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Research Article

Input-to-State Stability of Singularly Perturbed Control Systems with Delays

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We study the input-to-state stability of singularly perturbed control systems with delays. By using the generalized Halanay inequality and Lyapunov functions, we derive the input-to-state stability of some classes of linear and nonlinear singularly perturbed control systems with delays.

1. Introduction

The stability properties of control systems are an important research field. The concept of input-to-state stability (ISS) of the control systems was proposed by Sontag [1]. Since then, the ISS of the control systems has been widely studied (cf. [2–12]), and most of the obtained results are often based on the Lyapunov functions.

Singularly perturbed control systems are a special class of control systems which is characterized by small parameters multiplying the highest derivates. Recently, many attentions have been devoted to the study of singularly perturbed systems, in particular, to their stability properties. Saberi and Khalil [13] investigated the asymptotic and exponential stability of nonlinear singularly perturbed systems. They obtained a quadratic-type Lyapunov function as a weighted sum of quadratic-type Lyapunov functions of the reduced and the boundary-layer systems. They used the composite Lyapunov function to estimate the degree of exponential stability and the domain of attraction of stable equilibrium point. Corless and Glielmo [14] obtained some results and properties related to exponential stability of singularly perturbed systems. They pointed out that, if both the reduced and the boundary-layer systems are exponentially stable, then, provided that some further regularity conditions are satisfied, the full-order system is exponentially stable for sufficiently small value ϵ. Liu et al. [15] derived the exponential stability criteria of singularly perturbed systems with time delay. Christofides and Teel [11] obtained a type of total stability for the input-to-state stability property with respect to singular perturbations under the assumptions that the reduced system is ISS and the boundary-layer system is uniformly globally asymptotically stable. Tian [16, 17] discussed the analytic and numerical dissipativity and exponential stability of singularly perturbed delay differential equations. There are some results about the stability of numerical methods for control systems (cf. [18, 19]).

The previous studies have mainly focused on the exponential stability of singularly perturbed systems with or without delays and the ISS of singularly perturbed control systems without delay. There are no results about the ISS of delay singulary perturbed control systems. In this paper, we study the ISS of some classes of delay singularly perturbed control systems. By using the generalized Halanay inequality and the Lyapunov functions, we obtain the sufficient conditions under which these delay singularly perturbed control systems are input-to-state stable.

2. Preliminary

We introduce the following symbols (cf. [8, 11, 15]).

(1) $\|\cdot\|$ denotes the standard Euclidean norm of a vector, $\|\|x_t\|\| = \sup_{\sigma\in[t-\tau,t]}\|x(\sigma)\|$ and $\|A\| = \max_{1\le j\le m}(\sum_{i=1}^n a_{ij}^2)^{1/2}$ denotes the norm of an $n\times m$ matrix $A=(a_{ij})$. A matrix $A(t)$ is bounded means that $\|A(t)\|<\infty$.

(2) A^T denotes the transpose of the matrix A, $\lambda_i(A)$ denotes the ith eigenvalue of the matrix A, and $\operatorname{Re}\lambda_i(A)$ denotes the real part of $\lambda_i(A)$.

(3) The matrix $A>0$ means that A is positive-definite. The vector $v=(v_1,v_2,\ldots,v_n)^T\ge 0$ (>0) means each component $v_i\ge 0$ (>0), $i=1,2,\ldots,n$.

(4) A real $n\times n$ matrix $A=(a_{ij})$ with $a_{ij}\le 0$ for all $i\ne j$ is an M-matrix if A is nonsingular and $A^{-1}>0$.

(5) For any measurable locally essentially bounded function $u: R_{\ge 0}\to R^d$, $\|u_{[0,t)}\| = \sup_{t\ge 0}\{\|u(t)\|\}$.

(6) A function γ: $R_{\ge0}\to R_{\ge0}$ is a κ-function if it is continuous, strictly increasing, and $\gamma(0)=0$.

(7) A function β: $R_{\ge0}\times R_{\ge0}\to R_{\ge0}$ is a $\kappa\varphi$-function if, for each fixed $t\ge 0$, the function $\beta(\cdot,t)\in\kappa$, and for each fixed $s\ge0$, the function $\beta(s,\cdot)$ is decreasing and $\beta(s,t)\to 0$ as $t\to\infty$.

Lemma 1 (see [20]). *Let $A(t)$ be an $n\times n$ matrix whose elements are continuous functions defined on the time interval $J=[0,\infty)$ and the following assumptions hold:*

$$\begin{aligned}&(i)\ \operatorname{Re}\lambda(A(t))\le -c_1<0,\quad \forall t\in J,\\ &(ii)\ \|A(t)\|\le c_2,\quad \forall t\in J,\\ &(iii)\ \|A'(t)\|\le c_2,\quad \forall t\in J.\end{aligned}\tag{1}$$

Then there exists a positive-definitive matrix $P(t)$ such that the following algebraic Lyapunov equation holds:

$$A^T(t)P(t)+P(t)A(t)=-I,\tag{2}$$

where c_1, c_2 are constants, I is the identity matrix, and $P(t)$ is bounded.

The following generalized Halanay inequality will play a key role in studying the ISS for the system (9).

Lemma 2 (generalized Halanay inequality (see [16, 17])). *Suppose*

$$w'(t)\le\lambda(t)-\alpha(t)w(t)+\beta(t)\sup_{t-\tau\le\sigma\le t}w(\sigma),\quad \text{for } t\ge t_0.\tag{3}$$

Here $w(t)$ is a non-negative real-value continuous function, $\tau\ge 0$, $\lambda(t)$, $\alpha(t)$, and $\beta(t)$ are continuous with $0\le\lambda(t)\le\lambda^*$, $\alpha(t)\ge\alpha_0>0$, and $0<\beta(t)\le q\alpha(t)$ for $t\ge t_0$ and $0\le q<1$. Then

$$w(t)\le\frac{\lambda^*}{(1-q)\alpha_0}+Ge^{-\mu^*(t-t_0)},\quad t\ge t_0,\tag{4}$$

where $G=\sup_{t_0-\tau\le\sigma\le t_0}|w(\sigma)|$ and $\mu^*>0$ is defined as

$$\mu^*=\inf_{t\ge t_0}\left\{\mu(t):\mu(t)-\alpha(t)+\beta(t)e^{\mu(t)\tau}=0\right\}.\tag{5}$$

Lemma 3. *Let $A(t)$ and $B(t)$ be $n\times n$ matrix-value functions, $v(t)=(v_1(t),v_2(t),\ldots,v_n(t))^T$, and let $\Gamma(t)=(\Gamma_1(t),\Gamma_2(t),\ldots,\Gamma_n(t))^T$ be vector functions of dimensions n. Assume that*

(i) *$\lambda_i(A(t)+A^T(t))\le -a(t)<0$ $(i=1,2,\ldots,n)$, $B(t)$ is bounded;*

(ii) *$-qa(t)+(1+q)\|B(t)\|+q\le 0$ with $0\le q<1$;*

(iii) *$a(t)-\|B(t)\|-1\ge a_0^*>0$;*

(iv) *$v'(t)\le\Gamma(t)+A(t)v(t)+B(t)\sup_{t-\tau\le\sigma\le t}v(\sigma)$, $\|\Gamma(t)\|\le\Gamma^*$,*

for $t\ge t_0$, where $\sup_{t-\tau\le\sigma\le t}v(\sigma)=(\sup_{t-\tau\le\sigma\le t}v_1(\sigma), \sup_{t-\tau\le\sigma\le t}v_2(\sigma),\ldots,\sup_{t-\tau\le\sigma\le t}v_n(\sigma))^T$. Then the following estimate holds

$$\|v\|\le\frac{\Gamma^*}{\sqrt{(1-q)a_0^*}}+\|\|v_{t_0}\|\|e^{-\gamma^*(t-t_0)},\quad \text{for } t\ge t_0,\tag{6}$$

where γ^* is defined as

$$2\gamma^*=\inf\left\{\gamma(t):\gamma(t)-(a(t)-\|B(t)\|-1)+\|B(t)\|e^{\gamma(t)\tau}\right\}.\tag{7}$$

Proof. Let $V(t)=\|v\|^2=v^Tv$. Then

$$\begin{aligned}V'(t)&=v'^Tv+v^Tv'\\&\le 2v^T\Gamma(t)+v^T\left(A(t)^T+A(t)\right)v(t)\\&\quad+2v^TB(t)\sup_{t-\tau\le\sigma\le t}v(\sigma)\\&\le 2\|v(t)\|\|\Gamma(t)\|-a(t)\|v(t)\|^2\\&\quad+2\|B(t)\|\|v(t)\|\|\|v_t\|\|\\&\le\|\Gamma(t)\|^2+\|v(t)\|^2-a(t)\|v(t)\|^2\\&\quad+\|B(t)\|\left(\|v(t)\|^2+\|\|v_t\|\|^2\right)\\&\le\|\Gamma(t)\|^2-(a(t)-\|B(t)\|-1)\|v(t)\|^2\\&\quad+\|B(t)\|\|\|v_t\|\|^2.\end{aligned}\tag{8}$$

Moreover, by the conditions (i)–(iii), the estimate (6) can be derived as a consequence of (3)–(5) and (8). □

Consider the delay singularly perturbed control systems

$$\begin{aligned}x'(t)&=f(t,x(t),x(t-\tau),y(t),y(t-\tau),u(t)),\quad t\ge 0,\\ \epsilon y'(t)&=g(t,x(t),x(t-\tau),y(t),u(t)),\quad 0<\epsilon\ll 1,\\ x(t)&=\varphi(t),\quad y(t)=\psi(t),\quad t\in[-\tau,0],\end{aligned}\tag{9}$$

where $t \in R$ is the "time," $x \in R^m$ and $y \in R^n$ are the state variables, $u(t) \in R^d$ is the control input which is locally essentially bounded, ϵ is the singular perturbation parameter, and τ is a constant time delay. The sufficiently smooth mapping $f : R \times R^m \times R^m \times R^n \times R^n \times R^d \rightarrow R^m$, $g : R\times R^m\times R^m\times R^n\times R^n\times R^d \rightarrow R^n$ has bounded derivatives and $f(t,0,0,0,0,0) = g(t,0,0,0,0,0) = 0$. $\varphi \in R^m$ and $\psi \in R^n$ are given vector-functions and the derivative of ψ exists.

Definition 4. The delay singularly perturbed control system (9) is ISS if there exist $\kappa\varphi$-functions β_1, β_2: $R_{\geq 0} \times R_{\geq 0} \rightarrow R_{\geq 0}$ and κ-functions γ_1, γ_2 such that, for any initial functions $\varphi(t), \psi(t)$ and each essentially bounded input $u(t)$, the solution of (9) satisfy

$$\begin{aligned} \|x(t,\varphi,\psi,u,\epsilon)\| &\leq \beta_1(\xi,t) + \gamma_1(\|u_{[0,t)}\|), \\ \|y(t,\varphi,\psi,u,\epsilon)\| &\leq \beta_2(\xi,t) + \gamma_2(\|u_{[0,t)}\|), \end{aligned} \tag{10}$$

where $x(t,\varphi,\psi,u,\epsilon)$, $y(t,\varphi,\psi,u,\epsilon)$ are the solutions of (9), $\xi = \sup_{-\tau\leq t\leq 0}\|\varphi(t)\| + \sup_{-\tau\leq t\leq 0}\|\psi(t)\| + \sup_{-\tau\leq t\leq 0}\|\psi'(t)\|$.

3. Linear Systems

In this section, we are concerned with ISS of the following linear delay singularly perturbed control systems as a special class of (9):

$$\begin{aligned} x' &= A_{11}(t)x + A_{12}(t)x_t + B_{11}(t)y \\ &\quad + B_{12}(t)y_t + C_1(t)u(t), \quad t \geq 0, \\ \epsilon y' &= A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)y \\ &\quad + C_2(t)u(t), \quad 0 < \epsilon \ll 1, \\ x(t) &= \varphi(t), \quad y(t) = \psi(t), \quad t \in [-\tau, 0]. \end{aligned} \tag{11}$$

Here we let $x = x(t)$, $y = y(t)$, $x_t = x(t-\tau)$, $y_t = y(t-\tau)$, and $u = u(t)$ for simplicity; $A_{1j}(t) \in R^{m\times m}$, $A_{2j}(t) \in R^{n\times m}$, $B_{1j}(t) \in R^{m\times n}$ $(j = 1,2)$, $B_{21}(t) \in R^{n\times n}$, $C_1(t) \in R^{m\times d}$, and $C_2(t) \in R^{n\times d}$ are smooth matrix functions of t, and $B_{21}(t)$ is nonsingular for every t. Now, we introduce some assumptions.

Assumption 5. There exist positive constants c_1 and c_2 such that, for for all $t \in J = [0, +\infty)$,

$$\begin{gathered} \operatorname{Re}\lambda(A_{11}(t)) \leq -c_1, \qquad \|A_{11}(t)\| \leq c_2, \qquad \|A'_{11}(t)\| \leq c_2, \\ \operatorname{Re}\lambda(B_{21}(t)) \leq -c_1, \qquad \|B_{21}(t)\| \leq c_2, \qquad \|B'_{21}(t)\| \leq c_2, \\ \|B_{21}^{-1}(t)\| \leq c_2, \qquad \|B_{21}^{-1}(t)A_{21}(t)\| \leq c_2, \\ \|B_{21}^{-1}(t)A_{22}(t)\| \leq c_2, \qquad \|B_{21}^{-1}(t)C_2(t)\| \leq c_2. \end{gathered} \tag{12}$$

From Assumption 5 and Lemma 1, we can easily show that there exist the differentiable positive-definite matrices $P_1(t)$ and $P_2(t)$ such that

$$A_{11}^T(t)P_1(t) + P_1(t)A_{11}(t) = -I_m, \tag{13a}$$

$$B_{21}^T(t)P_2(t) + P_2(t)B_{21}(t) = -I_n, \tag{13b}$$

where I_m, I_n are $m\times m$, $n\times n$ identity matrices, respectively, [21] shows that Assumption 5 guarantees that Reference, for every $t \geq t_0$, (13a), (13b) have unique positive-definite solutions $P_1(t)$ and $P_2(t)$ given by

$$\begin{aligned} P_1(t) &= \int_0^\infty e^{A_{11}^T(t)\theta}e^{A_{11}(t)\theta}d\theta, \\ P_2(t) &= \int_0^\infty e^{B_{21}^T(t)\theta}e^{B_{21}(t)\theta}d\theta, \end{aligned} \tag{14}$$

respectively. It follows from the boundness and the positive-definiteness of $P_1(t)$ and $P_2(t)$ that there exist positive constants M_i, α_i, β_i $(i = 1,2)$ such that

$$\begin{aligned} M_1 &\leq \|P_i(t)\| \leq M_2, \quad i = 1,2, \\ \alpha_1\|x\|^2 &\leq x^T P_1(t)x \leq \beta_1\|x\|^2, \\ \alpha_2\|y\|^2 &\leq y^T P_2(t)y \leq \beta_2\|y\|^2. \end{aligned} \tag{15}$$

Assumption 6. There exist bounded functions $a_{ij}(t)$, $b_{ij}(t)$, and $c_i(t)$ $(i,j = 1,2)$ such that

$$\begin{aligned} &2x^T P_1(t)(A_{12}(t)x_t + B_{11}(t)y + B_{12}(t)y_t + C_1(t)u(t)) \\ &\quad + x^T P'_1(t)x \\ &\quad \leq c_1(t)\|u(t)\|^2 + a_{11}(t)\|x\|^2 + a_{12}(t)\|\|x_t\|\|^2 \\ &\quad + b_{11}(t)\|y-h\|^2 + b_{12}(t)\|\|(y-h)_t\|\|^2, \\ &-2(y-h)^T P_2(t)h' + (y-h)^T P'_2(t)(y-h) \\ &\quad \leq c_2(t)\|u(t)\|^2 + a_{21}(t)\|x\|^2 + a_{22}(t)\|\|x_t\|\|^2 \\ &\quad + b_{21}(t)\|y-h\|^2 + b_{22}(t)\|\|(y-h)_t\|\|^2, \end{aligned} \tag{16}$$

where

$$h = \begin{cases} -B_{21}^{-1}(t) \\ \quad \times [A_{21}(t)x + A_{22}(t)x_t + C_2(t)u(t)], & t \geq 0 \\ \psi(t) - \epsilon B_{21}^{-1}(t)\psi'(t), & t \in [-\tau, 0]. \end{cases} \tag{17}$$

Assumption 7. (1) There exists a positive number ϵ_0 such that $-\widetilde{A}(t)$ is an M-matrix;

(2) $\lambda_i(\widetilde{A}(t) + \widetilde{A}^T(t)) \leq -a(t) < 0$ $(i = 1,2)$;

(3) $-qa(t) + (1+q)\|\widetilde{B}(t)\| + q \leq 0$ with $0 \leq q < 1$;

(4) $a(t) - \|\widetilde{B}(t)\| - 1 \geq a_0^* > 0$,

where

$$\widetilde{A}(t) = \begin{pmatrix} -\frac{1-a_{11}(t)}{\beta_1} & \frac{b_{11}(t)}{\alpha_2} \\ \frac{a_{21}(t)}{\alpha_1} & -\frac{1-\epsilon_0 b_{21}(t)}{\epsilon_0 \beta_2} \end{pmatrix},$$
$$\widetilde{B}(t) = \begin{pmatrix} \frac{a_{12}(t)}{\alpha_1} & \frac{b_{12}(t)}{\alpha_2} \\ \frac{a_{22}(t)}{\alpha_1} & \frac{b_{22}(t)}{\alpha_2} \end{pmatrix}. \tag{18}$$

Theorem 8. *If Assumptions 5–7 hold, then the delay singularly perturbed control system* (11) *is input-to-state stable for* $\epsilon \in (0, \epsilon_0]$.

Proof. Let $V(t,x) = x^T P_1(t) x$, $W(t,x,y) = (y-h)^T P_2(t)(y-h)$. For the derivative of $V(t,x)$ along the trajectory of (11), we have

$$\begin{aligned} V'(t,x) &= [A_{11}(t)x + A_{12}(t)x_t + B_{11}(t)y \\ &\quad + B_{12}(t)y_t + C_1(t)u]^T P_1(t)x \\ &\quad + x^T P_1(t)[A_{11}(t)x + A_{12}(t)x_t \\ &\quad + B_{11}(t)y + B_{12}(t)y_t + C_1(t)u] \\ &\quad + x^T P_1'(t)x \\ &= x^T \left(A_{11}^T(t)P_1(t) + P_1(t)A_{11}(t)\right)x \\ &\quad + 2x^T P_1(t)[A_{12}(t)x_t + B_{11}(t)y \\ &\quad + B_{12}(t)y_t + C_1(t)u] \\ &\quad + x^T P_1'(t)x \\ &\le -(1-a_{11}(t))\|x\|^2 + a_{12}(t)\left\|\|x_t\|\right\|^2 \\ &\quad + b_{11}(t)\|y-h\|^2 + b_{12}(t)\left\|\|(y-h)_t\|\right\|^2 + c_1(t)\|u\|^2. \end{aligned} \tag{19}$$

For the derivative of $W(t,x,y)$ along the trajectory of (11), we have

$$\begin{aligned} W'(t,x,y) &= \left[\frac{1}{\epsilon}(A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)y + C_2(t)u) - h'\right]^T \\ &\quad \times P_2(t)(y-h) + (y-h)^T P_2(t) \\ &\quad \times \left[\frac{1}{\epsilon}(A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)y + C_2(t)u) - h'\right] \\ &\quad + (y-h)^T P_2'(t)(y-h) \\ &= \frac{1}{\epsilon}(y-h)^T\left[B_{21}^T(t)P_2(t) + P_2(t)B_{21}(t)\right]^T \\ &\quad - 2(y-h)^T(t)h' + (y-h)^T P_2'(t)(y-h) \\ &\le -\frac{1}{\epsilon}(1-\epsilon b_{21}(t))\|y-h\|^2 + a_{21}(t)\|x\|^2 \\ &\quad + a_{22}(t)\left\|\|x_t\|\right\|^2 + b_{22}(t)\left\|\|(y-h)_t\|\right\|^2 + c_2(t)\|u\|^2. \end{aligned} \tag{20}$$

From (1) of Assumption 7, we can derive $(1-\epsilon_0 b_{21}(t))/\epsilon_0\beta_2 > 0$ and the following inequalities for $\epsilon \in (0,\epsilon_0]$ and $t \ge 0$:

$$\begin{aligned} V' &\le -\frac{1-a_{11}(t)}{\beta_1}V + \frac{b_{11}(t)}{\alpha_2}W \\ &\quad + \frac{a_{12}(t)}{\alpha_1}\left\|\|V_t\|\right\| + \frac{b_{12}(t)}{\alpha_2}\left\|\|W_t\|\right\| + c_1(t)\|u\|^2, \\ W' &\le \frac{a_{21}(t)}{\alpha_1}V - \frac{1-\epsilon b_{21}(t)}{\epsilon\beta_2}W + \frac{a_{22}(t)}{\alpha_1}\left\|\|V_t\|\right\| \\ &\quad + \frac{b_{22}(t)}{\alpha_2}\left\|\|W_t\|\right\| + c_2(t)\|u\|^2 \\ &\le \frac{a_{21}(t)}{\alpha_1}V - \frac{1-\epsilon_0 b_{21}(t)}{\epsilon_0\beta_2}W + \frac{a_{22}(t)}{\alpha_1}\left\|\|V_t\|\right\| \\ &\quad + \frac{b_{22}(t)}{\alpha_2}\left\|\|W_t\|\right\| + c_2(t)\|u\|^2. \end{aligned} \tag{21}$$

It follows from Lemma 3 that there exist positive constants λ, a_0^*, and c_0^* such that

$$\begin{aligned} V &\le \left(\left\|\|V_{t_0}\|\right\| + \left\|\|W_{t_0}\|\right\|\right)e^{-2\lambda t} + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)a_0^*}}, \\ W &\le \left(\left\|\|V_{t_0}\|\right\| + \left\|\|W_{t_0}\|\right\|\right)e^{-2\lambda t} + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)a_0^*}}, \end{aligned} \tag{22}$$

where $\|(c_1(t), c_2(t))^T\| \le c_0^*$ and λ is defined by

$$4\lambda = \inf_{t\ge 0}\left\{\lambda(t) : \lambda(t) - \left(a(t) - \left\|\widetilde{B}(t)\right\| - 1\right) + \left\|\widetilde{B}(t)\right\| e^{\lambda(t)\tau}\right\}. \tag{23}$$

By the definition of $V(t,x)$ and the positive-definiteness of $P_1(t)$, we have

$$\begin{aligned}
\alpha_1\|x\|^2 &\le V \\
&\le \left(\left\|V_{t_0}\right\| + \left\|W_{t_0}\right\|\right)e^{-2\lambda t} + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)\,a_0^*}} \\
&\le \left(\beta_1\|x_0\|^2 + \beta_2\|(y-h)_0\|^2\right)e^{-2\lambda t} \\
&\quad + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)\,a_0^*}} \\
&\le \left(\beta_1\|x_0\|^2 + \beta_2\|y_0\|^2 + \beta_2\|h_0\|^2\right)e^{-2\lambda t} \\
&\quad + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)\,a_0^*}} \\
&\le \left(\beta_1\|x_0\|^2 + 2\beta_2\|y_0\|^2 + \beta_2\epsilon_0^2c_2^2\|\psi_0'\|^2\right)e^{-2\lambda t} \\
&\quad + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)\,a_0^*}}, \\
\|x\|^2 &\le K_1^2\left(\|\varphi_0\|^2 + \|\psi_0\|^2 + \|\psi_0'\|^2\right)e^{-2\lambda t} + K_2^2\|u_{[0,t)}\|^2,
\end{aligned} \tag{24}$$

where $K_1^2 = \max\{\beta_1/\alpha_1, 2\beta_2/\alpha_1, \beta_2\epsilon_0^2c_2^2/\alpha_1\}$ and $K_2^2 = c_0^*/\sqrt{(1-q)a_0^*}\,\alpha_1$. Moreover,

$$\|x\| \le K_1\left(\|\varphi_0\| + \|\psi_0\| + \|\psi_0'\|\right)e^{-\lambda t} + K_2\left\|u_{[0,t)}\right\|. \tag{25}$$

Thus, (25) and the inequality

$$\|y\| - \|h\| \le \|(y-h)(t)\| \le (\alpha_2)^{-1/2}W^{1/2} \tag{26}$$

imply that

$$\begin{aligned}
\|y\| &\le \|h\| + (\alpha_2)^{-1/2}W^{1/2} \\
&\le 2c_2\left(K_1\left(\|\varphi_0\| + \|\psi_0\| + \|\psi_0'\|\right)e^{-\lambda(t-\tau)} + K_2\left\|u_{[0,t)}\right\|\right) \\
&\quad + c_2\left\|u_{[0,t)}\right\| + (\alpha_2)^{-1/2}W^{1/2} \\
&\le \left(2c_2 + \sqrt{\frac{\alpha_1}{\alpha_2}}\right)K_1\left(\|\varphi_0\| + \|\psi_0\| + \|\psi_0'\|\right)e^{-\lambda(t-\tau)} \\
&\quad + \left(\left(2c_2 + \sqrt{\frac{\alpha_1}{\alpha_2}}\right)K_2 + c_2\right)\left\|u_{[0,t)}\right\| \\
&= K_1^*\left(\|\varphi_0\| + \|\psi_0\| + \|\psi_0'\|\right)e^{-\lambda(t-\tau)} + K_2^*\left\|u_{[0,t)}\right\|,
\end{aligned} \tag{27}$$

where $K_1^* = (2c_2 + \sqrt{\alpha_1/\alpha_2})K_1$, $K_2^* = (2c_2 + \sqrt{\alpha_1/\alpha_2})K_2 + c_2$. The proof is complete. □

4. Nonlinear Systems

In this section, we are concerned with ISS of the following nonlinear delay singularly perturbed control systems as a special class of (9):

$$\begin{aligned}
x' &= A(t)x + f(t,x,x_t,y,y_t,u), \quad t \ge 0, \\
\epsilon y' &= B(t)y + g(t,x,x_t,u), \quad 0 < \epsilon \ll 1, \\
x(t) &= \varphi(t), \quad y(t) = \psi(t), \quad t \in [-\tau, 0],
\end{aligned} \tag{28}$$

where $A(t) \in R^{m\times m}$, $B(t) \in R^{n\times n}$, $f(t,0,0,0,0,0) = 0$, and $g(t,0,0,0) = 0$. Assume that (28) has a unique equilibrium at the origin and the functions f and g are smooth enough and the derivative of ψ exists.

Assumption 9. There exist positive constants c_1, c_2 for all $t \in J$ such that

$$\begin{aligned}
&\operatorname{Re}\lambda(A(t)) \le -c_1, \quad \|A(t)\| \le c_2, \quad \left\|A'(t)\right\| \le c_2, \\
&\operatorname{Re}\lambda(B(t)) \le -c_1, \quad \|B(t)\| \le c_2, \quad \left\|B'(t)\right\| \le c_2.
\end{aligned} \tag{29}$$

If Assumption 9 holds, then there exist the differentiable positive-definite matrices $P_1(t)$ and $P_2(t)$ such that

$$\begin{aligned}
A^T(t)P_1(t) + P_1(t)A(t) &= -I_m, \\
B^T(t)P_2(t) + P_2(t)B(t) &= -I_n,
\end{aligned} \tag{30}$$

where I_m, I_n are $m\times m, n\times n$ identity matrices, respectively. It follows from the boundness and the positive-definiteness of $P_1(t)$ and $P_2(t)$ that there exist positive constants $M_i, \alpha_i,$ and β_i $(i = 1,2)$ such that

$$\begin{aligned}
&M_1 \le \left\|P_i(t)\right\| \le M_2, \quad i = 1,2, \\
&\alpha_1\|x\|^2 \le x^TP_1(t)x \le \beta_1\|x\|^2, \\
&\alpha_2\|y\|^2 \le y^TP_2(t)y \le \beta_2\|y\|^2.
\end{aligned} \tag{31}$$

Assumption 10. There exist bounded functions $a_{ij}(t)$, $b_{ij}(t)$, and $c_i(t)$ $(i, j = 1,2)$ such that

$$\begin{aligned}
&2x^TP_1(t)f(t,x,x_t,y,y_t,u) + x^TP_1'(t)x \\
&\quad \le c_1(t)\|u\|^2 + a_{11}(t)\|x\|^2 + a_{12}(t)\|x_t\|^2 \\
&\qquad + b_{11}(t)\|y-h\|^2 + b_{11}(t)\|(y-h)_t\|^2, \\
&-2(y-h)^TP_2(t)h' + (y-h)^TP_2'(t)(y-h) \\
&\quad \le c_2(t)\|u\|^2 + a_{21}(t)\|x\|^2 + a_{22}(t)\|x_t\|^2 \\
&\qquad + b_{21}(t)\|y-h\|^2 + b_{21}(t)\|(y-h)_t\|^2,
\end{aligned} \tag{32}$$

where

$$h = \begin{cases} -B^{-1}(t)\, g(t, x, x_t, u(t)), & t \geq 0 \\ \psi(t) - \epsilon B^{-1}(t)\, \psi'(t), & t \in [-\tau, 0]. \end{cases} \tag{33}$$

Assumption 11. (1) There exist a positive number ϵ_0 such that $-\widetilde{A}(t)$ is an M-matrix;

(2) $\lambda_i(\widetilde{A}(t) + \widetilde{A}^T(t)) \leq -a(t) < 0$ $(i = 1, 2)$;

(3) $-qa(t) + (1+q)\|\widetilde{B}(t)\| + q \leq 0$ with $0 \leq q < 1$;

(4) $a(t) - \|\widetilde{B}(t)\| - 1 \geq a_0^* > 0$,

where

$$\widetilde{A}(t) = \begin{pmatrix} -\dfrac{1 - a_{11}(t)}{\beta_1} & \dfrac{b_{11}(t)}{\alpha_2} \\ \dfrac{a_{21}(t)}{\alpha_1} & -\dfrac{1 - \epsilon_0 b_{21}(t)}{\epsilon_0 \beta_2} \end{pmatrix}, \qquad \widetilde{B}(t) = \begin{pmatrix} \dfrac{a_{12}(t)}{\alpha_1} & \dfrac{b_{12}(t)}{\alpha_2} \\ \dfrac{a_{22}(t)}{\alpha_1} & \dfrac{b_{22}(t)}{\alpha_2} \end{pmatrix}. \tag{34}$$

Theorem 12. *If Assumptions 9–11 hold, then the delay singularly perturbed control system* (28) *is input-to-state stable for* $\epsilon \in (0, \epsilon_0]$.

Proof. Let $V(t,x) = x^T P_1(t) x$, $W(t,x,y) = (y-h)^T P_2(t)(y-h)$. For the derivative of $V(t,x)$ along the trajectory of (28), we have

$$\begin{aligned} V'(t,x) &= [A(t)x + f(t,x,x_t,y,y_t,u)]^T P_1(t)x \\ &\quad + x^T P_1(t)[A(t)x + f(t,x,x_t,y,y_t,u)] + x^T P_1'(t)x \\ &= x^T\left(A^T(t)P_1(t) + P_1(t)A(t)\right)x \\ &\quad + 2x^T P_1(t) f(t,x,x_t,y,y_t,u) + x^T P_1'(t)x \\ &\leq -(1 - a_{11}(t))\|x\|^2 + a_{12}(t)\|\|x_t\|\|^2 \\ &\quad + b_{11}(t)\|y-h\|^2 + b_{12}(t)\|\|(y-h)_t\|\|^2 + c_1(t)\|u\|^2. \end{aligned} \tag{35}$$

For the derivative of $W(t,x,y)$ along the trajectory of (28), we have

$$\begin{aligned} W'(t,x,y) &= \left[\frac{1}{\epsilon}(B(t)y + g(t,x,x_t,u)) - h'\right]^T P_2(t)(y-h) \\ &\quad + (y-h)^T P_2(t)\left[\frac{1}{\epsilon}(B(t)y + g(t,x,x_t,u)) - h'\right] \\ &\quad + (y-h)^T P_2'(t)(y-h) \\ &= \frac{1}{\epsilon}(y-h)^T\left[B^T(t)P_2(t) + P_2(t)B(t)\right]^T(y-h) \\ &\quad - 2(y-h)^T P_2(t)h' + (y-h)^T P_2'(t)(y-h) \\ &\leq -\frac{1}{\epsilon}(1 - \epsilon b_{21}(t))\|y-h\|^2 + a_{21}(t)\|x\|^2 + a_{22}(t)\|\|x_t\|\|^2 \\ &\quad + b_{22}(t)\|\|(y-h)_t\|\|^2 + c_2(t)\|u\|^2. \end{aligned} \tag{36}$$

From (1) of Assumption 11, we can derive $(1 - \epsilon_0 b_{21}(t))/\epsilon_0\beta_2 > 0$ and the following inequalities for $\epsilon \in (0, \epsilon_0]$:

$$\begin{aligned} V' &\leq -\frac{1 - a_{11}(t)}{\beta_1}V + \frac{b_{11}(t)}{\alpha_2}W + \frac{a_{12}(t)}{\alpha_1}\|\|V_t\|\| \\ &\quad + \frac{b_{12}(t)}{\alpha_2}\|\|W_t\|\| + c_1(t)\|u\|^2, \\ W' &\leq \frac{a_{21}(t)}{\alpha_1}V - \frac{1 - \epsilon b_{21}(t)}{\epsilon\beta_2}W + \frac{a_{22}(t)}{\alpha_1}\|\|V_t\|\| \\ &\quad + \frac{b_{22}(t)}{\alpha_2}\|\|W_t\|\| + c_2(t)\|u\|^2 \\ &\leq \frac{a_{21}(t)}{\alpha_1}V - \frac{1 - \epsilon_0 b_{21}(t)}{\epsilon_0\beta_2}W + \frac{a_{22}(t)}{\alpha_1}\|\|V_t\|\| \\ &\quad + \frac{b_{22}(t)}{\alpha_2}\|\|W_t\|\| + c_2(t)\|u\|^2. \end{aligned} \tag{37}$$

It follows from Lemma 3 that there exist positive constants λ, a_0^*, and c_0^* such that

$$\begin{aligned} V &\leq \left(\|\|V_{t_0}\|\| + \|\|W_{t_0}\|\|\right)e^{-2\lambda t} + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)\,a_0^*}}, \\ W &\leq \left(\|\|V_{t_0}\|\| + \|\|W_{t_0}\|\|\right)e^{-2\lambda t} + \frac{c_0^*\|u_{[0,t)}\|^2}{\sqrt{(1-q)\,a_0^*}}, \end{aligned} \tag{38}$$

where $\|(c_1(t), c_2(t))^T\| \leq c_0^*$ and λ is defined as in (23).

By the definitions of $V(t,x)$, $W(t,x,y)$, the positive-definiteness of $P_1(t), P_2(t)$, and the similar proof process to that of Theorem 8, we can obtain

$$\|x\| \le K_1\left(\|\|\varphi_0\|\| + \|\|\psi_0\|\| + \|\|\psi_0'\|\|\right)e^{-\lambda t} + K_2\|u_{[0,t)}\|,$$
$$\|y\| \le K_1^*\left(\|\|\varphi_0\|\| + \|\|\psi_0\|\| + \|\|\psi_0'\|\|\right)e^{-\lambda(t-\tau)} + K_2^*\|u_{[0,t)}\|, \tag{39}$$

where $K_1 = \sqrt{\max\{\beta_1/\alpha_1, 2\beta_2/\alpha_1, \beta_2\epsilon_0^2 c_2^2/\alpha_1\}}$, $K_2 = \sqrt{c_0^*/\sqrt{(1-q)a_0^*}\alpha_1}$, $K_1^* = (2c_2 + \sqrt{\alpha_1/\alpha_2})K_1$ and $K_2^* = (2c_2 + \sqrt{\alpha_1/\alpha_2})K_2 + c_2$. The proof is complete. □

5. Examples

Example 1. Consider the following linear delay system as an application of Theorem 8:

$$x'(t) = -5x(t) + y(t-\tau) + u(t),$$
$$\epsilon y'(t) = 3x(t) - 5y(t). \tag{40}$$

Let $V = x^2/10$, $W = (y-h)^2/10$, $h = 3x(t)/5$. Then

$$V'(t,x) = \frac{1}{5}x(-5x + y_t + u)$$
$$\le -\frac{37}{5}V + \frac{3}{5}\|\|V_t\|\| + \|\|W_t\|\| + \frac{1}{5}\|u\|^2, \tag{41}$$

$$W'(t,x,y) = \frac{1}{5}(y-h)\left(\frac{1}{\epsilon}(3x-5y) - \frac{3}{5}(-5x + y_t + u)\right)$$
$$\le 3V + \left(-\frac{10}{\epsilon} + \frac{114}{25}\right)W$$
$$+ \frac{9}{25}\|\|V_t\|\|\frac{3}{5}\|\|W_t\|\| + \frac{3}{50}\|u\|^2. \tag{42}$$

So we obtain the matrices

$$\widetilde{A}(t) = \begin{pmatrix} -\frac{37}{5} & 0 \\ 3 & -\frac{10}{\epsilon} + \frac{114}{25} \end{pmatrix},$$
$$\widetilde{B}(t) = \begin{pmatrix} \frac{3}{5} & 1 \\ \frac{9}{25} & \frac{3}{5} \end{pmatrix}. \tag{43}$$

If we require that the constant ϵ_0 satisfies $-10/\epsilon_0 + 114/25 \le -37/5$; that is, $\epsilon_0 \le 250/299$, then, we can take $\epsilon_0 = 250/299$ such that it is easy to show that the conditions in Assumptions 5–7 will be satisfied for any $\epsilon \in (0, \epsilon_0]$. Moreover, by Theorem 8, the system (40) is ISS for $\epsilon \in (0, \epsilon_0]$.

Example 2. Consider the following nonlinear delay system as an application of Theorem 12:

$$x'(t) = -15x(t) + 5\ln\left(1 + x^2(t-\tau)\right) + \sin(y(t)) + u(t),$$
$$\epsilon y'(t) = 2x(t-\tau) - 6y(t). \tag{44}$$

Let $V = x^2/30$, $W = (y-h)^2/12$, and $h = x(t-\tau)/3$. Then

$$V'(t,x) = \frac{1}{15}x\left(-15x + 5\ln\left(1 + x_t^2\right) + \sin y + u\right)$$
$$\le \frac{-68}{3}V + \frac{2}{5}W + \frac{16}{3}\|\|V_t\|\| + \frac{1}{30}\|u\|^2,$$
$$W'(t,x,y) = \frac{1}{6}(y-h)\left(\frac{1}{\epsilon}(2x_t - 6y) - h'\right) \tag{45}$$
$$\le \left(-\frac{12}{\epsilon} + \frac{67}{9}\right)W + \frac{61}{9}\|\|V_t\|\|$$
$$+ \frac{1}{3}\|\|W_t\|\| + \frac{1}{36}\|u\|^2.$$

So we obtain the matrices

$$\widetilde{A}(t) = \begin{pmatrix} -\frac{68}{3} & \frac{2}{5} \\ 0 & -\frac{12}{\epsilon} + \frac{67}{9} \end{pmatrix},$$
$$\widetilde{B}(t) = \begin{pmatrix} \frac{16}{3} & 0 \\ \frac{61}{9} & \frac{1}{3} \end{pmatrix}. \tag{46}$$

If we require that the constant ϵ_0 satisfies $-12/\epsilon + 67/9 \le -68/3$; that is, $\epsilon_0 \le 108/271$, then, we can take $\epsilon_0 = 108/271$ such that it is easy to show that the conditions in Assumptions 9–11 will be satisfied for any $\epsilon \in (0, \epsilon_0]$. Moreover, by Theorem 12, the system (44) is ISS for $\epsilon \in (0, \epsilon_0]$.

6. Conclusion

In this paper, we have studied the input-to-state stability of two classes of the linear and nonlinear delay singularly perturbed control systems. The generalized Halanay inequality and the Lyapunov function play important roles in obtaining the stability results. The sufficient conditions of input-to-state stability for delay singularly perturbed control systems are given.

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