Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 732808, 7 pages
http://dx.doi.org/10.1155/2013/732808

Research Article

Lattices Generated by Two Orbits of Subspaces under Finite Singular Symplectic Groups

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Received 29 September 2012; Revised 18 November 2012; Accepted 21 November 2012

Academic Editor: P. G. L. Leach

In the paper titled "Lattices generated by two orbits of subspaces under finite classical group" by Wang and Guo. The subspaces in the lattices are characterized and the geometricity is classified. In this paper, the result above is generalized to singular symplectic space. This paper characterizes the subspaces in these lattices, classifies their geometricity, and computes their characteristic polynomials.

1. Introduction

In the following we recall some definitions and facts on ordered sets and lattices (see [1]).

Let P denote a finite set. A *partial order* on P is a binary relation $\le$ on P such that

(1) $a \le a$ for any $a \in P$.

(2) $a \le b$ and $b \le a$ implies $a = b$.

(3) $a \le b$ and $b \le c$ implies $a \le c$.

By a *partial ordered set* (or *poset* for short), we mean a pair $(P, \le)$, where P is a finite set and $\le$ is a partial order on P. As usual, we write $a < b$ whenever $a \le b$ and $a \ne b$. By abusing notation, we will suppress reference to $\le$, and just write P instead of $(P, \le)$.

Let P be a poset and let R be a commutative ring with the identical element. A binary function $\mu(a, b)$ on P with values in R is said to be the *Möbius function* of P if

$$\sum_{a \le c \le b} \mu(a, c) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For any two elements $a, b \in P$, we say a *covers* b, denoted by $b < \cdot a$, if $b < a$ and there exists no $c \in P$ such that $b < c < a$. An element m of P is said to be *minimal*, (resp., *maximal*) whenever there is no element $a \in P$ such that $a < m$, (resp., $a > m$). If P has a unique minimal, (resp., maximal) element, then we denote it by 0, (resp., 1) and say that P is a poset with 0, (resp., 1). Let P be a finite poset with 0. By a *rank function* on P, we mean a function r from P to the set of all the nonnegative integers such that

(1) $r(0) = 0$,

(2) $r(a) = r(b) + 1$, whenever $b < \cdot a$.

Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a)\, x^{r(1) - r(a)}, \tag{2}$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset P is said to be a *lattice* if both $a \vee b := \sup\{a, b\}$ and $a \wedge b := \inf\{a, b\}$ exist for any two elements $a, b \in P$. Let P be a finite lattice with 0. By an *atom* in P, we mean an element in P covering 0. We say P is *atomic lattice* if any element in $P \setminus \{0\}$ is a union of atoms. A finite atomic lattice P is said to be a *geometric lattice* if P admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \le r(a) + r(b), \quad \forall a, b \in P. \tag{3}$$

In this section we will introduce the concepts of subspaces of type (m, s, k) in singular symplectic spaces. Notation and terminologies will be adopted from Wan's book [2].

We always assume that

$$K_{\nu,l} = \begin{pmatrix} 0 & I^{(\nu)} & \\ -I^{(\nu)} & 0 & \\ & & 0^{(l)} \end{pmatrix}. \tag{4}$$

Let F_q be a finite field with q elements, where q is a prime power, and let E denote the subspace of $F_q^{(2\nu+l)}$ generated by $e_{2\nu+1}, e_{2\nu+2}, \ldots, e_{2\nu+l}$, where e_i is the row vector in $F_q^{(2\nu+l)}$ whose ith coordinate is 1 and all other coordinates are 0.

The singular symplectic group of degree $2\nu + l$ over F_q, denoted by $Sp_{2\nu+l,2\nu}(F_q)$, consists of all $(2\nu + l) \times (2\nu + l)$ nonsingular matrices T over F_q satisfying $TK_lT^t = K_l$. The row vector space $F_q^{(2\nu+l)}$ together with the right multiplication action of $Sp_{2\nu+l,2\nu}(F_q)$ is called the $(2\nu + l)$-dimensional singular symplectic space over F_q. An m-dimensional subspace P in the $(2\nu+l)$-dimensional singular symplectic space is said to be of type (m, s, k), if PK_lP^t is of rank $2s$ and $\dim(P \cap E) = k$. In particular, subspaces of type $(m, 0, 0)$ are called m-dimensional totally isotropic subspaces. Clearly, singular symplectic group $Sp_{2\nu+l,2\nu}(F_q)$ is transitive on the set of all subspaces of the same type in $F_q^{(2\nu+l)}$, see [2, Theorems 3.22].

The results on the lattices generated by one orbit of subspaces under finite classical groups may be found in Gao and You [3], Huo et al. [4–6], Huo and Wan [7], Orlik and Solomon [8], Wang and Feng [9], Wang and Guo [10], and Wang and Li [11].

For $1 \le m_1 \le m_2 \le 2\nu - 1$, $0 \le k_1 \le k_2 \le l$, let $L_1(m_1, s_1, k_1; 2\nu + l, \nu)$, (resp., $L_2(m_2, s_2, k_2; 2\nu + l, \nu)$) denote the set of all subspaces which are sums (resp. intersections) of subspaces in $M(m_1, s_1, k_1; 2\nu + l, \nu)$, (resp., $M(m_2, s_2, k_2; 2\nu + l, \nu)$) such that $M(m_2, s_2, k_2; 2\nu + l, \nu) \subseteq L_1(m_1, s_1, k_1; 2\nu + l, \nu)$, (resp., $M(m_1, s_1, k_1; 2\nu + l, \nu) \subseteq L_2(m_2, s_2, k_2; 2\nu + l, \nu)$). Suppose $L(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu+l, \nu)$ denotes the intersection of $L_1(m_1, s_1, k_1; 2\nu+l, \nu)$ and $L_2(m_2, s_2, k_2; 2\nu+l, \nu)$ containing 0 and $F_q^{(2\nu+l)}$. By ordering $L(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ by ordinary or reverse inclusion, two families of atomic lattices are obtained, denoted by $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ or $L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$, respectively. Wang and Guo [12] discussed the geometricity of the two lattices when $l = 0$. In this paper, we generalized their result to general case, characterizes the subspaces in these lattices in Section 2, classify their geometricity in Section 3, and computes their characteristic polynomials in Section 4.

2. Characterization of Subspaces Contained in $L(m_1,s_1,k_1;m_2,s_2,k_2;2\nu+l,\nu)$

Lemma 1. *Let $2\nu + l > 0$, $0 \le k \le l$, $2s \le m - k \le \nu + s$ and $m - k \ge 1$. For any subspace P of type $(m + 1, s, k + 1)$, there are two subspaces P_1 and P_2 of type (m, s, k) such that $P = P_1 + P_2$.*

Proof. Assume that

$$PK_\nu P^T = \begin{array}{c} \begin{pmatrix} k_s & & \\ & 0 & \\ & & 0 \end{pmatrix} \end{array} \begin{array}{c} 2s \\ m + 1 - (k + 1) - 2s \\ k + 1 \end{array} . \tag{5}$$

Write $P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}\begin{matrix} m+1-(k+1) \\ k+1 \end{matrix}$ with column sizes $\begin{matrix} 2\nu & l \end{matrix}$, where $P_{22} = \begin{pmatrix} P'_{22} \\ z_1 \\ z_2 \end{pmatrix}\begin{matrix} k-1 \\ 1 \\ 1 \end{matrix}$. Then $P_1 = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P'_{22} \\ 0 & z_1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P'_{22} \\ 0 & z_2 \end{pmatrix}\begin{matrix} m+1-(k+1) \\ k-1 \\ 1 \end{matrix}$ are subspace of type (m, s, k), such that $P = P_1 + P_2$. □

Lemma 2. *Let $2\nu + l > 0$, $0 \le k \le l$, $2s \le m - k \le \nu + s$ and $m \ne 2\nu + l$. For any subspace P of type $(m + 1, s + 1, k)$, there are two subspaces P_1 and P_2 of type (m, s, k) such that $P = P_1 + P_2$.*

Proof. Assume that $PK_\nu P^T = \begin{pmatrix} P_0 \\ x_1 \\ y_1 \end{pmatrix} K_\nu \begin{pmatrix} P_0 \\ x_1 \\ y_1 \end{pmatrix}^T = \begin{pmatrix} k_s & & & \\ & 0 & & \\ & & k_1 & \\ & & & 0 \end{pmatrix}\begin{matrix} 2s \\ m+1-2s-k-2 \\ 2 \\ k \end{matrix}$, write $P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P'_{22} \end{pmatrix}\begin{matrix} m+1-k \\ k \end{matrix}$ with column sizes $\begin{matrix} 2\nu & l \end{matrix}$, where $P_{11} = \begin{pmatrix} P_0 \\ x_1 \\ y_1 \end{pmatrix}\begin{matrix} m+1-k-2 \\ 1 \\ 1 \end{matrix}$, x_1 and y_1 are the $(m + 1 - (k + 1))$-th and $(m - k + 1)$-th row vectors of P_{11}, respectively.

$P_{12} = \begin{pmatrix} P'_0 \\ x'_1 \\ y'_1 \end{pmatrix}\begin{matrix} m+1-k-2 \\ 1 \\ 1 \end{matrix}$, x'_1 and y'_1 are the $(m + 1 - (k + 1))$-th and $(m - k + 1)$-th row vectors of P_{12}, respectively.

Then

$$P_1 = \begin{pmatrix} P_0 & P'_0 \\ x_1 & x'_1 \\ 0 & P_{22} \end{pmatrix} \begin{matrix} m + 1 - k - 2 \\ 1 \\ k, \end{matrix}$$
$$P_2 = \begin{pmatrix} P_0 & P'_0 \\ y_1 & y'_1 \\ 0 & P_{22} \end{pmatrix} \begin{matrix} m + 1 - k - 2 \\ 1 \\ k \end{matrix} \tag{6}$$

are subspace of type (m, s, k), such that $P = P_1 + P_2$. □

Theorem 3. *Let $2\nu + l > 0$, assume that (m, s, k), (m_1, s_1, k_1) satisfy $0 \le k \le l$, $2s \le m - k \le \nu + s$, $0 \le k_1 \le l$, $2s \le m_1 - k_1 \le \nu + s$. For any subspace P of type (m, s, k), there are subspace $P_1, P_2, \ldots, P_i$ of type (m_1, s_1, k_1) such that $P = P_1 + P_2 + \cdots + P_i$ if and only if*

$$\text{(i) } s = s_1 = \nu, \qquad 0 \le k_1 \le k \le l, \tag{7}$$

$$\begin{gathered} \text{(ii) } s < \nu, \qquad k_1 = k = l, \\ m - m_1 \ge s - s_1 \ge 0, \end{gathered} \tag{8}$$

$$\begin{gathered} \text{(iii) } s < \nu, \qquad 0 \le k_1 \le k \le l, \\ (m - k) - (m_1 - k_1) \ge s - s_1 \ge 0. \end{gathered} \tag{9}$$

Proof. Suppose that (m, s, k) and (m_1, s_1, k_1) satisfy condition (7). Let $k - k_1 = h (h \ge 0)$, since $s = s_1 = \nu$, $m = 2\nu + k$, $m_1 = 2\nu + k_1$, $m - m_1 = k - k_1 = h$, By Lemma 1 the desired result follows. Suppose (m, s, k) and (m_1, s_1, k_1) satisfy condition (8). Let $s - s_1 = t$, $m - m_1 = t + t'(t, t' \ge 0)$. By Lemma 1, any subspace of type (m, s, l) is the sum some subspaces of type $(m - t, s, l)$. By Lemma 2, any subspace of type $(m - t, s, l)$ is the sum some subspaces of type (m_1, s_1, l). Hence the desired result follows.

Suppose (m, s, k) and (m_1, s_1, k_1) satisfy condition (9). Let $s - s_1 = t$, $k - k_1 = h$, $m - m_1 = t + h + t'$ $(t, t', h \ge 0)$, By Lemma 1, we have any subspace of type (m, s, k) is the sum some subspace of type $(m - t', s, k)$. By Lemma 2, we have any subspace of type $(m - t', s, k)$ is the sum some subspace of type $(m - t' - t, s_1, k)$. By Lemma 1, we have any subspace of type $(m - t' - t, s_1, k)$ is the sum some subspace of type (m_1, s_1, k_1). Hence the desired result follows.

Conversely, If $s_1 = \nu$, then $s = \nu$, $k_1 \le l$. Let $Q \in M(m, \nu, k; 2\nu + l, \nu)$, there exists $P \in M(m_1, \nu, k_1; 2\nu + l, \nu)$, such that $P \subset Q$, hence $P \cap E \subset Q \cap E$, $k_1 = \dim(P \cap E) \leq \dim(Q \cap E) = k$. Therefore, $s = s_1 = \nu$, $0 \leq k_1 \leq k \leq l$, condition (7) hold.

If $s_1 < \nu$, let $P \in M(m_1, \nu, k_1; 2\nu + l, \nu)$, $Q \in M(m, \nu, k : 2\nu + l, \nu)$, such that $P \subset Q$, then $P \cap E \subset Q \cap E$, and $k_1 = \dim(P \cap E) \leq \dim(Q \cap E) = k$. If $k_1 = l$, then $k = l$ and $m_1 \geq l$, Assume $Q = \begin{array}{cc} \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} & \begin{array}{c} m-1 \\ l \end{array} \\ \begin{array}{cc} 2\nu & l \end{array} & \end{array}$ where rank $Q_{11} = m - l$, rank $Q_{22} = l$ and $Q_{11} K Q^t = M(m-l, s)$. Since $P \cap E \subset Q \cap E$, there exists a $l \times l$ matrix P_{22} with rank $P_{22} = l$, such that $P_{22} Q_{22}$ is a matrix representation of subspace $P \cap E$. Thus we can assume $P = \begin{array}{cc} \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix} & \begin{array}{c} m_1-1 \\ l \end{array} \\ \begin{array}{cc} 2\nu & l \end{array} & \end{array}$, where P_{11} is a matrix with rank $m_1 - l$. Because P is a subspaces of type (m_1, s_1, l), we can assume $P_{11} M(m-l, s) P_{11}^t = M(m_1 - l, s)$. Then Q_{11} can be considered as a subspace of type $(m_1 - l, s_1)$ in singular symplectic space $F_q^{(2s+(m-2s-l))}$. Hence $m_1 - l - s - s_1 \leq m - 2s - l$, condition (8) hold. Similarly we also can prove condition (9) hold. □

Theorem 4. *Let* $2\nu + l > 0$, *assume that* (m, s, k) *and* (m_2, s_2, k_2) *satisfy* $0 \leq k \leq l$, $2s \leq m - k \leq \nu + s$, $0 \leq k_2 \leq l$, $2s_2 \leq m_2 - k_2 \leq \nu + s$. *For any subspace* P *of type* (m, s, k), *there are subspace* $P_1, P_2, \ldots, P_j$ *of type* (m_2, s_2, k_2) *such that* $P = P_1 \cap P_2 \cap \cdots \cap P_j$ *if and only if*

(i) $s = s_2 = \nu$, $0 \leq k_2 \leq k \leq l$,

(ii) $s < \nu$, $k_2 = k = l$ *and* $m_2 - m \geq s_2 - s \geq 0$,

(iii) $s < \nu$, $k \leq k_2 \leq l$ *and* $(m_2 - k_2) - (m - k) \geq s_2 - s \geq 0$.

Proof. By [3], it is directed. □

Theorem 5. *For* $1 \leq m_1 \leq m_2 \leq 2\nu + l$, $L(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ *consist of* $\{0\}$, $F_q^{(2\nu+l)}$ *and all subspaces of type* (m, s, k) *in* $F_q^{(2\nu+l)}$ *such that*

(i) $s = s_1 = s_2 = \nu$, $0 \leq k_1 \leq k_2 \leq k \leq l$,

(ii) $s < \nu$, $k_1 = k = k_2 = l$ *and* $m - m_1 \geq s - s_1 \geq 0$, $m_2 - m \geq s_2 - s \geq 0$,

(iii) $s < \nu$, $k_1 \leq k \leq k_2 \leq l$ *and* $(m_2 - k_2) - (m - k) \geq s_2 - s \geq 0$, $(m - k) - (m_1 - k_1) \geq s - s_1 \geq 0$.

Proof. By Theorems 3 and 4, it is directed. □

3. The Geometricity of Lattices $L_O(m_1,s_1,k_1;m_2,s_2,k_2;2\nu + l,\nu)$ and $L_R(m_1,s_1,k_1;m_2,s_2,k_2;2\nu + l,\nu)$

Lemma 6 (see [3]). *If* $0 < k < l$, *then*

(i) $L_R(2\nu+k, \nu, k; 2\nu+l, \nu) \simeq L_R(k, l)$, $L_O(2\nu+k, \nu, k; 2\nu+l, \nu) \simeq L_O(k, l)$,

(ii) $L_R(k, 0, k; 2\nu + l, \nu) \simeq L_R(k, l)$, $L_O(k, 0, k; 2\nu + l, \nu) \simeq L_O(k, l)$.

Lemma 7 (see [3]). *If* $0 \leq s < \nu$ *and* $2s \leq m - l \leq \nu + s$, *then*

(i) $L_R(m, s, l; 2\nu + l, \nu) \simeq L_R(m - l, s; 2\nu)$, $L_O(m, s, l; 2\nu + l, \nu) \simeq L_O(m - l, s; 2\nu)$,

(ii) $L_R(m, s, 0; 2\nu + l, \nu) \simeq L_R(m, s; 2\nu)$, $L_O(m, s, 0; 2\nu + l, \nu) \simeq L_O(m, s; 2\nu)$.

Theorem 8. *Let* $2\nu + l > 0$. *Assume that* (m_1, s_1, k_1), (m_2, s_2, k_2) *satisfies* $0 \leq k_1 \leq l$, $2s_1 \leq m_1 - k_1 \leq \nu + s_1$, $0 \leq k_2 \leq l$, $2s_2 \leq m_2 - k_2 \leq \nu + s_2$ *and* $1 \leq m_1 \leq m_2 < 2\nu + l$. *Then*

(i) $L_O(k + 1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ *is a finite geometric lattice if and only if* $k = 0, l$,

(ii) $L_O(k, 0, k; 2\nu + k, \nu, k; 2\nu + l, \nu)$ *is a finite geometric lattice if and only if* $k = 1, l - 1$,

(iii) $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ *is not a geometric lattice when* $2 \leq m_1 - k_1 \leq m_2 - k_2 \leq 2\nu - 2$.

Proof. For any $X \in L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$, define

$$r_O(X) = \begin{cases} 0, & \text{if } X = 0, \\ m_2 - m_1 + 2, & \text{if } X = F_q^{(2\nu+l)}, \\ \dim X - m_1 + 1, & \text{otherwise.} \end{cases} \quad (10)$$

Then r_O is the rank function of $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$.

(i) For lattice $L_O(k + 1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$.

If $k = 0$, by Lemma 7, $L_O(1, 0, 0; 2\nu - l, \nu - 1, 0; 2\nu + l, \nu) \simeq L_O(1, 0, 2\nu - 1, \nu - 1; 2\nu)$, by [12], $L_O(k + 1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ is a finite geometric lattice.

If $k = l$, by Lemma 7, $L_O(l + 1, 0, l; 2\nu - l + l, \nu - 1, l; 2\nu + l, \nu) \simeq L_O(1, 0, 2\nu - 1, \nu - 1; 2\nu)$, by [12], $L_O(k + 1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ is a finite geometric lattice.

If $0 < k < l$. Let $U = \langle e_1, e_{2\nu+1}, \ldots, e_{2\nu+k} \rangle$ and $W = \langle e_{\nu+1}, e_{2\nu+2}, \ldots, e_{2\nu+k+1} \rangle$, then U and W both are of type $(k + 1, 0, k)$, $\langle U, W \rangle$ is of type $(k + 3, 1, k + 1)$, $\langle U, W \rangle \in L_O(k + 1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$, $U \cap W$ is of type $(k - 1, 0, k - 1)$, hence $r_O(U \wedge W) = 0$, $r_O(U \vee W) = 3$, $r_O(U) = r_O(W) = 1$. We have

$$r_O(U \wedge W) + r_O(U \vee W) = 3 > r_O(U) + r_O(W) = 2. \quad (11)$$

That is $L_O(k + 1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ is not a geometric lattice when $0 < k < l$.

(ii) For lattice $L_O(k, 0, k; 2\nu + k, \nu, k; 2\nu + l, \nu)$, by Lemma 6 $L_O(k, 0, k; 2\nu + k, \nu, k; 2\nu + l, \nu) \simeq L_O(k, l)$.

If $k = 1, l - 1$, $L_O(1, l)$ and $L_O(l - 1, l)$ is a geometric lattice.

If $2 \leq k \leq l - 2$, let $v_1, v_2, \ldots, v_l$ be a basis of $F_q^{(l)}$. Since $2 \leq k \leq l - 2$, we can take $U = \langle v_1, v_2, \ldots, v_k \rangle$, $W = \langle v_3, v_4, \ldots, v_{k+2} \rangle \in L_O(k, l)$. Hence $U \wedge W = \{0\}$, $U \vee W = \langle U, W \rangle$, $r_O(U \wedge W) = 0$, $r_O(U \vee W) = 3$, $r_O(U) = r_O(W) = 1$. We have

$$r_O(U \wedge W) + r_O(U \vee W) = 3 > r_O(U) + r_O(W) = 2. \quad (12)$$

That is $L_O(k, 0, k; 2\nu + k, \nu, k; 2\nu + l, \nu)$ is not a geometric lattice when $2 \leq k \leq l - 2$.

(iii) For lattice $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$, when $2 \leq m_1 - k_1 \leq m_2 - k_2 \leq 2\nu - 2$.

Case (a). $k_1 \le k_2 \le l$.

(a_1) $s_2 > 0$, Let

$$U = \left(\begin{array}{ccccccccc} I^{(s_2-1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s_2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & I^{(m_2-k_2-2s_2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_2)} & 0 \\ s_2-1 & 1 & m_2-k_2-2s_2 & \nu+s_2-m_2+k_2 & s_2 & m_2-k_2-2s_2 & \nu+s_2-m_2+k_2 & k_2 & l-k_2 \end{array}\right),$$

$$W = \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_2)} & 0 \\ s_2-1 & 1 & m_2-k_2-2s_2 & 1 & \nu+s_2-m_2+k_2 & s_2 & m_2-k_2-2s_2 & \nu+s_2-m_2+k_2 & k_2 & l-k_2 \end{array}\right). \quad (13)$$

Then U is a of type (m_2-1, s_2-1, k_2), W is a of type $(k_2+1, 0, k_2)$, $\langle U, W\rangle$ is a of type (m_2, s_2-1, k_2), $r_O(U \vee W) = r_O(F_q^{(2\nu+l)}) = m_2 - m_1 + 2$, $r_O(U \wedge W) = k_2 - m_1 + 1$, $r_O(U) = m_2 - 1 - m_1 + 1 = m_2 - m_1$, $r_O(W) = k_2 + 1 - m_1 + 1 = k_2 - m_1 + 2$, $r_O(U \vee W) + r_O(U \wedge W) = m_2 - 2m_1 + k_2 + 3$, $r_O(U) + r_O(W) = m_2 - 2m_1 + k_2 + 2$. We have

$$r_O(U \vee W) + r_O(U \wedge W) > r_O(U) + r_O(W). \quad (14)$$

Hence, $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ is not a geometric lattice when (a_1).

(a_2) $s_2 > 0$, $m_2 - k_2 = \nu + s_2$ or $s_2 = 0$. When $s_2 = 0$, we have $m_2 - k_2 - 2s_2 \ge 1$, when $s_2 > 0$, $m_2 - k_2 = \nu + s_2$, we have $\nu - 2 \ge s_2$, $m_2 - k_2 - 2s_2 - 1 = \nu - s_2 - 1 \ge 1$.

Let

$$U = \left(\begin{array}{cccccccccc} I^{(s_2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I^{(s_2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & I^{(m_2-k_2-2s_2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_2)} & 0 \\ s_2 & m_2-k_2-2s_2-1 & 1 & \nu+s_2-m_2+k_2 & s_2 & m_2-k_2-2s_2-1 & 1 & \nu+s_2-m_2+k_2 & k_2 & l-k_2 \end{array}\right),$$

$$W = \left(\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_2)} & 0 \\ s_2 & 1 & \nu - s_2 - 1 & s_2 & 1 & \nu - s_2 - 1 & k_2 & l - k_2 \end{array}\right). \quad (15)$$

Then U is a of type (m_2-1, s_2, k_2), W is a of type $(k_2+1, 0, k_2)$, $\langle U, W\rangle$ is a of type (m_2, s_2+1, k_2), clearly, we have

$$r_O(U \vee W) + r_O(U \wedge W) > r_O(U) + r_O(W). \quad (16)$$

Hence, $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ is not a geometric lattice when (a_2).

Case (b). $k_1 = k_2 = l$.

In this case $L_O(m_1, s_1, l; m_2, s_2, l; 2\nu + l, \nu) \simeq L_O(m_1 - l, s_1; m_2 - l, s_2; 2\nu)$, $L_O(m_1, s_1, l; m_2, s_2, l; 2\nu + l, \nu)$ is not a geometric lattice.

Hence $L_O(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ is not a geometric lattice when $2 \le m_1 - k_1 \le m_2 - k_2 \le 2\nu - 2$. ☐

Theorem 9. *Let $2\nu + l > 0$. Assume that (m_1, s_1, k_1), (m_2, s_2, k_2) satisfies $0 \le k_1 \le l$, $2s_1 \le m_1 - k_1 \le \nu + s_1$, $0 \le k_2 \le l$, $2s_2 \le m_2 - k_2 \le \nu + s_2$ and $1 \le m_1 \le m_2 < 2\nu + l$. Then*

(i) $L_R(k+1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ *is a finite geometric lattice if and only if* $k = 0, l$,

(ii) $L_R(k, 0, k; 2\nu + k, \nu, k; 2\nu + l, \nu)$ *is a finite geometric lattice if and only if* $k = 1, l - 1$,

(iii) $L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ *is not a geometric lattice when* $2 \le m_1 - k_1 \le m_2 - k_2 \le 2\nu - 2$.

Proof. (i) For $X \in L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$, define

$$r_{R(X)} = \begin{cases} 0, & \text{if } X = F_q^{((2\nu+l))}, \\ m_2 - m_1 + 2, & \text{if } X = 0, \\ m_2 + 1 - \dim X, & \text{otherwise.} \end{cases} \quad (17)$$

For lattice $L_R(k+1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$.

If $k = 0$, by Lemma 7, $L_R(1, 0, 0; 2\nu - l, \nu - 1, 0; 2\nu + l, \nu) \simeq L_R(1, 0, 2\nu - 1, \nu - 1; 2\nu)$, by [12], $L_R(k+1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ is a finite geometric lattice.

If $k = l$, by Lemma 7, $L_R(l+1, 0, l; 2\nu - l + l, \nu - 1, l; 2\nu + l, \nu) \simeq L_R(1, 0, 2\nu - 1, \nu - 1; 2\nu)$, by [12], $L_R(k+1, 0, k; 2\nu - 1 + k, \nu - 1, k; 2\nu + l, \nu)$ is a finite geometric lattice.

If $0 < k < l$, let $U = \langle e_1, \ldots, e_{\nu-1}, e_{\nu+1}, \ldots, e_{2\nu}, e_{2\nu+1}, \ldots, e_{2\nu+k}\rangle$, $W = \langle e_2, \ldots, e_\nu, e_{\nu+2}, \ldots, e_{2\nu+1}, e_{2\nu+2}, \ldots, e_{2\nu+k}\rangle$. Then U, W both are of type $(2\nu - 1 + k, \nu - 1, k)$, $U \wedge W = \langle U, W\rangle$ is of type $(2\nu + 1 + k, \nu, k+1)$, $U \vee W = U \cap W$ is of type $(2\nu - 3 + k, \nu, k)$. We have

$$r(U \wedge W) + r(U \vee W) = 3 > r(U) + r(W) = 2. \quad (18)$$

Hence, $L_R(k+1,0,k;2\nu-1+k,\nu-1,k;2\nu+l,\nu)$ is not a geometric lattice when $0<k<l$.

(ii) if $k=0,l-1$, by Lemma 7 $L_R(k,0,k;2\nu+k,\nu,k;2\nu+l,\nu)\simeq L_R(k,l)$ when $k=1,l-1$, $L_R(k,0,k;2\nu+k,\nu,k;2\nu+l,\nu)$ is a geometric lattice.

If $2\le k\le l-2$, by [7], $L_R(k,l)$ is not a geometric lattice.

(iii) Case (a). if $m_1-k_1<\nu+s_1$, then $\nu+s_1-m_1+k_1-1\ge 0$. Let

$$U=\left(\begin{array}{cccccccccc}
I^{(s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I^{(s_1)} & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I^{(m_1-k_1-2s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_1)} & 0\\
s_1 & 1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1-1 & s_1 & 1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1 & k_1 & l-k_1
\end{array}\right),$$

$$W=\left(\begin{array}{cccccccccc}
0 & I^{(s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I^{(s_1)} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I^{(m_1-k_1-2s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_1)} & 0\\
1 & s_1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1-1 & 1 & s_1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1-1 & k_1 & l-k_1
\end{array}\right). \tag{19}$$

Then U,W are of type (m_1+1,s_1,k_1), $U\cap W$ is of type (m_1,s_1-1,k_1), $\langle U,W\rangle$ is of type (m_1+1,s_1,k_1), $\langle U,W\rangle\in L_R(m_1,s_1,k_1;m_2,s_2,k_2;2\nu+l,\nu)$, $U\cap W\notin L_R(m_1,s_1,k_1;m_2,s_2,k_2;2\nu+l,\nu)$.

So $r_R(U\vee W)=m_2-m_1+2$, $r_R(U\wedge W)=m_2-m_1-1$, $r_R(U)=m_2+1-(m_1+1)=m_2-m_1=r_R(W)$. We have

$$\begin{aligned}
r_R(U\wedge W)+r_R(U\vee W)&=2m_2-2m_1+1\\
&>r_R(U)+r_R(W)\\
&=2m_2-2m_1.
\end{aligned} \tag{20}$$

Hence $L_R(m_1,s_1,k_1;m_2,s_2,k_2;2\nu+l,\nu)$ is not a geometric lattice when (a).

Case (b). $m_1-k_1=\nu+s_1$, from $m_1-k_1=\nu+s_1$, and $2\le m_1-k_1\le 2\nu-2$, we have $\nu-2\ge s_1$, $m_1-k_1-2s_1-2=\nu-s_1-2\ge 0$.

Let

$$U=\left(\begin{array}{cccccccccccc}
I^{(s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I^{(s_1)} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I^{(m_1-k_1-2s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_1)} & 0\\
s_1 & 1 & 1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1 & s_1 & 1 & 1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1 & k_1 & l-k_1
\end{array}\right),$$

$$W=\left(\begin{array}{cccccccccccc}
I^{(s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I^{(s_1)} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I^{(m_1-k_1-2s_1)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I^{(k_1)} & 0\\
s_1 & 1 & 1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1 & s_1 & 1 & 1 & m_1-k_1-2s_1 & \nu+s_1-m_1+k_1 & k_1 & l-k_1
\end{array}\right). \tag{21}$$

Then U, W are of type (m_1+1,s_1+1,k_1), $U\cap W$ is of type (m_1,s_1+1,k_1), and $\langle U,W\rangle$ is of type (m_1+2,s_1+2,k_1).

So $r_R(U\vee W)=m_2-m_1$, $r_R(U\wedge W)=m_2+1-(m_1+2)=m_2-m_1-1$, $r_R(U\wedge W)+r_R(U\vee W)=2m_2-2m_1+1>r_R(U)+r_R(W)=2m_2-2m_1$.

Hence $L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu+l, \nu)$ is not a geometric lattice when (b).

Hence $L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu+l, \nu)$ is not a geometric lattice when $2 \le m_1 - k_1 \le m_2 - k_2 \le 2\nu - 2$. □

4. Characteristic Polynomial of Lattice $L_R(m_1,s_1,k_1;m_2,s_2,k_2;2\nu+l,\nu)$

In this section we compute the characteristic polynomial of the lattice $L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$.

Theorem 10. *Let $2\nu + l > 0$, Assume that (m, s, k) satisfies $0 \le k \le l$, $2s \le m - k \le \nu + s$ and $0 < m < 2\nu + l$. Then*

$$\begin{aligned}\chi&\left(L_R\left(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu\right), t\right)\\ &= t^{m_2-m_1+2}\\ &\quad - \sum_{k=k_1}^{k}\sum_{s=s_1}^{s_2}\sum_{m=m_2-(k_2-k)+(s_2-s)+1}^{m_1-(k-k_1)+(s-s_1)} N\left(m, s, k; 2\nu + l, \nu\right) g_m(t)\\ &\quad + \sum_{k=0}^{k_1}\sum_{s=0}^{s_1-1}\sum_{m=2s+k}^{\nu+s+k} N\left(m, s, k; 2\nu + l, \nu\right) g_m(t)\\ &\quad + \sum_{k=k_2}^{l}\sum_{s=s_2+1}^{\nu}\sum_{m=2s+k}^{\nu+s+k} N\left(m, s, k; 2\nu + l, \nu\right) g_m(t),\end{aligned} \tag{22}$$

where $g_m(t) = \prod_{i=0}^{m-1}(t - q^i)$.

Proof. Define

$$r_R(X) = \begin{cases} 0, & \text{if } X = F_q^{(2\nu+l)}, \\ m_2 - m_1 + 2, & \text{if } X = \{0\}, \\ m_2 + 1 - \dim(X), & \text{otherwise.} \end{cases} \tag{23}$$

Then r_R is the rank function on $L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$. Let $V = F_q^{(2\nu+l)}$, $L_O = L_R(2\nu + l, \nu)$, $L = L_R(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu)$ we have

$$\chi(L_O, t) = \prod_{i=0}^{2\nu+l-1}\left(t - q^i\right). \tag{24}$$

For any $P \in L$, define

$$\begin{aligned} L^P &= \{Q \in L \mid Q \subset P\} = \{Q \in L \mid Q \ge P\},\\ L_O^P &= \{Q \in L_O \mid Q \subset P\} = \{Q \in L_O \mid Q \ge P\}.\end{aligned} \tag{25}$$

Clearly, $L^V = L$, $L^P = L_O^P$, when $P \ne \{0\}$, $P \ne V$

$$\begin{aligned}\chi\left(L^V, t\right) &= \chi(L, t) = \sum_{P\in L}\mu(0, P)\, t^{r(1)-r(P)}\\ &= \sum_{P\in L}\mu(0, P)\, t^{r(0)-r(P)}.\end{aligned} \tag{26}$$

By inversion to Möbius we have

$$\chi(L, t) = \chi\left(L^V, t\right) = t^{m_2-m_1+2} - \sum_{P\in L\setminus\{V\}}\chi\left(L^P, t\right), \tag{27}$$

by Theorem 5 we have

$$\begin{aligned}\chi&\left(L_R\left(m_1, s_1, k_1; m_2, s_2, k_2; 2\nu + l, \nu\right), t\right)\\ &= t^{m_2-m_1+2}\\ &\quad - \sum_{k=k_1}^{k_2}\sum_{s=s_1}^{s_2}\sum_{m=m_2-(k_2-k)+(s_2-s)+1}^{m_1-(k-k_1)+(s-s_1)} N\left(m, s, k; 2\nu + l, \nu\right) g_m(t)\\ &\quad + \sum_{k=0}^{k_1}\sum_{s=0}^{s_1-1}\sum_{m=2s+k}^{\nu+s+k} N\left(m, s, k; 2\nu + l, \nu\right) g_m(t)\\ &\quad + \sum_{k=k_2}^{l}\sum_{s=s_2+1}^{\nu}\sum_{m=2s+k}^{\nu+s+k} N\left(m, s, k; 2\nu + l, \nu\right) g_m(t).\end{aligned} \tag{28}$$

□

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant no. 61179026 and Supported by the Fundamental Research Funds for the Central Universities (ZXH2012 K003).

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