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Research Article

Some Applications of $(C_0, 1)$-Semigroups

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This study is originally motivated by the thermal balance equations for the gas and solid interphase heat-transfer for the fast-igniting catalytic converter of automobiles. Instead of solving this problem directly, we proved some results concerning the existence and uniqueness for abstract semilinear initial value problem by means of $(C_0, 1)$-semigroup theories on locally convex topological space. The most enjoyable here is that these results not only can be applied to solve the interphase heat-transfer for the fast-igniting catalytic converter of automobile under the situation of preheating at the entry edge of converter, but also can be applied to some other practical problems.

1. Introduction

Monolithic catalytic reactors are used in a variety of environmental and industrial applications. There has been a great deal of research in catalytic converter technology since in the mid-1970s. The initial few minutes after starting a car when the converter is still cold is very impotent, since during this period the converter is not able to perform its role of converting exhaust carbon monoxide and unburned hydrocarbons to carbon dioxide and water due to low converter temperatures. From an environment point of view how to cope with motor vehicle exhaust emission is an increasing concern in automobile engineering. This concern and many others lead to various mathematical models for the study of interphase heat transfer problem in catalytic converter. There have been suggestions made on how to decrease noxious gas, such as introducing a heater at the inlet. In this study we consider the thermal balance equations for the gas and solid interphase heat-transfer for the fast-igniting catalytic converter of automobiles. This problem can be simplified to the following mathematical model, which was original proposed by Leighton and Chang [1]:

$$\begin{aligned} &\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = cv, \quad 0 < t < T,\ 0 < x \le l, \\ &\frac{\partial v}{\partial t} + bv = bu + \lambda e^{v}, \quad 0 < t < T,\ 0 < x \le l, \\ &u(t,0) = \eta, \quad v(t,0) = 0, \quad 0 < t < T, \\ &u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 < x \le l, \end{aligned} \tag{1.1}$$

where u_0 and v_0 are continuous functions on $[0,l]$ with $u_0(0) = \eta$ and $v_0(0) = 0$. The physical meaning of the functions and parameters in this system was given in Chang et al. [2]. The additional initial condition $v(t,0) = 0$ is understood in the situation of preheating at the entry edge of converter. Instead of solving system (1.1) directly, we proved some results concerning the existence and uniqueness for the classical local solution of the semilinear initial value problem:

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t,u), \quad t \ge 0, \\ u(0) &= u_0 \in D(A), \end{aligned} \tag{1.2}$$

where A is a generator of a $(C_0,1)$-semigroup on a locally convex topological space. Before we consider the semilinear initial value problem (1.2), we consider following abstract Cauchy problem on complete locally convex topological linear space firstly:

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t), \quad t \ge 0, \\ u(0) &= u_0 \in D(A). \end{aligned} \tag{1.3}$$

It is well known that as long as A is a generator of a C_0-semigroup $\{T(t)\}_{t\ \ge 0}$ on a Banach space X, there exists a nonnegative real number ω such that $\{e^{-\omega t}T(t)x; t \ge 0\}$ being bounded in X for every given $x \in X$. But this is not true in general, for example, if X is a complex Hausdorff locally convex topological linear space and $\{T(t)\}_{t\ge 0}$ is a C_0-semigroup on X, then $\{e^{-\omega t}T(t)x;\ t \ge 0\}$ is not bounded for any nonnegative real number ω. Sometimes, complex Hausdorff locally convex topological linear space (hereafter, we will denote it by *lcs*) being a more natural domain for some partial differential operators. Partial differential equations are being currently studied on *lcs*, for example, the spaces of continuous functions, test functions and distributions with nonnormable *lcs* topology. Moreover, in a norm space endow with locally convex topology, using the locally convex topology sometimes is also better than using the norm in certain cases. Certain example was given in [3].

2. Preliminarie

Babalola [4] considers the operator $A = x(\partial/\partial x)$ for the Cauchy problem (1.3), the author showed that A generates a $(C_0,1)$-semigroup $\{T(t)\}_{t\ge 0}$, when X is a *lcs*. Roughly speaking, the $(C_0,1)$-semigroup is a C_0-semigroup on X which can be characterized as having the property that for each continuous seminorm p on X there exists a positive number σ_p (which depends on the seminorm p) and a continuous seminorm q on X such that

$$p(T(t)x) \le e^{\sigma_p t} q(x) \quad \forall x \in X,\ t \ge 0. \tag{2.1}$$

Precise definition of $(C_0, 1)$-semigroup will be given in Definition 2.1. A semigroup $\{T(t)\}_{t\ge0}$ is called an equicontinuous semigroup if $\sigma_p = 0$ in the above inequality, and it is called a quasiequicontinuous semigroup if $e^{\sigma_p t}$ is replaced by $e^{\omega t}$ for some positive constant ω (independent of seminorm p). Follows from these definitions, $(C_0, 1)$-semigroup is a more general than the quasi-equicontinuous semigroup. However, $(C_0, 1)$-semigroup lacks some property which the equicontinuous semigroup has. For instance, the resolvent operator $R(\lambda : A)$ exist for some $\lambda > 0$ as $\{T(t)\}_{t\ge0}$ is an equicontinuous C_0-semigroup but it is not true, while $\{T(t)\}_{t\ge0}$ is a $(C_0, 1)$-semigroup. In fact, let X $= S(R)$ be the Schwartz set of functions with topology determined by the seminorms $\{p_{mn}\}$. The seminorm p_{mn} on X is defined by

$$p_{mn}(f) = \|M^m D^n f\| \text{ for every pair of nonnegative integer } m, n, \tag{2.2}$$

where $(Mf)(x) = xf(x)$, $(Df)(x) = (d/dx)f(x)$, and $\|f\| = (\int_R |f(x)|^2 dx)^2$ for every $f \in$ X. Let the group $G = \{S(\xi) : -\infty < \xi < \infty\}$ be defined by

$$(S(\xi)f)(x) = f\left(e^{\xi} x\right), \quad \forall f \in \text{X}. \tag{2.3}$$

Then $p_{mn}(S(\xi)f) = e^{(n-m-1/2)} p_{mn}(f)$, and hence G is a $(C_0, 1)$ group with generator $A = x(\partial/\partial x)$. However, it is impossible to find any positive number ω such that the group $\{e^{-\omega|\xi|} S(\xi) : |\xi| \ge 0\}$ is equicontinuous. This shows that the resolvent operator $R(\lambda : A)$ does not exist for any $\lambda > 0$.

In view of above evidences, we would like to consider the abstract semilinear initial value problem:

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t, u), \quad t \ge 0, \\ u(0) &= u_0 \in D(A), \end{aligned} \tag{2.4}$$

where A is a generator of $(C_0, 1)$-semigroup in a *lcs* X and $f : [0, \infty) \times \text{X} \to \text{X}$ is a continuous function. For discussing the $(C_0, 1)$-semigroup and its underlying space X, we will use the following terminologies and lemmas in this paper. We say a family of continuous seminorms $\{p_\alpha; \alpha \in \Gamma\}$ is saturated if for any pair α_1,$\alpha_2 \in \Gamma$, there exists a $\alpha_0 \in \Gamma$ such that

$$p_{\alpha_i}(x) \le p_{\alpha_0}(x) \quad (i = 1, 2)\ \forall x \in \text{X}. \tag{2.5}$$

We always assume that X is a *lcs* endowed with a saturated family of continuous seminorms $\{p_\alpha; \alpha \in \Gamma\}$ such that the family $\{\varepsilon V_\alpha; \alpha \in \Gamma,\ \varepsilon > 0\}$ is a base of neighborhoods at the origin for the topology of X, where V_α is the set $\{x \in X : p_\alpha(x) \le 1\}$ for every $\alpha \in \Gamma$. We denote $L_\Gamma(X)$ by the family of linear operators $T : \text{X} \to \text{X}$ such that for each $\alpha \in \Gamma$ there exist positive number $\lambda_{\alpha,T}$ with

$$TV_\alpha \subset \lambda_{\alpha,T} V_\alpha, \quad \forall \alpha \in \Gamma. \tag{2.6}$$

It is easy to see that $T \in L_\Gamma(X)$ is continuous and satisfied:

$$p_\alpha(Tx) \le \lambda_{\alpha,T} p_\alpha(x), \quad \forall x \in X, \ \alpha \in \Gamma. \tag{2.7}$$

Furthermore, with addition of defined pointwise and multiplication by composition, $L_\Gamma(X)$ becomes an algebra, also we can define a topology on $L_\Gamma(X)$. For each $p_\alpha \in \{p_\alpha; \alpha \in \Gamma\}$, the real-valued function P_α on $L_\Gamma(X)$, defined by

$$\begin{aligned} P_\alpha(T) &= \inf\{\lambda_{\alpha,T} : p_\alpha(Tx) \le \lambda_{\alpha,T} p_\alpha(x) \ \forall x \in X\} \\ &= \sup_{x \in X, p_\alpha(x) \le 1} p_\alpha(Tx) \end{aligned} \tag{2.8}$$

is a seminorm with the additional properties that

$$\begin{gathered} P_\alpha(T_1T_2) \le P_\alpha(T_1)P_\alpha(T_2), \quad \forall T_1, T_2 \in L_\Gamma(X), \ \alpha \in \Gamma, \\ P_\alpha(Tx) \le P_\alpha(T)p_\alpha(x), \quad \forall T \in L_\Gamma(X), \ \alpha \in \Gamma. \end{gathered} \tag{2.9}$$

The topology on $L_\Gamma(X)$ is defined by the family $\{P_\alpha : \alpha \in \Gamma\}$ of seminorms on $L_\Gamma(X)$. Under this topology, $L_\Gamma(X)$ becomes a Hausdroff locally multiplicatively convex topological algebra, and $L_\Gamma(X)$ is complete whenever X does.

An *lcs* can be regarded as a projective limit of Banach spaces (see, e.g., [5, page 231]). Firstly, we express the definition of a projective system of spaces and homomorphism. Let $(I, \le)$ be a directed set, and let $(X_\alpha)_{\alpha \in I}$ be a family of normed spaces. Suppose we have a family of homomorphism for all with following properties:

(1) $f_{\alpha\alpha}$ is the identity in X_α;

(2) $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ for all $\alpha \le \beta \le \gamma$.

Then, the set of pair $(X_\alpha, f_{\alpha\beta})$ is called a projective system of normed spaces and homomorphism over I.

Hence, we can define the projective limit (it is also called inverse limit) of the projective system as a particular subspace of the direct product of the X_α's:

$$\lim_{\leftarrow} X_\alpha = \left\{ (X_\alpha) \in \prod_{\alpha \in I} X_\alpha : x_\alpha = f_{\alpha\beta}(x_\beta) \ \forall \alpha \le \beta \right\}. \tag{2.10}$$

For a given *lcs* X, we consider the normed space X_α and the set of all linear operators $L(X_\alpha)$ as follows. For each $\alpha \in \Gamma$, let be the coset of x in the quotient space $X/p_\alpha^{-1}(0)$, and let

$$\|x_\alpha\|_\alpha = p_\alpha(x) \quad \text{for each } x \in X. \tag{2.11}$$

Then, $\|\cdot\|_\alpha$ is a norm on the quotient space $X/p_\alpha^{-1}(0)$. Under the topology induced by $\|\cdot\|_\alpha$, the normed linear space formed by the elements of $X/p_\alpha^{-1}(0)$ is denoted by X_α and denoting its completion by $\overline{X_\alpha}$. For each $\alpha \in \Gamma$, the natural homomorphism $\pi_\alpha : X \to X_\alpha$, defined by $\pi_\alpha(x) = x_\alpha$ for each $x \in X$ is continuous and onto.

Now we can relate $L_\Gamma(X)$ to the uniform norm algebra $L(X_\alpha)$. Let $\alpha \in \Gamma$ and $S \in L_\Gamma(X)$, then the operator $S_\alpha : X_\alpha \to X_\alpha$ defined by $S_\alpha x_\alpha = (Sx)_\alpha$. It is easy to see S_α that in $L(X_\alpha)$

(for detail, please see [4, Proposition 1.12]). Moreover, S_α is uniquely extensible to a bounded linear operator $\overline{S_\alpha}$ on $\overline{X_\alpha}$ such that

$$\left\|\overline{S_\alpha}\right\|_\alpha = \|S_\alpha\|_\alpha = \sup_{\|x\|_\alpha \le 1} \|S_\alpha x_\alpha\|_\alpha = P_\alpha(S). \tag{2.12}$$

To identify $L_\Gamma(X)$ with projective families of operators, we order the index set Γ by $\alpha \ge \alpha'$ if and only if $V_\alpha \subset V_{\alpha'}$. Define for $\alpha \ge \alpha'$, an operator $\pi_{\alpha'\alpha} : X_\alpha \to X_{\alpha'}$ by $\pi_{\alpha'\alpha}(x_\alpha) = x_{\alpha'}$ is a continuous homomorphism also it can be extended to $\overline{X_\alpha}$. The set $(\overline{X_\alpha}, \pi_{\alpha'\alpha})$ consists of a projective system of spaces and homomorphism over Γ. Denote the project limit by $\lim_{\leftarrow} \overline{X_\alpha} = X$. We call a projective family of Banach spaces is saturated if every finite product of members is still a member. Throughout this paper we always assume that our projective family of Banach spaces $\{\overline{X_\alpha} : \alpha \in \Gamma\}$ is saturated, X is a complete *lcs*, and X_α is a Banach space for each $\alpha \in \Gamma$. Let S_α be a linear operator from $D(S_\alpha) \subset X_\alpha$ into $X_\alpha (\alpha \in \Gamma)$. We call $\{S_\alpha : \alpha \in \Gamma\}$ a (saturated) projective family of operators if and only if $S_\alpha(\pi_{\beta\alpha} x_\beta) = \pi_{\beta\alpha}(S_\beta x_\beta)$ for $x_\beta \in D(S_\beta)$ and $\beta \ge \alpha$. For such a family, we can define a linear operator S on the project limit $D(S)$ of $\{S_\alpha : \alpha \in \Gamma\}$ by $\pi_\alpha(Sx) = S_\alpha(\pi_\alpha x)$ for $x \in D(S)$ and $\alpha \in \Gamma$, and we call the operator S be the project limit of the family of operators $\{S_\alpha : \alpha \in \Gamma\}$. If $S_\alpha \in L(X_\alpha)$ for each $\alpha \in \Gamma$, then $S \in L_\Gamma(X)$. Moreover, the family $\{\overline{S_\alpha} : \alpha \in \Gamma\}$ associated with S $\in L_\Gamma(X)$ is projective, and its limit is S.

Now, we are ready to define the $(C_0, 1)$-semigroup on X.

Definition 2.1. The family of continuous linear operators $\{T(t)\}_{t\ge 0} \subset L(X)$ is called a C_0-semigroup if and only if:

(1) $T(t+s)x = T(t)(T(s)x)$ for all $s, t \ge 0$ and $x \in X$;

(2) $T(0)x = x$ for all $x \in X$;

(3) $T(t)x \to x$ as $t \downarrow 0$ for every $x \in X$.

It is called an $L_\Gamma(X)$-operator semigroup of class $(C_0, 1)$ ($(C_0, 1)$-semigroup for shot) if and only if, in addition, for each α and each positive δ, there exists a positive number $\lambda = \lambda(\alpha, \{T(t) : 0 \le t \le \delta\})$ such that $T(t)V_\alpha \subset \lambda V_\alpha$, for all $0 \le t \le \delta$, where $V_\alpha = \{x \in X : p_\alpha(x) \le 1\}$ for every $\alpha \in \Gamma$.

Definition 2.2. We call an operator $S : D(S) \subset X \to X$ s a compartmentalized operator (with respective to $\{p_\alpha : \alpha \in \Gamma\}$ if for each $\alpha \in \Gamma$, the operator $S_\alpha : \pi_\alpha(D(S)) \to X_\alpha$, given by $S_\alpha x_\alpha = (Sx)_\alpha$ for $x_\alpha \in \pi_\alpha(D(S)))$, is well defined. Follows from the definitions of $L_\Gamma(X)$ and compartmentalized operator. It is easy to see that every operator S in $L_\Gamma(X)$ is a compartmentalized operator.

The following results linked the $(C_0, 1)$-semigroup in *lcs* with the well-know C_0-semigroup in Banach space. For details, please see the reference [4].

Lemma 2.3 (see [4, Theorem 2.5]). *There is a 1-1 correspondence between $(C_0, 1)$-semigroup $\{S(\xi) : \xi \ge 0\}$ on complete lcs and projective families of C_0-semigroup$\{S_\alpha(\xi) = \pi_\alpha(S(\xi)) : \xi \ge 0\}$ on Banach space X_α such that if A is the generator of a $(C_0, 1)$ semigroup, and $\{A_\alpha\}$ is the family of generators associated with the corresponding C_0-semigroup on $\{X_\alpha : \alpha \in \Gamma\}$, then $\{A_\alpha\}$ is the projective family, and its limit is A.*

Babalola (see [4, Theorem 2.6]) show that the C_0-semigroup $\{S(\xi) : \xi \geq 0\}$ is a $(C_0, 1)$-semigroup on a locally convex space X if and only if there exist sets $\{M_\alpha; \alpha \in \Gamma\}$ and $\{\sigma_\alpha; \alpha \in \Gamma\}$ of real numbers such that

$$p_\alpha(S(\xi)x) \leq M_\alpha e^{\sigma_\alpha \xi} p_\alpha(x) \quad \forall \xi \geq 0,\ x \in X. \tag{2.13}$$

The author also showed that, by choosing a suitable seminorm q_α on X, the last inequality is equivalent to

$$p(S(\xi)x) \leq e^{\sigma_p \xi} q(x) \quad \forall x \in X,\ \xi \geq 0. \tag{2.14}$$

For details, please see [4, page 171].

From this property, Definition 2.1 is equivalent to the following definition.

Definition 2.4. Let $\{S(\xi) : \xi \geq 0\} \subset L(X)$ be a family of continuous linear operators on X, it is a $(C_0, 1)$-semigroup if and only if it satisfies following conditions

(1) $\{S(\xi) : \xi \geq 0\}$ is a semigroup of class (C_0) in X;

(2) for each continuous seminorm p on X, there exist a nonnegative number σ_p and a continuous seminorm q on X such that

$$p(S(\xi)x) \leq e^{\sigma_p \xi} q(x) \quad \forall \xi \geq 0,\ x \in X. \tag{2.15}$$

Now we can start to consider the abstract semilinear initial value problem (2.4). Suppose X is a complete *lcs* which is the saturated projective limit of Banach spaces $\{X_\alpha; \alpha \in \Gamma\}$. We are searching suitable conditions for the function f such that (2.4) has a mild solution. At first, we consider the function f in (2.4) only depends on the variable t. That is $f : [0, t] \to X$. For any $t_0 \in [0, t]$, in as much as X is the projective limit of X_α, for each $\alpha \in \Gamma$, there is a function $f_\alpha : [0, t] \to X_\alpha$ such that $f_\alpha(t_0) = \pi_\alpha(f(t_0)) \in X_\alpha$, and it satisfies

$$f(t_0) = \varprojlim f_\alpha(t_0) = \varprojlim \pi_\alpha(f(t_0)). \tag{2.16}$$

We assume that for every $\alpha \in \Gamma$, $f_\alpha : [0, t] \to X_\alpha$, is Bochner integrable (integrable for short), and the integration is denoted by $\int_0^t f_\alpha(s)ds$. Since X is the saturated projective limit space of $\{X_\alpha; \alpha \in \Gamma\}$, if $f : [0, t] \to X$ satisfies that for every $\alpha \in \Gamma$, $f_\alpha = \pi_\alpha(f)$ is integrable and $\int_0^t f_\alpha(s)ds = z_\alpha \in X_\alpha$, then

$$z_{\alpha'} = \int_0^t f_{\alpha'}(s)ds = \int_0^t \pi_{\alpha'\alpha}(f_\alpha(s))ds = \pi_{\alpha'\alpha}\left(\int_0^t f_\alpha(s)ds\right) = \pi_{\alpha'\alpha}(z_\alpha), \quad (\alpha \geq \alpha'). \tag{2.17}$$

The same reason also implies that $\{z_\alpha \in X_\alpha; \alpha \in \Gamma\}$ has a limit $z = \lim_{\leftarrow} z_\alpha$ in X. In this case, we say f is integrable and denotes the integration of f on $[0,t]$ by $\int_0^t f(s)ds = z$. Since π_α is the projection from X onto X_α and $f(s) = (f_\alpha(s))_{\alpha\in\Gamma}$, we have

$$\begin{aligned} \int_0^t f(s)ds &= z = \lim_{\leftarrow} z_\alpha = \lim_{\leftarrow} \int_0^t f_\alpha(s)ds = \lim_{\leftarrow} \int_0^t \pi_\alpha(f(s))ds \\ &= \lim_{\leftarrow} \pi_\alpha\left(\int_0^t f(s)ds\right), \end{aligned} \tag{2.18}$$

$$\int_0^t f(s)ds = \int_0^t \lim_{\leftarrow} f_\alpha(s)ds = \int_0^t \lim_{\leftarrow} \pi_\alpha(f(s))ds. \tag{2.19}$$

Combine (2.18) and (2.19), we have

$$\int_0^t \lim_{\leftarrow} \pi_\alpha(f(s))ds = \int_0^t f(s)ds = \lim_{\leftarrow} \pi_\alpha\left(\int_0^t f(s)ds\right). \tag{2.20}$$

We say $f : [0,T] \times X \to X$ is uniformly Lipschitz continuous corresponding to $\alpha \in \Gamma$ with positive Lipschitz constant L_α (independent of s, u, v) if $f_\alpha = \pi_\alpha(f)$ satisfies Lipschitz continuous condition and $0 < \sup_{\alpha\in\Gamma} L_\alpha < \infty$. Moreover, if there exists a constant c (independent of $\alpha \in \Gamma$) such that the Lipschitz constant $L_\alpha(t,c)$ of f_α depends on (t,c), satisfies locally Lipschitz condition and $0 < \sup_{\alpha\in\Gamma} L_\alpha(t,c) < \infty$. Then, we say f is locally Lipschitz continuous.

Definition 2.5. For each fixed $\alpha \in \Gamma$ and positive real number T, we define a norm $\|.\|_{\alpha,\infty}$ on the function space $\{u : [0,T] \to X : u$ is continuous$\}$ by$\|u\|_{\alpha,\infty} = \|\pi_\alpha(u)\|_{\alpha,\infty} = \sup_{s\in[0,T]} \|u_\alpha(s)\|_\alpha$.

3. Main Result

Firstly, we consider the initial value problem:

$$\begin{aligned} \frac{d}{dt}u(t) &= Au(t) + f(t), \quad t \geq 0, \\ u(0) &= u_0 \in D(A), \end{aligned} \tag{3.1}$$

where A is a generator of a $(C_0, 1)$-semigroup on a *lcs* and $f : [0,T] \times X \to X$. We found that (3.1) has a unique mild solution provide the function f is integrable. Furthermore, if f is continuous, then this mild solution is also the solution of the differential equation.

Namely, we have following theorem.

Theorem 3.1. *If A is the generator of a $(C_0, 1)$-semigroup $\{T(t)\}_{t\geq 0}$, and f is integrable, then* (3.1) *has a unique mild solution:*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds. \tag{3.2}$$

Moreover, if f is continuous then $u(t)$ is a classical solution of (3.1).

Proof. For fixed $\alpha \in \Gamma$, we first consider the initial value problem (in X_α):

$$\frac{d}{dt}u_\alpha(t) = A_\alpha u_\alpha(t) + f_\alpha(t), \quad t \geq 0, \\ u_\alpha(0) = u_{0,\alpha} \in D(A_\alpha), \tag{3.3}$$

where A_α is defined as in Lemma 2.3, $u_{0,\alpha} = \pi_\alpha(u_0)$, and $f_\alpha(s) = \pi_\alpha(f(s))$ is integrable. Since X_α is a Banach space, from C_0-semigroup theorems, it is well known that (3.3) has a unique mild solution which is given by

$$u_\alpha(t) = T_\alpha(t)u_{0,\alpha} + \int_0^t T_\alpha(t-s)f_\alpha(s)ds \quad \text{for every } \alpha \in \Gamma. \tag{3.4}$$

Since X is the projective limit of $\{X_\alpha; \alpha \in \Gamma\}$, the projective limit of $\{u_\alpha; \alpha \in \Gamma\}$ exists for each fixed t. We denote it by $u(t)$. Furthermore, since

$$\begin{aligned} u(t) = \varprojlim u_\alpha(t) &= \varprojlim \left(T_\alpha(t)u_{0,\alpha} + \int_0^t T_\alpha(t-s)f_\alpha(s)ds \right) \\ &= \varprojlim \pi_\alpha \left(T(t)u_0 + \int_0^t T(t-s)f(s)ds \right) \\ &= T(t)u_0 + \int_0^t T(t-s)f(s)ds. \end{aligned} \tag{3.5}$$

This shows the projective limit $u(t)$ satisfies (3.2), and hence $u(t)$ is a unique mild solution of (3.1). To see $u(t)$ is a classical solution of (3.1) for f is a continuous function, we need to check $u'(t)$ exists for all $t > 0$ and satisfies (3.1). In fact, we have

$$\begin{aligned} u'(t) &= \lim_{h\to 0}\frac{1}{h}(u(t+h) - u(t)) \\ &= \lim_{h\to 0}\frac{1}{h}\left(T(t+h)u_0 + \int_0^{t+h} T(t+h-s)f(s)ds - T(t)u_0 - \int_0^t T(t-s)f(s)ds \right) \\ &= \lim_{h\to 0}\frac{1}{h}(T(t+h)u_0 - T(t)u_0) \\ &\quad + \lim_{h\to 0}\frac{1}{h}\left(\int_0^{t+h} T(t+h-s)f(s)ds - \int_0^t T(t-s)f(s)ds \right) \\ &= AT(t)u_0 + \lim_{h\to 0}\frac{1}{h}(T(h) - I)\int_0^t T(t-s)f(s)ds \\ &\quad + \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} T(t+h-s)f(s)ds \\ &= AT(t)u_0 + A\int_0^t T(t-s)f(s)ds + f(t) \\ &= Au(t) + f(t). \end{aligned} \tag{3.6}$$

Thus, the derivative of $u(t)$ exists and it satisfies (3.1) as we claimed.

Since global (or local) equicontinuous semigroup is a special case of $(C_0, 1)$-semigroup, we have the following corollary immediately. □

Corollary 3.2. *If A is the generator of a global (locally) equicontinuous semigroup $\{T(t)\}_{t\geq 0}$, and f is integrable, then* (3.1) *has a unique mild solution $u(t)$. Furthermore, if f is continuous, then the mild solution $u(t)$ is a classical solution of* (3.1). *Next we consider the semilinear initial value problem*

$$\frac{d}{dt}u(t) = Au(t) + f(t, u(t)), \quad t \geq 0, \qquad u(0) = u_0 \in D(A), \tag{3.7}$$

where A is a generator of a $(C_0, 1)$-semigroup on a lcs and $f : [0, T] \times X \to X$. We found that (3.7), *having a unique solution, providing the function f; is uniformly Lipschitz continuous. Namely, we have the following theorem.*

Theorem 3.3. *Suppose A is the generator of a $(C_0, 1)$ semigroup $\{T(t)\}_{t\geq 0}$, $f : [t_0, T] \times X \to X$ is continuous in t in the interval $[t_0, T]$, and it satisfies uniformly Lipschitz continuous condition, then* (3.7) *has a uniqe mild solution*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds. \tag{3.8}$$

Proof. For a given $u_0 \in X$, we defined a mapping

$$F : C([t_0, T] : X) \longrightarrow C([t_0, T] : X), \tag{3.9}$$

by

$$Fu(t) = T(t-t_0)u_0 + \int_{t_0}^t T(t-s)f(s, u(s))ds, \quad t_0 \leq t \leq T. \tag{3.10}$$

The projective family of Banach space $\{X_\alpha : \alpha \in \Gamma\}$ is defined in Section 2, then for each $\alpha \in \Gamma$ the mapping

$$F_\alpha = \pi_\alpha \circ F : C([t_0, T] : X_\alpha) \longrightarrow C([t_0, T] : X_\alpha) \tag{3.11}$$

is well defined and satisfies

$$F_\alpha u_\alpha(t) = T_\alpha(t-t_0)u_{0,\alpha} + \int_{t_0}^t T_\alpha(t-s)f_\alpha(s, u_\alpha(s))ds, \quad t_0 \leq t \leq T. \tag{3.12}$$

From (3.12)

$$\begin{aligned}
\|F_\alpha u_\alpha(t) - F_\alpha v_\alpha(t)\|_\alpha &= \|\pi_\alpha(Fu(t) - Fv(t))\|_\alpha \\
&= \left\| \int_{t_0}^{t} T_\alpha(t-s)(f_\alpha(s,u_\alpha(s)) - f_\alpha(s,v_\alpha(s)))ds \right\|_\alpha \\
&\le \int_{t_0}^{t} \left\| T_\alpha(t-s)(f_\alpha(s,u_\alpha(s)) - f_\alpha(s,v_\alpha(s))) \right\|_\alpha ds \\
&\le M_\alpha e^{\sigma_\alpha t} L_\alpha \|u_\alpha - v_\alpha\|_{\alpha,\infty}(t-t_0) \\
&\le K_\alpha L_\alpha \|u_\alpha - v_\alpha\|_{\alpha,\infty}(t-t_0),
\end{aligned} \tag{3.13}$$

where $K_\alpha = M_\alpha e^{\sigma_\alpha T}$. Using (3.12) and (3.13), by induction on n, it follows easily that

$$\begin{aligned}
\|F_\alpha^n u_\alpha(t) - F_\alpha^n v_\alpha(t)\|_\alpha &\le \frac{1}{n!}\,(K_\alpha L_\alpha(t-t_0))^n \|u_\alpha - v_\alpha\|_{\alpha,\infty} \\
&\le \frac{1}{n!}\,(K_\alpha L_\alpha T)^n \|u_\alpha - v_\alpha\|_{\alpha,\infty} \quad \text{for every } \alpha \in \Gamma,
\end{aligned} \tag{3.14}$$

for n large enough $(1/n!)(K_\alpha L_\alpha T)^n < 1$. By a well-known extension of the contraction principle, for every $\alpha \in \Gamma$, F_α has a unique fixed point u_α in $C([t_0,T] : X_\alpha)$ which satisfies

$$u_\alpha(t) = T_\alpha(t-t_0)u_{0,\alpha} + \int_{t_0}^{t} T_\alpha(t-s) f_\alpha(s,u_\alpha(s))ds. \tag{3.15}$$

Since X is the projective limit space of $\{X_\alpha : \alpha \in \Gamma\}$, $\lim_{\leftarrow} u_\alpha(t)$ exists for each fixed $t \in [t_0,T]$. Denote the projective limit by $u(t)$, then it satisfies

$$\begin{aligned}
u(t) = \lim_{\leftarrow} u_\alpha(t) &= \lim_{\leftarrow} \left(T_\alpha(t-t_0)u_{0,\alpha} + \int_{t_0}^{t} T_\alpha(t-s) f_\alpha(s,u_\alpha(s))ds \right) \\
&= T(t-t_0)u_\alpha + \int_{t_0}^{t} T(t-s) f(s,u(s))ds.
\end{aligned} \tag{3.16}$$

This shows that $u(t)$ is the mild solution of (3.1). The uniqueness of $u(t)$ followed from $u_\alpha(t)$ is unique in X_α for each $\alpha \in \Gamma$ and $u(t)$ is the projective limit of $u_\alpha(t)$. Appling the same method as we used in the proof of Theorem 3.1, one may show that $u'(t)$ exists and satisfies the differential equation in (3.7) for all $t > 0$, and hence (3.8) is the solution of (3.7). □

Remark 3.4. Let $f : [t_0,T] \times X \to X$ be a continuous function for t in $[t_0,T]$ and satisfies locally Lipschitz condition uniformly for t on bounded intervals. If A is the generator of a $(C_0,1)$-semigroup $\{T(t)\}_{t\ge 0}$ on X, then for every $u_0 \in X$ there is a $t_{\max} \le \infty$ such that (3.7) has a unique solution $u(t)$ on $[0,t_{\max})$. Furthermore, if $t_{\max}$ is a finite number, then $\lim_{t\to t_{\max}^-} \|u(t)\|_\alpha = \infty$ for some $\alpha \in \Gamma$. This implies that the solution of (3.7) blows up in finite time.

Corollary 3.5. *Suppose A is the generator of a global (locally) equicontinuous semigroup $\{T(t)\}_{t\geq 0}, f : [t_0, T] \times X \to X$ is continuous in t on the interval $[t_0, T]$, and it satisfies uniformly Lipschitz continuous condition, then* (3.7) *has a unique solution $u(t)$.*

4. Applications

Example 4.1. We consider the thermal balance equations for the gas and solid interphase heat-transfer for the fast-igniting catalytic converter of automobiles:

$$\begin{gathered}
\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} + cu = cv, \quad 0 < t < T,\ 0 < x \leq 1, \\
\frac{\partial v}{\partial t} + bv = bu + \lambda e^{v}, \quad 0 < t < T,\ 0 < x \leq l, \\
u(t,0) = \eta, \quad v(t,0) = 0, \quad 0 < t < T, \\
u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad 0 < x \leq l,
\end{gathered} \tag{4.1}$$

where $a,b,c,\lambda,\eta,T > 0$, and $l \geq 0$ are arbitrary given constants; $u_0(x)$ and $v_0(x)$ are known continuous functions on $[0,l]$. Let $Q = [0,T] \times (0,l]$, and let $C^1(Q)$ be the set of functions that are continuously differentiable in Q. To solve the boundary-initial value problem (4.1), we are looking for a pair of functions u, v in $C^1(Q)$, which satisfy the boundary and initial conditions. Denote $Q_\alpha = [0,T] \times [1/\alpha, l]$ for every $\alpha \in N$, then $Q = \cup_{\alpha=1}^{\infty} Q_\alpha$. Let X_α be the space $C^1(Q_\alpha) \times C^1(Q_\alpha)$ endow with the norm

$$\begin{aligned}
\left\| \begin{bmatrix} u_{1\alpha} \\ u_{2\alpha} \end{bmatrix} \right\|_\alpha = \max\Big\{ &\sup\nolimits_{(t,x)\in Q_\alpha}(|u_{i\alpha}(t,x)|), \sup\nolimits_{(t,x)\in Q_\alpha}\left(\left|\frac{\partial}{\partial t}u_{i\alpha}(t,x)\right|\right), \\
&\sup\nolimits_{(t,x)\in Q_\alpha}\left(\left|\frac{\partial}{\partial x}u_{i\alpha}(t,x)\right|\right) \quad \text{for } i = 1,2 \Big\}.
\end{aligned} \tag{4.2}$$

Then X_α is a Banach space for every $\alpha \in N$. We consider a topological space $X = C^1(Q) \times C^1(Q)$ with the seminorms $\{p_\alpha\}_{\alpha\in N}$ which is defined as

$$p_\alpha\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \left\| \begin{bmatrix} u_{1\alpha} \\ u_{2\alpha} \end{bmatrix} \right\|_\alpha \quad \forall \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in X, \ (\text{where } u_{i\alpha} = u_i|_{X_\alpha},\ i = 1,2). \tag{4.3}$$

Then X is a complete topological locally convex space. Let a vector value function $\vec{U}(t,x) = \begin{bmatrix} u(t,x) \\ v(t,x) \end{bmatrix}$ for all $(t,x) \in Q$, then (4.1) can be rewritten as

$$\begin{gathered}
\frac{d}{dt}\vec{U} = A\vec{U} + B\vec{U} + F\left(t,\vec{U}\right), \quad (t,x) \in Q, \\
\vec{U}(0,x) = \begin{bmatrix} u_0(x) \\ v_0(x) \end{bmatrix} \quad 0 < x < l,
\end{gathered} \tag{4.4}$$

where $A = \begin{bmatrix} -a(\partial/\partial x) & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -c & c \\ b & -b \end{bmatrix}$, and $F(t,\vec{U}) = \begin{bmatrix} 0 \\ \lambda e^{v} \end{bmatrix}$ with domain of A as

$$D(A) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \in X : \begin{bmatrix} u(t,0) \\ v(t,0) \end{bmatrix} = \begin{bmatrix} \eta \\ 0 \end{bmatrix} \right\}. \tag{4.5}$$

Let $u_\alpha = u_i|_{Q_\alpha}, v_\alpha = v_i|_{Q_\alpha}$, $\vec{U}_\alpha = \vec{U}(t,x)|_{Q_\alpha} = \begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix} \in X_\alpha$, $B_\alpha = B|_{X_\alpha}$, $F_\alpha(t,\vec{U}) = \begin{bmatrix} 0 \\ \lambda e^{v_\alpha} \end{bmatrix}$ and $A_\alpha = A|_{X_\alpha}$ with domain

$$D(A_\alpha) = \left\{ \begin{bmatrix} u_\alpha \\ v_\alpha \end{bmatrix} \in X : \begin{bmatrix} u\left(t, \dfrac{1}{\alpha}\right) \\ v\left(t, \dfrac{1}{\alpha}\right) \end{bmatrix} = \begin{bmatrix} \eta \\ 0 \end{bmatrix} \right\}. \tag{4.6}$$

It is well known that the Cauchy problem

$$\begin{gathered} \frac{d}{dt} g = -a \frac{d}{dx} g, \\ g(0,x) = g_0(x), \end{gathered} \tag{4.7}$$

on Banach space $C^1(Q_\alpha)$ has a unique solution $g(t,x) = T(t)g_0(x) = g_0(x-at)$, where $\{T(t)\}_{t\geq 0}$ is a C_0-semigroups on $C^1(Q_\alpha)$ generated by $-a(\partial/\partial x)$. This implies that A_α is a generator of a C_0-semigroups on X_α for every $\alpha \in N$. It is obviously that B_α is a bounded operator on X_α, and hence $A_\alpha + B_\alpha$ generates a C_0-semigroup on X_α for every $\alpha \in N$. Follows [4, Theorem 2.6], $A+B$ is a generator of a $(C_0, 1)$-semigroup on X. According to Theorem 3.3, (4.1) has a unique solution on Q as long as $F(t,\vec{U})$ satisfies the local Lipschitz condition. To see this, we apply the identity

$$e^{v_1} - e^{v_2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left((v_1)^n - (v_2)^n \right) = (v_1 - v_2) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n-1} (v_1)^{(n-1)-k} (v_2)^k \right). \tag{4.8}$$

Suppose $\|v_{1\alpha}\|_\alpha, \|v_{2\alpha}\|_\alpha \leq c$ for some constant c and for all $\alpha \in N$, where $\|\cdot\|_\alpha$ is the norm on $C^1(Q_\alpha)$ defined as

$$\|v\|_\alpha = \max\left\{ \sup_{(t,x)\in Q_\alpha} (|v(t,x)|), \sup_{(t,x)\in Q_\alpha} \left(\left| \frac{\partial}{\partial t} v(t,x) \right| \right), \sup_{(t,x)\in Q_\alpha} \left(\left| \frac{\partial}{\partial x} v(t,x) \right| \right) \right\}, \tag{4.9}$$

for every v in $C^1(Q_\alpha)$, then

$$\begin{aligned}\|e^{v_{1\alpha}} - e^{v_{2\alpha}}\|_\alpha &\leq \left\|(v_{1\alpha} - v_{2\alpha})\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n-1}(v_{1\alpha})^{(n-1)-k}(v_{2\alpha})^k\right)\right\|_\alpha \\ &\leq \|(v_{1\alpha} - v_{2\alpha})\|_\alpha\sum_{n=0}^{\infty}\frac{1}{n!}\left\|\left(\sum_{k=0}^{n-1}(v_{1\alpha})^{(n-1)-k}(v_{2\alpha})^k\right)\right\|_\alpha \\ &\leq \|(v_{1\alpha} - v_{2\alpha})\|_\alpha\sum_{n=0}^{\infty}\frac{1}{n!}(c)^n \leq e^c\|(v_{1\alpha} - v_{2\alpha})\|_\alpha.\end{aligned} \tag{4.10}$$

This shows that $F_\alpha(t,\vec{U})$ satisfies local Lipschitz condition on $C^1(Q_\alpha) \times C^1(Q_\alpha)$ for every $\alpha \in N$. It is easy to check that $F(t,\vec{U})$ satisfies local Lipschitz condition on X, and hence, by Theorem 3.3, (4.1) has a unique local solution in X.

Example 4.2. The Lasota equation

$$\frac{\partial}{\partial t}u + x\frac{\partial}{\partial x}u = \lambda u, \tag{4.11}$$

describes the process of reproduction and differentiation of a population of red blood cells. Lasota's equation can be solved by ergodic method (please see Rudnicki [6]). However, we like to apply our Theorem 3.3 to solve this problem. For this purpose, let $A = -x(\partial/\partial x)$ and $f(t,u) = \lambda u$, then the Lasota equation is a special case of (2.4). Let $S(R)$ be the set of all rapidly decreasing test functions whose topology is determined by the seminorms $\{p_{mn}\}$ which is defined as in Section 2. Consider the Lasota equation with the initial value $u(0,x) = u_0(x) \in X = S(R)$. It is an example of Theorem 3.3. Since the function $f(t,u(t)) = \lambda u(t)$ obviously satisfies uniformly Lipschitz condition corresponding to p_{mn} with Lipschitz constant λ (independent of all nonnegative integers m and n), the Lasota equation has a unique solution:

$$u(t,x) = T(t)u_0(x) + \int_0^t T(t-s)f(s,u(s,x))ds. \tag{4.12}$$

In fact, the corresponding $(C_0,1)$-group $\{T(t)\}$ generated by $-x(\partial/\partial x)$ is given by for any $(T(t)u_0)x = u_0(e^{-t}x)$ for $f \in X$. Moreover, we may consider more general initial value problem

$$\begin{gathered}\frac{d}{dt}u = x\frac{d}{dx}u + \varphi(x)u + f(t,u(t,x)), \quad t>0,\ x \in R, \\ u(0,x) = u_0(x), \quad x \in R,\ u_0 \in X,\end{gathered} \tag{4.13}$$

where φ is any given C^∞ function on R possessing bounded derivatives of all orders, and $f(t,u(t,.)) = e^{-t}u^2(t,.)$ for all $u \in X$. Let $Bu = \varphi(x)u$ for every $u \in X$. Babalola [4] showed that $(x(d/dx)+B)$ generates a $(C_0,1)$-semigroup. For any given constant c and nonnegative integers m and n, if $p_{mn}(u)\le c$ and $p_{mn}(v)\le c$, then

$$\begin{aligned} p_{mn}\Big(f\Big(t,u^2(t,x)\Big)-f\Big(t,v^2(t,x)\Big)\Big) &= p_{mn}\Big(e^{-t}u^2(t,x)-e^{-t}v^2(t,x)\Big) \\ &= e^{-t}p_{mn}(u(t,x)-v(t,x))(u(t,x)+v(t,x)) \\ &\le p_{mn}(u(t,x)+v(t,x))p_{mn}(u(t,x)-v(t,x)) \\ &\le 2cp_{mn}(u(t,x)-v(t,x)). \end{aligned} \tag{4.14}$$

This shows that f is a locally Lipschitz continuous function with Lipschitz constant $L_{mn}(t,c)=2c$. Notice that the Lipschitz constant $2c$ is independent of t, m, and n. According to Theorem 3.3, (4.13) has a unique local solution.

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