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Research Article

Analytic Solutions of an Iterative Functional Differential Equations Near Regular Points

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The existence of analytic solutions of an iterative functional differential equation is studied when the given functions are all analytic and when the given functions have regular points. By reducing the equation to another functional equation without iteration of the unknown function an existence theorem is established for analytic solutions of the original equation.

1. Introduction

Functional differential equations with state-dependent delay have attracted the attentions of many authors in the last few years (see [1–8]). In [6–8], analytic solutions of the state-dependent functional differential equations

$$\begin{gathered} x'(z) = x(az + bx(z)), \\ \alpha z + \beta x'(z) = x\left(az + bx'(z)\right), \\ f^m(x) = G\left(\sum_{k=0}^{m-1} a_k f^k(x)\right) + F(x), \quad m \geq 2, \ x \in \mathbb{C} \end{gathered} \tag{1}$$

are found. In this paper, we will be concerned with analytic solutions of the functional differential equation

$$\alpha z + \beta x'(z) = F\left(\sum_{l=0}^{m} c_l x^l(z)\right) + G(z), \quad z \in \mathbb{C}, \tag{2}$$

where α, β, $c_1, c_2, \ldots, c_m$ are complex numbers, $\beta \neq 0$, $\sum_{l=0}^{m} |c_l| < 1$, and $x^l(z) = x(x^{l-1}(z))$ that denote the nth iterate of a map x. In general, F, G are given complex-valued functions of a complex variable.

In this paper, analytic solutions of nonlinear iterative functional differential equations are investigated. Existence of locally analytic solutions and their construction is given in the case that all given functions exist regular points. As well as in previous work [6–8], we still reduce this problem to find analytic solutions of a differential-difference equation and a functional differential equation with proportional delay. The existence of analytic solutions for such equation is closely related to the position of an indeterminate constant μ depending on the eigenvalue of the linearization of x at its fixed point 0 in the complex plane. For technical reasons, in [6, 7], only the situation of μ off the unit circle in $\mathbb{C}$ and the situation of μ on the circle with the Diophantine condition, "$|\mu| = 1$, μ is not a root of unity, and $\log(1/|\mu^n - 1|) \leq T \log n$, $n = 2, 3, \ldots$ for some positive constant T", are discussed. The Diophantine condition requires μ to be far from all roots of unity that the fixed point 0 is irrationally neutral. In this paper, besides the situation that μ is the inside of the unit circle S^1, we break the restriction of the Diophantine condition and study the situations that the constant μ in (5) (or $\mu = b^{-\lambda}$, b is a complex constant, and λ is in (4)) is resonance and a root of unity in the complex plane $\mathbb{C}$ near resonance under the Brjuno condition.

2. Discussion on Auxiliary Equations

In this section we assume that both F and G are analytic functions in a neighborhood of the origin, that is, 0 is a regular point, and have power series expansions

$$\begin{gathered} F(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad G(z) = \sum_{n=0}^{\infty} d_n z^n, \\ a_0 \neq 0, \quad d_0 \neq 0, \quad z \in \mathbb{C}. \end{gathered} \tag{3}$$

If there exists a complex constant λ and an invertible function $\psi(z)$ such that $\psi(\psi^{-1}(z)+\lambda)$ is well defined, then letting $x(z)=\psi(\psi^{-1}(z)+\lambda)$, we can formally transform (2) into the differential-difference equation

$$\alpha\psi(z)\psi'(z)+\beta\psi'(z+\lambda)=F\left(\sum_{l=0}^{m}c_l\psi(z+l\lambda)\right)\psi'(z)+G(\psi(z))\psi'(z). \quad (4)$$

The indeterminate constant λ will be discussed in the following cases:

(I1) $\Re\lambda>0$;

(I2) $\mu=b^{-\lambda}=e^{2\pi i\theta}$, and $\theta\in\mathbb{R}\setminus\mathbb{Q}$ is a Brjuno number [9, 10]; that is, $B(\theta)=\sum_{k=0}^{\infty}(\log(q_{k+1})/q_k)<\infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ, said to satisfy the *Brjuno condition*;

(I3) $\mu=b^{-\lambda}=e^{2\pi iq/p}$ for some integers $p\in\mathbb{N}$ with $p\geq 2$ and $q\in\mathbb{Z}\setminus\{0\}$, and $\alpha\neq e^{2\pi il/k}$ for all $1\leq k\leq p-1$ and $l\in\mathbb{Z}\setminus\{0\}$.

Take notations $S_p:=\{z\in\mathbb{C}:\Re z>-\ln\rho/\ln|b|,\ -\infty<\Im z<+\infty\}$.

A change of variable further transforms (4) into the functional differential equation

$$\alpha\varphi(z)\varphi'(z)+\beta\mu\varphi'(\mu z)=F\left(\sum_{l=0}^{m}c_l\varphi(\mu^l z)\right)\varphi'(z)+G(\varphi(z))\varphi'(z), \quad (5)$$

where μ is a complex constant. The solution of this equation has properties similar to those of (4). If (5) has an invertible solution $\varphi(z)$, which satisfies the initial value conditions

$$\varphi(0)=0,\qquad \varphi'(0)=\tau\neq 0, \quad (6)$$

then we can show that $x(z)=\varphi(\mu\varphi^{-1}(z))$ is an analytic solution of (2).

Theorem 1. *Suppose that (I1) holds, then* (5) *has an analytic solution of the form*

$$\psi(z)=\sum_{n=1}^{\infty}b_n b^{-nz}, \quad (7)$$

in the half plane S_ρ for a certain constant $\rho>0$, which satisfies $\lim_{\Re z\to+\infty}\psi(z)=0$.

Proof. Since F and G are analytic in a neighborhood of the origin and have the power series expansion (3), there exists a positive ϱ such that

$$|a_n|\leq\varrho^{n-1},\quad |d_n|\leq\varrho^{n-1},\quad n=2,3,\ldots. \quad (8)$$

Without loss of generality, we can assume that $\varrho=1$; that is, $|a_n|\leq 1$, $|d_n|\leq 1$ for $n=2,3,\ldots$. In fact, let $\widetilde{F}(z)=\varrho F(\varrho^{-1}z)$, $\widetilde{G}(z)=\varrho G(\varrho^{-1}z)$, and put $y=\varrho z$, $\widetilde{\beta}=\varrho\beta$, $u=\varrho\lambda$, and $\widetilde{\psi}(z)=\varrho\psi(\varrho^{-1}z)$. Then (4) can be rewritten as

$$\alpha\widetilde{\psi}(y)\widetilde{\psi}'(y)+\widetilde{\beta}\widetilde{\psi}'(y+u)=\widetilde{F}\left(\sum_{l=0}^{m}c_l\widetilde{\psi}(y+lu)\right)\widetilde{\psi}'(y)+\widetilde{G}(\widetilde{\psi}(y))\widetilde{\psi}'(y), \quad (9)$$

in the same form as (4) and $|a_n\varrho^{1-n}|\leq 1$, $|d_n\varrho^{1-n}|\leq 1$ for $n=2,3,\ldots$ by (8).

Consider a solution $\psi(z)$ of (4) in the formal Dirichlet series (7); that is, $\psi(z)=\sum_{n=1}^{\infty}b_nb^{-nz}$, where b is a complex constant and $|b|>1$. Substituting series (3) and (7) of F, G, and ψ in (4) and comparing coefficients we obtain that

$$\left[\beta b^{-\lambda}-(a_0+d_0)\right]\ln b\, b_1=0, \quad (10)$$

$$\left[\beta b^{-n\lambda}-(a_0+d_0)\right]nb_n=-\alpha\sum_{i=1}^{n-1}ib_ib_{n-i}+\sum_{i=1}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}i\left[d_t+a_t\prod_{k=1}^{t}\left(\sum_{l=0}^{m}c_lb^{l_kl\lambda}\right)\right]b_ib_{l_1}b_{l_2}\cdots b_{l_t},\quad n\geq 2. \quad (11)$$

If $b_1=\tau=0$, then (4) has a trivial solution $\psi(z)=0$. Assume that $b_1=\tau\neq 0$, because $|b|>1$; from (10) we have $a_0+d_0=\beta b^{-\lambda}$. From (11) we obtain that

$$\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)nb_n=-\alpha\sum_{i=1}^{n-1}ib_ib_{n-i}+\sum_{i=1}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}i\left[d_t+a_t\prod_{k=1}^{t}\left(\sum_{l=0}^{m}c_lb^{-l_kl\lambda}\right)\right]b_ib_{l_1}b_{l_2}\cdots b_{l_t},\quad n\geq 2. \quad (12)$$

The sequence $\{b_n\}_{n=2}^{\infty}$ is successively determined by (12) in a unique manner.

In what follows we need to prove that the series (7) is convergent in a right-half plane. Since $\Re\lambda>0$, so we have

$$\lim_{n\to\infty}\left|\frac{\alpha i}{n\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)}\right|\leq\lim_{n\to\infty}\frac{|\alpha|}{\left|\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)\right|}=\frac{|\alpha|}{|\beta|\left|b^{-\lambda}\right|},\quad \Re\lambda>0,$$

$$\lim_{n\to\infty}\left|\frac{i\left[d_t + a_t\prod_{k=1}^{t}\left(\sum_{l=0}^{m} c_l b^{-l_k l\lambda}\right)\right]}{n\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)}\right|$$
$$\leq \lim_{n\to\infty}\frac{|d_t| + |a_t|\prod_{k=1}^{t}\left(\sum_{l=0}^{m}|c_l|\right)}{\left|\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)\right|}$$
$$\leq \frac{2}{|\beta|\left|b^{-\lambda}\right|}, \quad \Re\lambda > 0. \tag{13}$$

This implies that there exists a constant $M > 0$ such that

$$\lim_{n\to\infty}\left|\frac{\alpha i}{n\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)}\right| \leq M,$$
$$\lim_{n\to\infty}\left|\frac{i\left[d_t + a_t\prod_{k=1}^{t}\left(\sum_{l=0}^{m} c_l b^{l_k l\lambda}\right)\right]}{n\beta b^{-\lambda}\left(b^{-(n-1)\lambda}-1\right)}\right| \leq M, \tag{14}$$
$$\forall n \geq 2, \quad \Re\lambda > 0.$$

Therefore, from (12) we obtain

$$|b_n| \leq M\left(\sum_{i=1}^{n-1}|b_i||b_{n-i}| + \sum_{i=1}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}|b_i|\left|b_{l_1}\right|\left|b_{l_2}\right|\cdots\left|b_{l_t}\right|\right), \quad n \geq 2. \tag{15}$$

In order to construct a majorant series of (7), we consider the implicit functional equation

$$H(z) = |\tau| z + M\left[H^2(z) + \frac{H^2(z)}{1-H(z)}\right]. \tag{16}$$

Define the function

$$\omega(z,H) := \omega(z,H,\tau,M) = H - |\tau| z - M\left(H^2 + \frac{H^2}{1-H}\right) \tag{17}$$

for (z,H) in a neighborhood of the origin. Then $\omega(0,0) = 0$, $\omega'_H(0,0) = 1 \neq 0$. Thus, there exists a unique function $H(z)$ analytic in a neighborhood of zero; that is, there is a constant $\delta_1 > 0$, as $|z| < \delta_1$, the function $H(z)$ is analytic, such that $H(0) = 0$, $H'(0) = -\omega'_z(0,0)/\omega'_H(0,0) = |\tau|$ and $\omega(z,H(z)) = 0$. Since $H(0) = 0$, there is a constant $\delta_2 > 0$, such that $|H(z)| < 1$ for $|z| < \delta_2$. Therefore, as $|z| < \delta := \min\{\delta_1,\delta_2\}$, the function $H(z)$ satisfies the equation

$$\omega(z,H(z)) = H(z) - |\tau| z - M\left[H^2(z) + \frac{H^2(z)}{1-H(z)}\right] = 0. \tag{18}$$

Choosing $B_1 = |\tau|$ and putting

$$H(z) = \sum_{n=1}^{\infty} B_n b^{-nz} \tag{19}$$

in (18), we can determine all coefficients recursively by $B_1 = |\tau|$ and

$$B_n = M\left(\sum_{i=1}^{n-1}B_iB_{n-i} + \sum_{i=1}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}B_iB_{l_1}B_{l_2}\cdots B_{l_t}\right), \quad n \geq 2. \tag{20}$$

Moreover, it is easy to see from (15)

$$|b_n| \leq B_n, \quad n = 1,2,\ldots. \tag{21}$$

It follows that the power series

$$\phi(z) = \sum_{n=1}^{\infty} b_n z^n \tag{22}$$

is also convergent as $|z| < \delta$. So there exists $\rho \leq \delta$ such that Dirichlet series (7) is convergent in S_ρ.

Furthermore, one has

$$\lim_{\Re z\to+\infty} b^{-z} = \lim_{\Re z\to+\infty} b^{-\Re z}\left[(\cos(\Im z\ln b)) - i\sin(\Im z\ln b)\right] = 0. \tag{23}$$

Thus $\lim_{\Re z\to+\infty}\psi(z) = \lim_{\Re z\to+\infty}\sum_{n=1}^{\infty} b_n b^{-nz} = 0$. The proof is complete. □

We observe that $\mu = b^{-\lambda}$ is inside the unit circle of (I1) but on the unit circle in the rest cases. Next we devote attention to the existence of analytic solutions of (4) under the Brjuno condition. To do this, we first recall briefly the definition of Brjuno numbers and some basic facts. As stated in [11], for a real number θ, we let $[\theta]$ denote its integer part, and $\{\theta\} = \theta - [\theta]$ its fractional part. Then every irrational number θ has a unique expression of the Gauss' continued fraction

$$\theta = a_0 + \theta_0 = a_0 + \frac{1}{a_1 + \theta_1} = \cdots, \tag{24}$$

denoted simply by $\theta = [a_0, a_1, \ldots, a_n, \ldots]$, where a_j's and θ_j's are calculated by the algorithm: (a) $a_0 = [\theta]$, $\theta_0 = \{\theta\}$ and (b) $a_n = [1/\theta_{n-1}]$, $\theta_n = \{1/\theta_{n-1}\}$ for all $n \geq 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$q_{-2} = 1, \quad q_{-1} = 0, \quad q_n = a_n q_{n-1} + q_{n-2},$$
$$p_{-2} = 0, \quad p_{-1} = 1, \quad p_n = a_n p_{n-1} + p_{n-2}. \tag{25}$$

It is easy to show that $p_n/q_n = [a_0, a_1, \ldots, a_n]$. Thus, for every $\theta \in \mathbb{R}\setminus\mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta) = \sum_{n\geq 0}(\log(q_{n+1})/q_n)$. We say that θ is a Brjuno number or that it satisfies Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $a_{n+1} \leq ce^{a_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [a_0, a_1, \ldots, a_n, \ldots]$ is a Brjuno number but is not a Diophantine number. So, the case (I2) contains both Diophantine condition and a part of $\mu = b^{-\lambda}$ "near" resonance. let

$$A_k = \left\{ n \geq 0 \mid \|n\theta\| \leq \frac{1}{8q_k} \right\}, \qquad E_k = \max\left(q_k, \frac{q_{k+1}}{4} \right),$$
$$\eta_k = \frac{q_k}{E_k}. \tag{26}$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left((1 + \eta_k) \frac{n}{q_k} - 2, (m_n \eta_k + n) \frac{1}{q_k} - 1 \right), \tag{27}$$

where $m_n = \max\{ j \mid 0 \leq j \leq n, \ j \in A_k^* \}$. We then define the function $h_k : \mathbb{N} \to \mathbb{R}_+$ as follows:

$$h_k(n) = \begin{cases} \dfrac{m_n + \eta_k n}{q_k} - 1, & \text{if } m_n + q_k \in A_k^*, \\ l_k(n), & \text{if } m_n + q_k \notin A_k^*. \end{cases} \tag{28}$$

Let $g_k(n) := \max(h_k(n), [n/q_k])$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is nondecreasing. Moreover, the function g_k is nonnegative. Then we are able to state the following result.

Lemma 2 (Davie's lemma [12]). *Let* $K(n) = n \log 2 + \sum_{j=0}^{k(n)} g_j(n) \log(2q_{j+1})$. *Then*

(a) *there is a universal constant $\gamma > 0$ (independent of n and θ) such that*

$$K(n) \leq n \left(\sum_{j=0}^{k(n)} \frac{\log q_{j+1}}{q_j} + \gamma \right), \tag{29}$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all n_1 and n_2, and*

(c) $-\log |b^{-\lambda n} - 1| \leq K(n) - K(n-1)$.

Now we state and prove the following theorem under Brjuno condition.

Theorem 3. *Suppose that (I2) holds. Then* (4) *has an analytic solution ψ of the form* (7) *in the half plane* $S_\rho = \{z \in \mathbb{C} : \Re z > -\ln\rho/\ln|b|, -\infty < \Im z < +\infty\}$ *for a certain constant $\rho > 0$, which satisfies* $\lim_{\Re z \to +\infty} \psi(z) = 0$.

Proof. As in the proof of Theorem 1, we find a solution in the form of the Dirichlet series (7). Using the same method as above mentioned, for chosen $b_1 = \tau$ we can uniquely determine the sequence $\{b_n\}_{n=2}^{\infty}$ recursively by (12). In fact, in view of (I2) we see that $\mu = b^{-\lambda}$ satisfies the conditions of Lemma 2, and from (12) we have

$$|b_n| \leq \frac{M_1}{|b^{-(n-1)\lambda} - 1|} \left(\sum_{i=1}^{n-1} |b_i| |b_{n-i}| + \sum_{i=1}^{n-1} \sum_{\substack{l_1 + l_2 + \cdots + l_t = n-i \\ t = 1,2,\ldots,n-i}} |b_i| |b_{l_1}| |b_{l_2}| \cdots |b_{l_t}| \right), \quad n \geq 2, \tag{30}$$

where $M_1 = \max\{|\alpha|/|\beta|, 2/|\beta|\} > 0$.

To construct a governing series of (7), we consider the implicit functional equation

$$\omega(z, U, \tau, M_1) = 0, \tag{31}$$

where ω is defined in (17). Similarly to the proof of Theorem 1, using the implicit function theorem we can prove that (31) has a unique analytic solution $U(z, \tau, M_1)$ in a neighborhood of the origin; that is, there is a constant $\delta_3 > 0$, as $|z| < \delta_3$, the function $U(z, \tau, M_1)$ is analytic such that $U(0, \tau, M_1) = 0$ and $U_z'(0, \tau, M_1) = |\tau|$. Thus $U(z, \tau, M_1)$ in (31) can be expanded into a convergent series

$$U(z, \tau, M_1) = \sum_{n=1}^{\infty} C_n z^n, \tag{32}$$

in a neighborhood of the origin. Replacing (32) into (31) and comparing coefficients, we obtain that $C_1 = |\tau|$ and

$$C_n = M_1 \left(\sum_{i=1}^{n-1} C_i C_{n-i} + \sum_{i=1}^{n-1} \sum_{\substack{l_1 + l_2 + \cdots + l_t = n-i \\ t = 1,2,\ldots,n-i}} C_i C_{l_1} C_{l_2} \cdots C_{l_t} \right), \quad n \geq 2. \tag{33}$$

Note that the series (32) converges in a neighborhood of the origin. So, there is a constant $T > 0$ such that

$$C_n < T^n, \quad n \geq 1. \tag{34}$$

Now, we can deduce, by induction, that $|b_n| \leq C_n e^{K(n-1)}$ for $n \geq 1$, where $K : \mathbb{N} \to \mathbb{R}$ is defined in Lemma 2. In fact $|b_1| = |\tau| = C_1$, for inductive proof, and we assume that $|b_j| \le C_j e^{K(j-1)}$, $j \le n$. From (30) and Lemma 2 we obtain

$$\begin{aligned}
|b_{n+1}| &\le \frac{M_1}{|b^{-n\lambda} - 1|} \times \left(\sum_{i=1}^{n} |b_i| |b_{n-i+1}| + \sum_{i=1}^{n} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i+1 \\ t=1,2,\ldots,n-i+1}} |b_i| |b_{l_1}| |b_{l_2}| \cdots |b_{l_t}| \right) \\
&\le \frac{M_1}{|b^{-n\lambda} - 1|} \left[\sum_{i=1}^{n} C_i C_{n-i+1} e^{K(i-1)+K(n-i)} + \sum_{i=1}^{n} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i+1 \\ t=1,2,\ldots,n-i+1}} C_i C_{l_1} C_{l_2} \cdots C_{l_t} \times e^{K(i-1)+K(l_1-1)+\cdots+K(l_t-1)} \right].
\end{aligned} \tag{35}$$

Note that

$$\begin{aligned}
&K(i-1) + K(n-i) \le K(n-1) \le K(n) + \log|b^{-\lambda n} - 1|, \\
&K(l_1-1) + K(l_2-1) + \cdots + K(l_t-1) \le K(n-i+1-t) \le K(n-i), \\
&K(i-1) + K(n-i) \le K(n-1) \le K(n) + \log|b^{-\lambda n} - 1|,
\end{aligned} \tag{36}$$

then

$$\begin{aligned}
|b_{n+1}| &\le \frac{e^{K(n-1)} M_1}{|b^{-n\lambda} - 1|} \times \left(\sum_{i=1}^{n} C_i C_{n-i+1} + \sum_{i=1}^{n} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i+1 \\ t=1,2,\ldots,n-i+1}} C_i C_{l_1} C_{l_2} \cdots C_{l_t} \right) \\
&\le C_{n+1} e^{K(n)}
\end{aligned} \tag{37}$$

as desired. Moreover, from Lemma 2, we know that $K(n) \le n(B(\theta) + \gamma)$ for some universal constant $\gamma > 0$. Then from (34) we have $|b_n| \le C_n e^{K(n-1)} \le T^n e^{(n-1)(B(\theta)+\gamma)}$; that is, $\lim_{n\to\infty} \sup(|b_n|^{1/n}) \le \lim_{n\to\infty} \sup(T e^{((n-1)/n)(B(\theta)+\gamma)}) = T e^{B(\theta)+\gamma}$, which shows that the series (7) converges for $|z| < \rho = \min\{\delta_3, (T e^{B(\theta)+\gamma})^{-1}\}$. So does series (22) in S_ρ. Similarly $\lim_{\Re z \to +\infty} \psi(z) = 0$, as proved in Theorem 1.

The next theorem is devoted to the case of (I3), where $b^{-\lambda}$ is not only on the unit circle in $\mathbb{C}$ but also a root of unity. In this case the Diophantine condition and Brjuno condition are not satisfied. The idea of our proof is acquired from [13]. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence defined by $A_1 = \tau$ and

$$A_n = \xi M_1 \left(\sum_{i=1}^{n-1} A_i A_{n-i} + \sum_{i=1}^{n-1} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i \\ t=1,2,\ldots,n-i}} A_i A_{l_1} A_{l_2} \cdots A_{l_t} \right), \quad n \ge 2, \tag{38}$$

where $\xi = \max\{1, |\alpha^i - 1|^{-1}, i = 1, 2\ldots, p-1\}$ and M_1 is defined in Theorem 3. □

Theorem 4. *Suppose that (I3) holds and p is given as above mentioned. Let $\{b_n\}_{n=1}^{\infty}$ be determined recursively by $b_1 = \tau$ and*

$$\beta b^{-\lambda} \left(b^{-(n-1)\lambda} - 1 \right) n b_n = \Omega(n, \lambda), \quad n = 2, 3, \ldots, \tag{39}$$

where

$$\begin{aligned}
\Omega(n,\lambda) &= -\alpha \sum_{i=1}^{n-1} i b_i b_{n-i} \\
&+ \sum_{i=1}^{n-1} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i \\ t=1,2,\ldots,n-i}} i \left[d_t + a_t \prod_{k=1}^{t} \left(\sum_{l=0}^{m} c_l b^{-l_k l \lambda} \right) \right] b_i b_{l_1} b_{l_2} \cdots b_{l_t}.
\end{aligned} \tag{40}$$

If $\Omega(\nu p + 1, \lambda) = 0$ for all $\nu = 1, 2, \ldots$, then (4) has an analytic solution $\psi(z) = \phi(b^{-z})$ in the half plane $S_\rho := \{z \in \mathbb{C} : \Re z > -\ln\rho/\ln|b|, -\infty < \Im z < +\infty\}$ for a certain $\rho > 0$, where ϕ is an analytic function of the form (22) in $U_\rho(0) = \{z \mid |z| < \rho\}$ such that $\phi(0) = 0$, and $\phi^{(\nu p+1)}(0) = (\nu p + 1)!\tau_{\nu p+1}$, for all $\nu = 0, 1, 2, \ldots$, where $\tau'_{\nu p+1}$s are arbitrary constants satisfying the inequality $|\tau_{\nu p+1}| \le A_{\nu p+1}$ and the sequence $\{A_n\}_{n=1}^{\infty}$ is defined in (38). The other derivatives at 0 satisfy that $\phi^{(i)}(0) = i! b_i$ for $i \ne \nu p + 1$. Otherwise, if $\Omega(\nu p + 1, \lambda) \ne 0$ for some $\nu = 1, 2, \ldots$, then (4) has no analytic solutions in the half plane $S_\rho := \{z \in \mathbb{C} : \Re z > -\ln\rho/\ln|b|, -\infty < \Im z < +\infty\}$ for any $\rho > 0$.

Proof. Analogously to the proof of Theorem 1, we seek for a solution of (4) in Dirichlet (7). Without loss of generality, as in the proof of Theorem 1 we still assume that $\rho = 1$ in (8). Taking (3) and (7) in (4) and defining $b_1 = \tau \ne 0$, we obtain (12) or (39). If $\Omega(\nu p + 1, \lambda) = 0$ for all natural numbers ν, then for each ν, $b^{-\nu p\lambda} - 1 = 0$, the corresponding $b_{\nu p+1}$ has infinitely many choices in $\mathbb{C}$; that is, the formal series solution

(7) or (22) defines a family of solutions with infinitely many parameters. Choose $b_{\nu p+1} = \tau_{\nu p+1}$ arbitrarily such that

$$\left|\tau_{\nu p+1}\right| \le A_{\nu p+1}, \quad \nu = 0, 1, 2, \dots, \tag{41}$$

where $A_{\nu p+1}$ is defined by (38). In what follows, we prove that the formal series solution (7) converges in a neighborhood of the origin. Observe that $|b^{-n\lambda} - 1|^{-1} \le \xi$ for $n \ne \nu p$. It follows from (12) or (39) that

$$\left|b_n\right| \le \xi M_1 \left(\sum_{i=1}^{n-1} \left|b_i\right| \left|b_{n-i}\right| + \sum_{i=1}^{n-1} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i \\ t=1,2,\dots,n-i}} \left|b_i\right| \left|b_{l_1}\right| \left|b_{l_2}\right| \cdots \left|b_{l_t}\right| \right) \tag{42}$$

for all $n \ne \nu p + 1$, $\nu = 0, 1, 2, \dots$, where M_1 is defined in Theorem 3. Let

$$V(z, \tau, \xi M_1) = \sum_{n=1}^{\infty} A_n z^n, \quad A_1 = |\tau|. \tag{43}$$

It is easy to check that there exists a constant $\rho > 0$, for $|z| < \rho$, and (43) satisfies the implicit functional equation

$$\omega(z, V, \tau, \xi M_1) = 0, \tag{44}$$

where ω is defined in (17). Moreover, similarly to the proof of Theorem 1, we can prove that (44) has a unique analytic solution $V(Z, \tau, \xi M_1)$ in a neighborhood of the origin such that $V(0, \tau, \xi M_1) = 0$ and $V_z'(0, \tau, \xi M_1) = |\tau| \ne 0$. Thus (43) converges in a neighborhood of the origin. Moreover, it is easy to show that, by induction,

$$\left|b_n\right| \le A_n, \quad n = 1, 2, \dots. \tag{45}$$

By inequality (45) we see that the series (22) converges in $U_\rho(0) = \{z \mid |z| < \rho\}$. Thus series (7) converges in S_ρ. This completes the proof. □

The following theorem shows that each analytic solution of (4) leads to an analytic solution of (5). We shall discuss (5) in the following cases:

(H1) $0 < |\mu| \ne 1$;

(H2) $\mu = e^{2\pi i\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a Brjuno number, $B(\theta) = \sum_{k=0}^{\infty} (\log(q_{k+1})/q_k) < \infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fraction of the continued fraction expansion of θ, said to satisfy the *Brjuno condition*;

(H3) $\mu = e^{2\pi iq/p}$ for some integers $p \in \mathbb{N}$ with $p \ge 2$ and $q \in \mathbb{Z} \setminus \{0\}$, and $\mu \ne e^{2\pi il/k}$ for all $1 \le k \le p-1$ and $l \in \mathbb{Z} \setminus \{0\}$.

Theorem 5. *Suppose that (H1) holds and that $a_0 \ne 0$, $d_0 \ne 0$. Then in a neighborhood of the origin (5) has an analytic solution φ satisfying $\varphi(0) = 0$, $\varphi'(0) = \eta$.*

Proof. Let

$$\varphi(z) = \sum_{n=1}^{\infty} b_n z^n, \quad b_1 = \eta \tag{46}$$

be the formal series of the solution φ for (5). We are going to determine $\{b_n\}_{n=1}^{\infty}$. Substituting (3) and (46) into (5) and comparing coefficients, we obtain

$$\left[\beta\mu - (a_0 + d_0)\right] b_1 = 0, \tag{47}$$

$$\begin{aligned}(n+1)&\left[\beta\mu^{n+1} - (a_0 + d_0)\right] b_{n+1} \\ &= -\alpha \sum_{i=0}^{n-1} (i+1) b_{i+1} b_{n-i} \\ &\quad + \sum_{i=0}^{n-1} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i \\ t=1,2,\dots,n-i}} (i+1) \left[d_t + a_t \prod_{k=1}^{t} \left(\sum_{l=0}^{m} c_l \mu^{ll_k}\right)\right] \\ &\quad \times b_{i+1} b_{l_1} b_{l_2} \cdots b_{l_t}, \quad n \ge 1.\end{aligned} \tag{48}$$

If $b_1 = \eta = 0$, then (5) has a trivial solution $\varphi(z) = 0$. Assume that $b_1 = \eta \ne 0$; from (47) we can choose $\beta\mu = a_0 + d_0$, then (48) can be changed into

$$\begin{aligned}(n+1)&\beta\mu(\mu^n - 1) b_{n+1} \\ &= -\alpha \sum_{i=0}^{n-1} (i+1) b_{i+1} b_{n-i} \\ &\quad + \sum_{i=0}^{n-1} \sum_{\substack{l_1+l_2+\cdots+l_t=n-i \\ t=1,2,\dots,n-i}} (i+1) \left[d_t + a_t \prod_{k=1}^{t} \left(\sum_{l=0}^{m} c_l \mu^{ll_k}\right)\right] \\ &\quad \times b_{i+1} b_{l_1} b_{l_2} \cdots b_{l_t}, \quad n \ge 1.\end{aligned} \tag{49}$$

From (49) the sequence $\{b_n\}_{n=2}^{\infty}$ is determined uniquely in the recursive way.

Now we show the convergence of series (46) near zero. Since the power series in (3) are both convergent for $|z| < \sigma$, for any fixed $r \in (0, \sigma)$ there exists a constant $M_2 > 0$ such that

$$\left|a_n\right| \le \frac{M_2}{r^n}, \qquad \left|d_n\right| \le \frac{M_2}{r^n}. \tag{50}$$

Note that since $1 \le i \le n-1$, then there exists some positive number M_3 as follows:

$$\begin{aligned}\left|\frac{(i+1)\prod_{k=1}^{t}\left(\sum_{l=0}^{m} c_l \mu^{ll_k}\right)}{(n+1)\beta\mu(\mu^n - 1)}\right| &\le \frac{1}{|\beta||\mu||\mu^n - 1|} \le M_3, \\ \left|\frac{i+1}{(n+1)\beta\mu(\mu^n - 1)}\right| &\le \frac{1}{|\beta||\mu||\mu^n - 1|} \le M_3, \\ \left|\frac{\alpha(i+1)}{(n+1)\beta\mu(\mu^n - 1)}\right| &\le \frac{|\alpha|}{|\beta||\mu||\mu^n - 1|} \le M_3,\end{aligned} \tag{51}$$

then we have

$$|b_{n+1}| \le L\left(\sum_{i=0}^{n-1}|b_{i+1}||b_{n-i}| + \sum_{i=0}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}\frac{1}{r^t}|b_{i+1}||b_{l_1}||b_{l_2}|\cdots|b_{l_t}|\right), \quad n\ge 1, \tag{52}$$

where $L := \max\{2M_2M_3, M_3\} > 0$. Let

$$\Theta(z,W) = W - |\eta| z - L\left(\frac{W^2/r}{1-(W/r)} + W^2\right) \tag{53}$$

for (z,W) from a neighborhood of the origin. Since $\Theta(0,0) = 0, \Theta'_W(0,0) = 1 \neq 0$, there exists a unique function $W(z)$, analytic in a neighborhood of the origin, such that $W(0)=0$, $W'(0) = |\eta| \neq 0$, and $\Theta(z, W(z)) = 0$. Then define a sequence $\{D_n\}_{n=1}^{\infty}$ by $D_1 = |\eta| \neq 0$ and

$$D_{n+1} = L\left(\sum_{i=0}^{n-1}D_{i+1}D_{n-i} + \sum_{i=0}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}\frac{1}{r^t}D_{i+1}D_{l_1}D_{l_2}\cdots D_{l_t}\right), \quad n\ge 1. \tag{54}$$

By (52) we see that

$$|b_n| \le D_n, \quad n \ge 1. \tag{55}$$

Let

$$P(z) = \sum_{n=1}^{\infty} D_n z^n, \quad D_1 = |\eta|, \tag{56}$$

with the recursive law of $\{D_n\}_{n=1}^{\infty}$. Then

$$\begin{aligned} P^2(z) &= \left(\sum_{n=0}^{\infty}D_{n+1}z^{n+1}\right)\left(\sum_{n=1}^{\infty}D_n z^n\right) \\ &= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}D_{i+1}D_{n-i}z^{n+1}, \end{aligned}$$

$$\begin{aligned} \frac{P^2(z)/r}{1-(P(z)/r)} &= P(z)\frac{P(z)/r}{1-(P(z)/r)} \\ &= \left(\sum_{n=0}^{\infty}D_{n+1}z^{n+1}\right)\left(\sum_{n=1}^{\infty}\left(\frac{P(z)}{r}\right)^n\right) \\ &= \left(\sum_{n=0}^{\infty}D_{n+1}z^{n+1}\right)\left(\sum_{n=1}^{\infty}\frac{1}{r^n}\left(\sum_{n=1}^{\infty}D_n z^n\right)^n\right) \\ &= \left(\sum_{n=0}^{\infty}D_{n+1}z^{n+1}\right)\left(\sum_{n=1}^{\infty}\sum_{\substack{l_1+l_2+\cdots+l_t=n\\ t=1,2,\ldots,n}}\frac{1}{r^t}D_{l_1}D_{l_2}\cdots D_{l_t}z^n\right) \\ &= \sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}\frac{1}{r^t}D_{i+1}D_{l_1}D_{l_2}\cdots D_{l_t}z^{n+1}. \end{aligned} \tag{57}$$

Then we have

$$\frac{P^2(z)/r}{1-(P(z)/r)} + p^2(z) = \frac{1}{L}\sum_{n=1}^{\infty}D_{n+1}z^{n+1} = \frac{1}{L}(P(z) - |\eta| z), \tag{58}$$

that is

$$P(z) - |\eta| z - L\left[\frac{P^2(z)/r}{1-(P(z)/r)} + p^2(z)\right] = 0. \tag{59}$$

This shows that $P(0) = 0, P'(0) = |\eta|$, and $\Theta(z, P(z)) = 0$. So we have $P(z) = W(z)$. It follows that the power series (56) converges in a neighborhood of the origin. Therefore, from (55) we see that (46) converges in a neighborhood of the origin. The proof is complete. □

In the case (H2) we obtain similarly an analogue to Theorem 3.

Theorem 6. *Suppose that (H2) holds and that $a_0 \neq 0$, $d_0 \neq 0$. Then in a neighborhood of the origin (5) has an analytic solution $\varphi(z)$ satisfying $\varphi(0) = 0, \varphi'(0) = \eta \neq 0$.*

In the case (H3) we also obtain similarly an analogue to Theorem 4.

Theorem 7. *Suppose that (H3) holds, $a_0 \neq 0$, $d_0 \neq 0$, and p is given as above mentioned. Let $\{b_n\}_{n=1}^{\infty}$ be determined recursively by $b_1 = \eta$ and*

$$(n+1)\beta\mu(\mu^n - 1)b_{n+1} = \Xi(n+1, \mu), \tag{60}$$

where

$$\Xi(n+1,\mu) = -\alpha\sum_{i=0}^{n-1}(i+1)\,b_{i+1}b_{n-i} + \sum_{i=0}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}(i+1)\left[d_t+a_t\prod_{k=1}^{t}\left(\sum_{l=0}^{m}c_l\mu^{ll_k}\right)\right] \times b_{i+1}b_{l_1}b_{l_2}\cdots b_{l_t},\quad n\geq 1. \tag{61}$$

If $\Xi(lp+1,\mu)=0$ *for all* $l=1,2,\ldots$, *then* (5) *has an analytic solution in a neighborhood of the origin such that* $\varphi(0)=0$, $\varphi'(0)=\eta\neq 0$, *and* $\varphi^{(lp+1)}(0)=(lp+1)!\eta_{lp+1}$, *where* $\eta_{lp+1}'s$ *are arbitrary constants satisfying the inequality* $|\eta_{lp+1}|\leq d_{lp+1}$, $l=1,2,\ldots$ *and the sequence* $\{d_n\}_{n=1}^{\infty}$ *is defined as follows:* $d_1=|\eta|$ *and*

$$d_{n+1} = \Gamma L_1\left(\sum_{i=0}^{n-1}d_{i+1}d_{n-i}+\sum_{i=0}^{n-1}\sum_{\substack{l_1+l_2+\cdots+l_t=n-i\\ t=1,2,\ldots,n-i}}\frac{1}{r^t}d_{i+1}d_{l_1}d_{l_2}\cdots d_{l_t}\right),\quad n\geq 1, \tag{62}$$

where $\Gamma=\max\{1,|\mu^i-1|^{-1},\ i=1,2\ldots,p-1\}$, *and* $L_1=\max\{2/|\beta|,|\alpha|/|\beta|\}$. *Otherwise, if* $\Xi(lp+1,\mu)\neq 0$ *for some* $l=1,2,\ldots$, *then* (5) *has no analytic solutions in any neighborhood of the origin.*

3. Analytic Solutions of the Original (2)

Having known analytic solutions of the auxiliary equation (4) and (5), we can give results to the original (2).

Theorem 8. *Suppose that the conditions of Theorems 1, 3, or 4 are satisfied. Then* (2) *has an analytic solution* $x(z)=\psi(\psi^{-1}(z)+\lambda)$ *in a neighborhood of the origin, where* $\psi(z)$ *is an analytic solution of* (4) *in the half plane* S_ρ.

Proof. In view of Theorems 1, 3, or 4, we may find a sequence $\{b_n\}_{n=1}^{\infty}$ such that the function $\psi(z)$ of the form (7) is an analytic solution of (4) in the half plane S_ρ. As in Theorem 1, $\psi(z)=\phi(b^{-z})$. Since $\phi'(0)\neq 0$, the function ϕ^{-1} is analytic in a neighborhood of the point $\phi(0)=0$. Thus $\psi^{-1}(z)=-\ln\phi^{-1}(z)/\ln b$ is analytic in a neighborhood of the origin. Define $x(z)=\psi(\psi^{-1}(z)+\lambda)$ which is analytic clearly. Note that

$$x'(z)=\frac{\psi'\left(\psi^{-1}(z)+\lambda\right)}{\psi'\left(\psi^{-1}(z)\right)},\qquad x^k(z)=\psi\left(\psi^{-1}(z)+k\lambda\right), \tag{63}$$

then from (4)

$$\begin{aligned}\alpha z+\beta x'(z) &= \alpha z+\frac{\beta\psi'\left(\psi^{-1}(z)+\lambda\right)}{\psi'\left(\psi^{-1}(z)\right)}\\ &= F\left(\sum_{l=0}^{m}c_l\psi\left(\psi^{-1}(z)+l\lambda\right)\right)+G\left(\psi\left(\psi^{-1}(z)\right)\right)\\ &= F\left(\sum_{l=0}^{m}c_lx^l(z)\right)+G(z).\end{aligned} \tag{64}$$

This shows that $x(z)=\psi(\psi^{-1}(z)+\lambda)$ satisfies (2). The proof is complete. □

Under the hypothesis of Theorem 1 the origin 0 is a hyperbolic fixed point of x, but under hypotheses of Theorems 3 and 4 it is not. Actually, when the constant $\mu=b^{-\lambda}$ satisfies the Brjuno condition, the norm of the eigenvalue of the linearized of x at 0 equals 1, but the eigenvalue is not a root of unity. Under (I3), the fixed point 0 of x is a resonance.

When $0<|b|<1$, the same method is applicable and a similarly result can be obtained; that is, there exists a constant $\rho>0$ such that (4) has an analytic solution in the left-half plane $\{z\in\mathbb{C}\mid \Re z<-\ln\rho/\ln|b|,\ -\infty<\Im z<+\infty\}$.

Theorem 9. *Under one of the conditions in Theorems 5, 6, or 7,* (2) *has an analytic solution of the form* $x(z)=\varphi(\mu\varphi^{-1}(z))$ *in a neighborhood of the origin, where* $\varphi(z)$ *is an analytic solution of* (5) *in a neighborhood of the origin.*

Proof. In Theorems 5–7 we have found a solution $\varphi(z)$ of (5) in the form (46), which is analytic near 0. Since $\varphi(0)=0$, $\varphi'(0)=\eta\neq 0$, the function φ^{-1} is also analytic near 0. Thus $x(z)=\varphi(\mu\varphi^{-1}(z))$ is analytic. Moreover,

$$x'(z)=\frac{\mu\varphi'\left(\mu\varphi^{-1}(z)\right)}{\varphi'\left(\varphi^{-1}(z)\right)},\qquad x^l(z)=\varphi\left(\mu^l\varphi^{-1}(z)\right), \tag{65}$$

so from (5) we have

$$\begin{aligned}\alpha z+\beta x'(z) &= \alpha z+\frac{\beta\mu\varphi'\left(\mu\varphi^{-1}(z)\right)}{\varphi'\left(\varphi^{-1}(z)\right)}\\ &= F\left(\sum_{l=0}^{m}c_l\varphi\left(\mu^l\varphi^{-1}(z)\right)\right)+G\left(\varphi\left(\varphi^{-1}(z)\right)\right)\\ &= F\left(\sum_{l=0}^{m}c_lx^l(z)\right)+G(z).\end{aligned} \tag{66}$$

Therefore, $x(z)=\varphi(\mu\varphi^{-1}(z))$ satisfies (2). The proof is complete. □

4. Example

Example 1. Consider the equation

$$z + x'(z) = 4e^{(1/2)x(x(z))} - e^z - \frac{17}{6}. \tag{67}$$

It is in the form of (2), where $\alpha = \beta = 1$, $c_0 = c_1 = 0$, $c_2 = 1/2$, $m = 2$, $F(z) = 4e^z - (13/3) = \sum_{n=1}^{\infty}(4z^n/n!) - (1/3)$, and $G(z) = (3/2) - e^z = -\sum_{n=1}^{\infty}(z^n/n!) + (1/2)$. Clearly both G and F are analytic near 0, $a_0 = F(0) = -1/3$, $d_0 = G(0) = 1/2$, $a_1 = F'(0) = 4$, and $d_1 = G'(0) = -1$. For arbitrary $|b| > 1$, let $\lambda = \ln 6/\ln|b|$, the $0 < |b^{-\lambda}| < 1$. By Theorem 1, there is a constant $\rho > 0$ such that the corresponding auxiliary equation

$$\begin{aligned} &\psi(z)\psi'(z) + \psi'(z+\lambda) \\ &\quad = F\left(\frac{1}{2}\psi(z+2\lambda)\right)\psi'(z) + G(\psi(z))\psi'(z), \end{aligned} \tag{68}$$

and (67) itself have an analytic solution in the half plane $S_\rho = \{z \mid \Re z > -(\ln\rho/\ln b), -\infty < \Im z < +\infty\}$. In the routine in the proofs of our theorems we can calculate the solutions

$$x(z) = \frac{1}{6}z - \frac{35}{36}z^2 - \frac{551}{1944}z^3 + \cdots. \tag{69}$$

Example 2. Consider the equation

$$z - x'(z) = \frac{1}{2}x(z) - \frac{1}{4}x(x(z)) + e^z - \frac{1}{4}. \tag{70}$$

It is in the form of (2), where $\alpha = 1$, $\beta = -1$, $c_0 = 0$, $c_1 = -1/2$, $c_2 = 1/4$, $m = 2$, $a_0 = 1/2$, $a_1 = -1$, $a_n = 0\,(n \ge 2)$, $d_0 = 1/4$, $d_1 = 1$, $F(z) = -z + (1/2)$, and $G(z) = e^z - (3/4) = \sum_{n=1}^{\infty}(z^n/n!) + (1/4)$. Clearly, $\sum_{l=0}^{2}|c_l| < 1$ and $\mu = (1/\beta)(a_0 + d_0) = -3/4$ such that $0 < |\mu| < 1$. By Theorem 5 the corresponding auxiliary equation

$$\begin{aligned} \varphi(z)\varphi'(z) - \mu\varphi'(\mu z) &= \left(\frac{1}{2}\varphi(\mu z) - \frac{1}{4}\varphi(\mu^2 z)\right)\varphi'(z) \\ &\quad + \left(e^{\varphi(z)} - \frac{1}{4}\right)\varphi'(z) \end{aligned} \tag{71}$$

and (70) itself have an analytic solution each in a neighborhood of the origin. Proofs of our theorems provide a method to calculate the solution

$$\begin{aligned} x(z) &= \mu z + \frac{1}{8}\mu(\mu-2)z^2 \\ &\quad + \frac{1}{3!}\left[\frac{1}{16}\mu(\mu-1)(\mu-2)(\mu+2) - 1\right]z^3 + \cdots. \end{aligned} \tag{72}$$

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