^{2j} y \to z_1$ as $j \to \infty$; for all $i \in \overline{p}$. Property (i) has been proven. Property (ii) was proven in Theorem 3.10, [12] for 2-cyclic k-contractive self- mappings in uniformly convex Banach spaces since they can be directly endowed with a norm-induced metric. The proof is valid here for a norm- induced distance in a uniformly convex Banach space since such distances are translation-invariant and homogeneous. It is also valid if the subsets are not closed with the fixed point then being in the nonempty intersection of their closures. Property (iii) follows directly from Lemma 3.5(v), which implies that $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ is k-contractive for all $i \in \overline{p}$, and the fact that all distances between the closures of all pairs of adjacent subsets are zero since (X, d) is a complete metric space since X is a Banach space. □

Theorem 3.8 also applies to the composed 2-cyclic self-mappings of k-contractive p-cyclic self-mappings. However, we have the following extension containing stronger results for such a case:

Theorem 3.9. *Let* $(X, \|\ \|)$ *be a uniformly convex Banach space endowed with the norm-induced translation-invariant and homogeneous metric* $d : X \times X \to \mathbf{R}_{0+}$ *with nonempty subsets* $A_i \subset X$, *for all* $i \in \overline{p}$ *of pair-wise disjoint closures. Let* $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ *be a* p*-cyclic* k*-contractive self-mapping so that the composed 2-cyclic self-mappings* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$, *for all* $i \in \overline{p}$, *are defined as* $\mathbf{T}_i x = T(T^{p-1}x)$; *for all* $x \in A_i \cup A_{i+1}$; *for all* $i \in \overline{p}$. *Assume also that* A_i *is convex and* $T(\mathrm{cl}\, A_i) \subseteq \mathrm{cl}\ A_{i+1}$; *for all* $i \in \overline{p}$. *Then, the following properties hold.*

(i) *As* $j \to \infty$, *the iterates* $T^j x$ *and* $T^j y$*; for all* $x \in A_i$, *for all* $y \in A_{i+1}$ *converge to best proximity points in* $\mathrm{cl}(A_i)$ *and* $\mathrm{cl}(A_{i+1})$ *which are also in* A_i *if* A_i *is closed, respectively, in* A_{i+1} *if* A_{i+1} *is closed for any* $i \in \overline{p}$. *Also, for any given* $i \in \overline{p}$ *such that the sets* A_i *and* A_{i+1} *are convex and closed, if any, then both best proximity points of* $T : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ *of Property (i) are unique and belong, respectively, to* A_i *and* A_{i+1}. *If, furthermore,* $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i \neq \emptyset$, *then the best proximity points of Property (i) become a unique fixed point for the* p*-cyclic* k*-contractive self-mapping* $T : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$. *Such a fixed point is in* $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i$ *(and also in* $\bigcap_{i\in\overline{p}} A_i$ *if all the subsets* $A_i \subset X$, $i \in \overline{p}$ *are closed).*

(ii) *All the composed 2-cyclic self-mappings* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$, *for all* $i \in \overline{p}$ *are* k*-contractive. Thus, the iterates* ${\mathbf{T}_i}^j x = T(T^{jp-1}x)$ *and* ${\mathbf{T}_i}^j y = T(T^{jp-1}y)$*; for all* $x \in A_i$, *for all* $y \in A_{i+1}$*converge as* $j \to \infty$ *to best proximity points in* $\mathrm{cl}(A_i)$ *and* $\mathrm{cl}(A_{i+1})$ *which are also in* A_i *if* A_i *is closed, respectively in* A_{i+1} *if* A_{i+1} *is closed. For any given* $i \in \overline{p}$ *such that the sets* A_i *and* A_{i+1} *are closed and convex, if any, then both best proximity points of* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ *of Property (i) are unique and belong, respectively, to* A_i *and* A_{i+1}. *If, furthermore,* $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i \neq \emptyset$, *then the best proximity points of Property (i) become a unique fixed point for all the composed 2-cyclic self-mappings* $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ *which are* k*-contractive;* $i \in \overline{p}$. *Such a fixed point is in* $\bigcap_{i\in\overline{p}} \mathrm{cl}\ A_i$ *(and also in* $\bigcap_{i\in\overline{p}} A_i$ *if all the subsets* A_i, $i \in \overline{p}$ *are closed).*

Outline of Proof

Property (ii) is the direct version of Theorem 3.8 applicable to the composed 2-cyclic self-mappings $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$ which are all k-contractive since $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic k-contractive self-mapping. Since p-cyclic contractive self-mappings are nonexpansive, all the distances between adjacent subsets are identical (Lemma 3.4) so that there is no mutual constraint on distances contrarily to Theorem 3.8(i). Property (i) is close to Property (ii) by taking into account that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is also k-contractive.

Definition 2.5 is extended to p-cyclic self-mappings as follows.

Definition 3.10. $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a 2-cyclic (α,β)-Kannan self-mapping for some real $\alpha \in [0,1/2)$ if it satisfies for some $\beta \in \mathbf{R}_+$:

$$d(Tx,Ty) \leq \alpha\bigl(d(x,Tx)+d(y,Ty)\bigr)+\beta(1-\alpha)\,\mathrm{dist}(A,B); \quad \forall x \in A,\ \forall y \in B. \tag{3.21}$$

Now, Theorem 2.9 and Theorems 2.18–2.21 for 2-cyclic accretive and Kannan self-mappings extend directly with direct replacements of their relevant parts as follows.

Theorem 3.11. *Let* $(X,\|\,\|)$ *be a Banach space so that* (X,d)*is its associate complete metric space endowed with a norm-induced translation-invariant and homogeneous metric* $d : X \times X \to \mathbf{R}_{0+}$. *Consider a self-mapping* $T : X \to X$ *which is also a* p*-cyclic* k*-contractive self-mapping if restricted t* $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$, *where* A_i *are nonempty convex subsets of* X*; for all* $i \in \overline{p}$. *Then, Theorem 2.9 holds "mutatis-mutandis" by replacing the subsets* A *and* B *for pairs of adjacent subsets* A_i *and* A_{i+1}, $i \in \overline{p}$, $A \cup B \to \bigcup_{i\in\overline{p}} A_i$, $\mathrm{cl}(A \cap B) \to \mathrm{cl}(\bigcap_{i\in\overline{p}} A_i)$, $A \cap B \to \bigcap_{i\in\overline{p}} A_i$ *and* $k \to k := \prod_{i=1}^{p}[k_i]$. *In the same way, Theorems 2.18, 2.19, and 2.21 still hold.*

The above result extends directly to each composed 2-cyclic self-mappings $\mathbf{T}_i : A_i \cup A_{i+1} \to A_i \cup A_{i+1}$; for all $i \in \overline{p}$ defined from the p-cyclic weak k-contractive self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ since $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$; $i \in \overline{p}$ are k-contractive.

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