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Research Article

Fekete-Szegő Inequality for a Subclass of p-Valent Analytic Functions

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The main object of this paper is to study Fekete-Szegő problem for the class of p-valent functions. Fekete-Szegő inequality of several classes is obtained as special cases from our results. Applications of the results are also obtained on the class defined by convolution.

1. Introduction and Preliminaries

Let A_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic in the open unit disk E. Also, $A_1 = A$, the usual class of analytic functions defined in the open unit disk $E = \{z : |z| < 1\}$. Let $f(z)$ and $g(z)$ be analytic in E. We say that the function f is subordinate to the function g and write $f(z) \prec g(z)$, if and only if there exists Schwarz function w, analytic in E such that $w(0) = 0$, $|w(z)| < 1$ for $z \in E$, and $f(z) = g(w(z))$. In particular, if g is univalent in E, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0), \qquad f(E) \subset g(E). \tag{2}$$

For any two analytic functions $f(z)$ of the form (1) and $g(z)$ with

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \quad z \in E, \tag{3}$$

the convolution (Hadamard product) is given by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in E. \tag{4}$$

Let $\phi(z)$ be an analytic function with positive real part on E with $\phi(0) = 1$, $\phi'(0) > 0$, which maps the unit disk E onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Denote by $S_p^*(\phi)$ the class of functions f analytic in E for which

$$\frac{zf'(z)}{pf(z)} \prec \phi(z), \quad z \in E. \tag{5}$$

The class $S_p^*(\phi)$ was defined and studied by Ali et al. [1]. They obtained the Fekete-Szegő inequality for functions in the class $S_p^*(\phi)$. The class $S_1^*(\phi)$ coincides with the class $S^*(\phi)$ discussed by Ma and Minda [2]. Owa [3] introduced a subclass of p-valently Bazilevic functions $H_p(A, B, \alpha, \beta)$. A function $f \in A_p$ is said to be in the class $H_p(A, B, \alpha, \beta)$ if and only if

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} \prec \frac{1+Az}{1+Bz}, \quad z \in E, \tag{6}$$

where $-1 \le B < A \le 1$, $\alpha \ge 0$, and $0 \le \beta \le 1$. We now define the following subclass of analytic functions.

Definition 1. Let $\phi(z)$ be a univalent starlike function with respect to 1 which maps the unit disk E onto a region in the right half-plane which is symmetric with respect to the real axis with $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in A_p$ is in the class $V_{p,b,\alpha,\beta}(\phi)$ if

$$1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}\right\} \prec \phi(z), \quad (7)$$

where $0 \leq \beta \leq 1$, $\alpha \geq 0$, and $b > 0$.

Definition 2. A function $f \in A_p$ is in the class $V_{p,b,\alpha,\beta,g}(\phi)$ if

$$1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{(f*g)(z)}{z^p}\right)^{\alpha} + \beta\frac{z(f*g)'(z)}{p(f*g)(z)}\left(\frac{(f*g)(z)}{z^p}\right)^{\alpha}\right\} \prec \phi(z), \quad (8)$$

where $0 \leq \beta \leq 1$, $\alpha \geq 0$, and $b > 0$. In other words, a function $f \in A_p$ is in the class $V_{p,b,\alpha,\beta,g}(\phi)$ if $(f*g)(z) \in V_{p,b,\alpha,\beta}(\phi)$.

We have the following special cases.

(i) $V_{p,2,1,1}(\phi)$ coincides with the class $S_p^*(\phi)$ introduced and studied by Ali et al. [1].

(ii) For $p = 1$, $b = 2$, and $\beta = 1$, we have the class $B_\alpha(\phi)$ introduced and studied by Ravichandran et al. [4].

(iii) For $b = 2$ and $\phi(z) = (1 + Az)/(1 + Bz)$, the class $V_{p,2,\alpha,\beta}(\phi)$ reduces to $H_p(A, B, \alpha, \beta)$ introduced and studied by Owa [3].

(iv) For $\phi(z) = (1 + (1 - 2\gamma)z)/(1 - z)$, the class $V_{p,2,\alpha,1}(\phi)$ reduces to the class $B_p(\alpha, \gamma)$ defined as

$$B_p(\alpha,\gamma) = \left\{f \in A_p : \operatorname{Re}\left(\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha}\right) > \gamma, \ 0 \leq \gamma < 1, \ z \in E\right\}. \quad (9)$$

(v) For $\phi(z) = (1 + (1 - 2\gamma)z)/(1 - z)$, the class $V_{p,2,\alpha,0}(\phi)$ is defined as

$$\left\{f \in A_p : \operatorname{Re}\left(\frac{f(z)}{z^p}\right)^{\alpha} > \gamma, \ 0 \leq \gamma < 1, \ z \in E\right\}. \quad (10)$$

(vi) $V_{1,2,0,1}(\phi) = S^*(\phi)$ is investigated by Ma and Minda [2].

(vii) For $\phi(z) = (1 + (1 - 2\gamma)z)/(1 - z)$, the class $V_{1,2,1,0}(\phi)$ reduces to the class

$$B_\gamma = \left\{f \in A_1 : \operatorname{Re}\frac{f(z)}{z} > \gamma\right\}. \quad (11)$$

studied by Chen [5].

We need the following results to obtain our main results.

Lemma 3 (see [1]). *Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, satisfying condition $|w(z)| < 1$. If $w \in \Omega$ and $w(z) = w_1 z + w_2 z^2 + \cdots$, $z \in E$, then*

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t, & t < -1, \\ 1, & -1 \leq t \leq 1, \\ t, & t > 1. \end{cases} \quad (12)$$

For $t \leq -1$ or $t \geq 1$, the equality holds, if and only if $w(z) = z$ or one of its rotation. For $-1 \leq t \leq 1$, the equality holds, if $w(z) = z^2$ or one of its rotation. the equality holds for $t = -1$, if and only if $w(z) = z((\lambda + z)/(1 + \lambda z))$ $(0 \leq \lambda \leq 1)$ or one of its rotation, while for $t = 1$, the equality holds, if and only if $w(z) = -z((\lambda + z)/(1 + \lambda z))$ $(0 \leq \lambda \leq 1)$ or one of its rotation. The above upper bound for $-1 < t < 1$ is sharp, and it can be improved as follows:

$$\begin{aligned} &\left|w_2 - t w_1^2\right| + (1+t)\left|w_1\right|^2 \leq 1, \quad -1 < t \leq 0, \\ &\left|w_2 - t w_1^2\right| + (1-t)\left|w_1\right|^2 \leq 1, \quad 0 < t < 1. \end{aligned} \quad (13)$$

Lemma 4 (see [6, (7), page 10]). *If $w \in \Omega$ and $w(z) = w_1 z + w_2 z^2 + \cdots$, $z \in E$, then*

$$\left|w_2 - t w_1^2\right| \leq \max\{1; |t|\}, \quad (14)$$

for any complex number t. The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Lemma 5 (see [7]). *If $w \in \Omega$, then for any real number q_1 and q_2, the following sharp estimate holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leq H(q_1, q_2), \quad (15)$$

where

$$H(q_1,q_2) = \begin{cases} 1, & (q_1,q_2) \in D_1 \cup D_2, \\ |q_2|, & (q_1,q_2) \in \bigcup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{1/2}, & (q_1,q_2) \in D_8 \cup D_9, \\ \frac{1}{3}q_2\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{1/2}, & (q_1,q_2) \in D_{10} \cup D_{11} \setminus \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{1/2}, & (q_1,q_2) \in D_{12}. \end{cases} \quad (16)$$

The extremal function up to the rotations is of the form

$$w(z) = z^3, \qquad w(z) = z,$$
$$w(z) = w_0(z) = \frac{z([(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2] z},$$
$$w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z},$$
$$w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z},$$
$$|\varepsilon_1| = |\varepsilon_2| = 1, \qquad \varepsilon_1 = t_0 - e^{-i\theta_0/2}(a \mp b),$$
$$\varepsilon_2 = -e^{-i\theta_0/2}(ia \pm b),$$
$$a = t_0 \cos\frac{\theta_0}{2}, \qquad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \qquad \lambda = \frac{b \pm a}{2b}, \tag{17}$$
$$t_0 = \left(\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right)^{1/2},$$
$$t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{1/2},$$
$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{1/2},$$
$$\cos\frac{\theta_0}{2} = \frac{q_1}{2}\left(\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right).$$

The sets D_k, $k = 1, 2, \ldots, 12$ are defined as follows:

$$D_1 = \left\{(q_1, q_2) : |q_1| \le \frac{1}{2},\ |q_2| \le 1\right\},$$
$$D_2 = \left\{(q_1, q_2) : \frac{1}{2} \le |q_1| \le 2,\ \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \le q_2 \le 1\right\},$$
$$D_3 = \left\{(q_1, q_2) : |q_1| \le \frac{1}{2},\ |q_2| \le -1\right\},$$
$$D_4 = \left\{(q_1, q_2) : |q_1| \ge \frac{1}{2},\ q_2 \le -\frac{2}{3}(|q_1|+1)\right\},$$
$$D_5 = \{(q_1, q_2) : |q_1| \le 2,\ q_2 \ge 1\},$$
$$D_6 = \left\{(q_1, q_2) : 2 \le |q_1| \le 4,\ q_2 \ge \frac{1}{12}(q_1^2+8)\right\},$$
$$D_7 = \left\{(q_1, q_2) : |q_1| \ge 4,\ q_2 \ge \frac{2}{3}(|q_1|-1)\right\},$$
$$D_8 = \left\{(q_1, q_2) : \frac{1}{2} \le |q_1| \le 2,\ -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\right\},$$
$$D_9 = \left\{(q_1, q_2) : |q_1| \ge 2,\ -\frac{2}{3}(|q_1|+1) \le q_2 \le \frac{2|q_1|(|q_1|+1)}{q_1^2 + 2|q_1| + 4}\right\},$$
$$D_{10} = \left\{(q_1, q_2) : 2 \le |q_1| \le 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2 + 2|q_1| + 4} \le q_2 \le \frac{1}{12}(q_1^2+8)\right\},$$
$$D_{11} = \left\{(q_1, q_2) : |q_1| \ge 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2 + 2|q_1| + 4} \le q_2 \le \frac{2|q_1|(|q_1|-1)}{q_1^2 - 2|q_1| + 4}\right\},$$
$$D_{12} = \left\{(q_1, q_2) : |q_1| \ge 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2 - 2|q_1| + 4} \le q_2 \le \frac{2}{3}(|q_1|-1)\right\}. \tag{18}$$

2. Main Results

Theorem 6. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, where B_n, s are real with $B_1 > 0$ and $B_2 \ge 0$. Let*

$$\sigma_1 = \frac{(\alpha p + \beta)^2}{bpB_1^2(\alpha p + 2\beta)} \times \left\{2(B_2 - B_1) - bpB_1^2 \frac{(\alpha-1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2}\right\},$$
$$\sigma_2 = \frac{(\alpha p + \beta)^2}{bpB_1^2(\alpha p + 2\beta)} \times \left\{2(B_2 + B_1) - bpB_1^2 \frac{(\alpha-1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2}\right\},$$
$$\sigma_3 = \frac{(\alpha p + \beta)^2}{bpB_1^2(\alpha p + 2\beta)} \times \left\{2B_2 - bpB_1^2 \frac{(\alpha-1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2}\right\},$$
$$\Phi(p, \alpha, \beta, \mu) = \frac{(\alpha p + 2\beta)(2\mu + \alpha - 1)}{2(\alpha p + \beta)^2}. \tag{19}$$

If $f(z)$ is of the form (1) *and belongs to the class $V_{p,b,\alpha,\beta}(\phi)$, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \begin{cases} \dfrac{bp}{2(\alpha p+2\beta)}\left(B_2-\dfrac{bpB_1^2}{2}\Phi(p,\alpha,\beta,\mu)\right), & \mu<\sigma_1, \\ \dfrac{bpB_1}{2(\alpha p+2\beta)}, & \sigma_1\leq\mu\leq\sigma_2, \\ -\dfrac{bp}{2(\alpha p+2\beta)}\left(B_2-\dfrac{bpB_1^2}{2}\Phi(p,\alpha,\beta,\mu)\right), & \mu>\sigma_2. \end{cases} \tag{20}$$

Furthermore, for $\sigma_1\leq\mu\leq\sigma_3$,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{1}{bpB_1}\left(2\left(1-\frac{B_2}{B_1}\right)\frac{(\alpha p+\beta)^2}{(\alpha p+2\beta)} + \frac{bpB_1}{2}(2\mu+\alpha-1)\right)\left|a_{p+1}\right|^2 \leq \frac{bpB_1}{2(\alpha p+2\beta)}, \tag{21}$$

and for $\sigma_3\leq\mu\leq\sigma_2$,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| + \frac{1}{bpB_1}\left(2\left(1+\frac{B_2}{B_1}\right)\frac{(\alpha p+\beta)^2}{(\alpha p+2\beta)} - \frac{bpB_1}{2}(2\mu+\alpha-1)\right)\left|a_{p+1}\right|^2 \leq \frac{bpB_1}{2(\alpha p+2\beta)}. \tag{22}$$

For any complex number μ,

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \leq \frac{bpB_1}{2(\alpha p+2\beta)}\max\left\{1,\left|\frac{bpB_1}{2}\Phi(p,\alpha,\beta,\mu)-\frac{B_2}{B_1}\right|\right\}. \tag{23}$$

Also,

$$\left|a_{p+3}\right| \leq \frac{bpB_1}{2(\alpha p+3\beta)}H(q_1,q_2), \tag{24}$$

where $H(q_1,q_2)$ is defined in Lemma 3 and

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1}+\frac{bpB_1}{2}\frac{(1-\alpha)(\alpha p+3\beta)}{(\alpha p+\beta)(\alpha p+2\beta)}, \\ q_2 &= \frac{B_3}{B_1}+\left(\frac{bpB_1}{2}\right)^2\frac{(\alpha-1)(2\alpha-1)(\alpha p+3\beta)}{6(\alpha p+\beta)^3} \\ &\quad+\frac{bpB_2}{2}\frac{(1-\alpha)(\alpha p+3\beta)}{(\alpha p+\beta)(\alpha p+2\beta)}. \end{aligned} \tag{25}$$

These results are sharp.

Proof. Since $f\in V_{p,b,\alpha,\beta}(\phi)$, therefore we have for a Schwarz function

$$w(z)=w_1z+w_2z^2+\cdots,\qquad z\in E \tag{26}$$

such that

$$1-\frac{2}{b}+\frac{2}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^\alpha+\beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^\alpha\right\} = \phi(w(z)). \tag{27}$$

Now,

$$\begin{aligned} &1-\frac{2}{b}+\frac{2}{b}\left\{(1-\beta)\left(\frac{f(z)}{z^p}\right)^\alpha+\beta\frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^\alpha\right\} \\ &\quad=1+\frac{2}{bp}(\alpha p+\beta)a_{p+1}z+\frac{1}{bp}(\alpha p+2\beta) \\ &\quad\times\left\{2a_{p+2}+(\alpha-1)a_{p+1}^2\right\}z^2+\frac{2(\alpha p+3\beta)}{bp} \\ &\quad\times\left\{a_{p+3}+(\alpha-1)a_{p+1}a_{p+2}+\frac{(\alpha-1)(\alpha-2)}{6}a_{p+1}^3\right\}z^3+\cdots. \end{aligned} \tag{28}$$

Also, we have

$$\begin{aligned} \phi(w(z)) &= 1+B_1w_1z+\left(B_1w_2+B_1w_1^2\right)z^2 \\ &\quad+\left(B_1w_3+2B_2w_1w_2+B_3w_1^3\right)z^3+\cdots. \end{aligned} \tag{29}$$

Comparing the coefficients of z, z^2, z^3 and after simple calculations, we obtain

$$\begin{aligned} a_{p+1} &= \frac{bpB_1w_1}{2(\alpha p+\beta)}, \\ a_{p+2} &= \frac{bpB_1}{2(\alpha p+2\beta)} \times\left\{w_2-\left(\frac{bpB_1}{2}\frac{(\alpha-1)(\alpha p+2\beta)}{2(\alpha p+\beta)^2}-\frac{B_2}{B_1}\right)w_1^2\right\}, \\ a_{p+3} &= \frac{bpB_1}{2(\alpha p+2\beta)}\left\{w_3+q_1w_1w_2+q_2w_1^3\right\}, \end{aligned} \tag{30}$$

where q_1 and q_2 are defined in (25). It can be easily followed from (30) that

$$a_{p+2} - \mu a_{p+1}^2 = \frac{bpB_1}{2(\alpha p + 2\beta)} \left\{ w_2 - \nu w_1^2 \right\}, \tag{31}$$

where

$$\nu = \frac{bpB_1}{2} \Phi(p, \alpha, \beta, \mu) - \frac{B_2}{B_1}. \tag{32}$$

The results from (20) to (22) are obtained by using Lemma 3, (23) by using Lemma 4, and (24) by using Lemma 5. To show that these results are sharp, we define the functions $K_{\phi n}(z)$, $F_\lambda(z)$, and $G_\lambda(z)$ such that

$$\begin{aligned} K_{\phi n}(0) &= \left[K_{\phi n}\right]'(0) - 1 = 0, \\ F_\lambda(0) &= F_\lambda'(0) - 1 = 0, \\ G_\lambda(0) &= G_\lambda'(0) - 1 = 0, \end{aligned} \tag{33}$$

with

$$\begin{aligned} &1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{K_{\phi n}(z)}{z^p}\right)^\alpha + \beta \frac{zK_{\phi n}'(z)}{pK_{\phi n}(z)}\left(\frac{K_{\phi n}(z)}{z^p}\right)^\alpha\right\} \\ &\quad \prec \phi\left(z^{n-1}\right), \\ &1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{F_\lambda(z)}{z^p}\right)^\alpha + \beta \frac{zF_\lambda'(z)}{pF_\lambda(z)}\left(\frac{F_\lambda(z)}{z^p}\right)^\alpha\right\} \\ &\quad \prec \phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right), \\ &1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{G_\lambda(z)}{z^p}\right)^\alpha + \beta \frac{zG_\lambda'(z)}{pG_\lambda(z)}\left(\frac{G_\lambda(z)}{z^p}\right)^\alpha\right\} \\ &\quad \prec \phi\left(-\frac{z(z+\lambda)}{1+\lambda z}\right). \end{aligned} \tag{34}$$

It is clear that the functions $K_{\phi n}, F_\lambda, G_\lambda \in V_{p,b,\alpha,\beta}(\phi)$. Let $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality occurs for the function K_ϕ or one of its rotations. For $\sigma_1 < \mu < \sigma_2$, the equality is attained, if and only if f is $K_{\phi 3}$ or one of its rotations. When $\mu = \sigma_1$, then the equality holds for the function F_λ or one of its rotations. If $\mu = \sigma_2$, then the equality is obtained for the function G_λ or one of its rotations. □

Corollary 7. *For $b = 2$, the results from* (20) *to* (24) *coincide with the results proved by Ramachandran et al. [8].*

Corollary 8. *For $b = 2$, $p = 1$, and $\beta = 1$, the results from* (20) *to* (22) *coincide with the results obtained by Ravichandran et al. [4] for the class $B_\alpha(\phi)$.*

Corollary 9. *For $b = 2$, $\alpha = 0$, and $\beta = 1$, the results from* (20) *to* (24) *coincide with the results obtained by Ali et al. [1] for the class $S_p^*(\phi)$.*

Corollary 10. *For $b = 2$, $p = 1$, $\alpha = 0$, and $\beta = 1$, the results from* (20) *to* (22) *coincide with the results obtained by Ma and Minda [2] for the class $S^*(\phi)$.*

2.1. Application of Theorem 6 to the Function Defined by Convolutions

Theorem 11. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, where B_n, s are real with $B_1 > 0$ and $B_2 \geq 0$. Let*

$$\begin{aligned} \sigma_1 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{(\alpha p + \beta)^2}{bpB_1^2(\alpha p + 2\beta)} \\ &\quad \times \left\{ 2(B_2 - B_1) - bpB_1^2 \frac{(\alpha - 1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2} \right\}, \\ \sigma_2 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{(\alpha p + \beta)^2}{bpB_1^2(\alpha p + 2\beta)} \\ &\quad \times \left\{ 2(B_2 + B_1) - bpB_1^2 \frac{(\alpha - 1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2} \right\}, \\ \sigma_3 &= \frac{g_{p+1}^2}{g_{p+2}} \frac{(\alpha p + \beta)^2}{bpB_1^2(\alpha p + 2\beta)} \\ &\quad \times \left\{ 2B_2 - bpB_1^2 \frac{(\alpha - 1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2} \right\}, \\ \Phi^*(p, \alpha, \beta, \mu) &= \frac{(\alpha p + 2\beta)\left(2\mu\left(g_{p+2}/g_{p+1}\right) + \alpha - 1\right)}{2(\alpha p + \beta)^2}. \end{aligned} \tag{35}$$

If $f(z)$ is of the form (1) *and belongs to the class $V_{p,b,\alpha,\beta,g}(\phi)$, then*

$$\begin{aligned} &\left| a_{p+2} - \mu a_{p+1}^2 \right| \\ &\leq \begin{cases} \dfrac{bp}{2g_{p+2}(\alpha p + 2\beta)} \left(B_2 - \dfrac{bpB_1^2}{2} \Phi^*(p, \alpha, \beta, \mu) \right), & \mu < \sigma_1, \\ \dfrac{bpB_1}{2g_{p+2}(\alpha p + 2\beta)}, & \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{bp}{2g_{p+2}(\alpha p + 2\beta)} \left(\dfrac{bpB_1^2}{2} \Phi^*(p, \alpha, \beta, \mu) - B_2 \right), & \mu > \sigma_2. \end{cases} \end{aligned} \tag{36}$$

Furthermore, for $\sigma_1 \le \mu \le \sigma_3$,

$$\begin{aligned} &\left|a_{p+2} - \mu a_{p+1}^2\right| \\ &\quad + \frac{g_{p+1}^2}{bpB_1 g_{p+2}} \left(2\left(1 - \frac{B_2}{B_1}\right)\frac{(\alpha p + \beta)^2}{(\alpha p + 2\beta)} + \frac{bpB_1}{2}\left(2\mu \frac{g_{p+2}}{g_{p+1}} + \alpha - 1\right)\right)\left|a_{p+1}\right|^2 \\ &\le \frac{bpB_1}{2 g_{p+2}(\alpha p + 2\beta)}, \end{aligned} \tag{37}$$

and for $\sigma_3 \le \mu \le \sigma_2$,

$$\begin{aligned} &\left|a_{p+2} - \mu a_{p+1}^2\right| \\ &\quad + \frac{g_{p+1}^2}{bpB_1 g_{p+2}} \left(2\left(1 + \frac{B_2}{B_1}\right)\frac{(\alpha p + \beta)^2}{(\alpha p + 2\beta)} - \frac{bpB_1}{2}\left(2\mu \frac{g_{p+2}}{g_{p+1}} + \alpha - 1\right)\right)\left|a_{p+1}\right|^2 \\ &\le \frac{bpB_1}{2 g_{p+2}(\alpha p + 2\beta)}. \end{aligned} \tag{38}$$

For any complex number μ,

$$\begin{aligned} &\left|a_{p+2} - \mu a_{p+1}^2\right| \\ &\le \frac{bpB_1}{2 g_{p+2}(\alpha p + 2\beta)} \max\left\{1, \left|\frac{bpB_1}{2}\Phi^*(p, \alpha, \beta, \mu) - \frac{B_2}{B_1}\right|\right\}. \end{aligned} \tag{39}$$

Also,

$$\left|a_{p+3}\right| \le \frac{bpB_1}{2(\alpha p + 3\beta) g_{p+3}} H(q_1, q_2), \tag{40}$$

where $H(q_1, q_2)$ *is defined in Lemma 5 and*

$$\begin{aligned} q_1 &= \frac{2B_2}{B_1} + \frac{bpB_1}{2}\frac{(1-\alpha)(\alpha p + 3\beta)}{(\alpha p + \beta)(\alpha p + 2\beta)}, \\ q_2 &= \frac{B_3}{B_1} + \left(\frac{bpB_1}{2}\right)^2 \frac{(\alpha - 1)(2\alpha - 1)(\alpha p + 3\beta)}{6(\alpha p + \beta)^3} \\ &\quad + \frac{bpB_2}{2}\frac{(1-\alpha)(\alpha p + 3\beta)}{(\alpha p + \beta)(\alpha p + 2\beta)}. \end{aligned} \tag{41}$$

These results are sharp.

Proof. Since $f \in V_{p,b,\alpha,\beta,g}(\phi)$, therefore we have for a Schwarz function w, such that

$$\begin{aligned} &1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{(f * g)(z)}{z^p}\right)^\alpha \right. \\ &\quad \left. + \beta \frac{z(f * g)'(z)}{p(f * g)(z)}\left(\frac{(f * g)(z)}{z^p}\right)^\alpha\right\} = \phi(w(z)). \end{aligned} \tag{42}$$

Now,

$$\begin{aligned} &1 - \frac{2}{b} + \frac{2}{b}\left\{(1-\beta)\left(\frac{(f * g)(z)}{z^p}\right)^\alpha \right. \\ &\quad \left. + \beta \frac{z(f * g)'(z)}{p(f * g)(z)}\left(\frac{(f * g)(z)}{z^p}\right)^\alpha\right\} \\ &= 1 + \frac{2}{bp}(\alpha p + \beta) a_{p+1} g_{p+1} z + \frac{1}{bp}(\alpha p + 2\beta) \\ &\quad \times \left\{2 a_{p+2} g_{p+2} + (\alpha - 1) a_{p+1}^2 g_{p+1}^2\right\} z^2 + \frac{2}{bp}(\alpha p + 3\beta) \\ &\quad \times \left\{\begin{matrix} a_{p+3} g_{p+3} + (\alpha - 1) a_{p+1} a_{p+2} g_{p+1} g_{p+2} \\ + \dfrac{(\alpha - 1)(\alpha - 2)}{6} a_{p+1}^3 g_{p+1}^3 \end{matrix}\right\} z^3 + \cdots. \end{aligned} \tag{43}$$

Also, we obtain

$$\begin{aligned} \phi(w(z)) &= 1 + B_1 w_1 z + \left(B_1 w_2 + B_1 w_1^2\right) z^2 \\ &\quad + \left(B_1 w_3 + 2B_2 w_1 w_2 + B_3 w_1^3\right) z^3 + \cdots. \end{aligned} \tag{44}$$

Comparing the coefficients of z, z^2, z^3 and after simple calculations, we obtain

$$\begin{aligned} a_{p+1} &= \frac{bpB_1 w_1}{2 g_{p+1}(\alpha p + \beta)}, \\ a_{p+2} &= \frac{bpB_1}{2 g_{p+2}(\alpha p + 2\beta)} \\ &\quad \times \left\{w_2 - \left(\frac{bpB_1}{2}\frac{(\alpha - 1)(\alpha p + 2\beta)}{2(\alpha p + \beta)^2} - \frac{B_2}{B_1}\right) w_1^2\right\}, \\ a_{p+3} &= \frac{bpB_1}{2 g_{p+3}(\alpha p + 2\beta)}\left\{w_3 + q_1 w_1 w_2 + q_2 w_1^3\right\}. \end{aligned} \tag{45}$$

The remaining proof of the theorem is similar to the proof of Theorem 6. □

Corollary 12. *For* $b = 2$, *the results from* (36) *to* (40) *coincide with the results proved by Ramachandran et al. [8] for the class* $R_{p,1,\alpha,\beta,g}(\phi)$.

Corollary 13. *For $b = 2$, $\alpha = 0$, and $\beta = 1$, the results from* (36) *to* (40) *coincide with the results obtained by Ali et al. [1] for the class $S^*_{p,g}(\phi)$.*

Corollary 14. *For $b = 2$, $p = 1$, $\alpha = 0$, and $\beta = 1$,*

$$g_2 = \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda},$$
$$g_3 = \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}, \tag{46}$$
$$B_1 = \frac{8}{\pi^2}, \qquad B_2 = \frac{16}{3\pi^2},$$

the results from (36) *to* (38) *coincide with the results obtained by Srivastava and Mishra [9].*

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