Research Article
Graph Operations and Neighbor Rupture Degree

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In a communication network, the vulnerability parameters measure the resistance of the network to disruption of operation after the failure of certain stations or communication links. A vertex subversion strategy of a graph G, say S, is a set of vertices in G whose closed neighborhood is removed from G. The survival subgraph is denoted by G/S. The neighbor rupture degree of G, $\mathrm{Nr}(G)$, is defined to be $\mathrm{Nr}(G) = \max\{w(G/S) - |S| - c(G/S) : S \subset V(G), w(G/S) \geq 1\}$, where S is any vertex subversion strategy of G, $w(G/S)$ is the number of connected components in G/S and $c(G/S)$ is the maximum order of the components of G/S (G. Bacak Turan, 2010). In this paper we give some results for the neighbor rupture degree of the graphs obtained by some graph operations.

1. Introduction

A network can be broke down completely or partially with unexpected reasons. If the data is not transmitted to the desired location that means there is a problem on the system. This problem can block a treaty of billions of euros or make a big problem for human's life. In these days the reliability and the vulnerability of networks are so important. For that reason graphs are taken as a model in the research area of reliability and vulnerability of the networks. Each network center is taken as a vertex and the connections of these vertices are edges of a graph.

A few questions can be asked at this point How can the reliability and the vulnerability of a network be determined? What are the factors of the reliability and the vulnerability? For example, what can be done if there is a problem on the way you are using every day to work? We have two choices; we may give up going to work although we have the risk of dismissal or we can look for another way to work. The question "if there is another way to reach work" may come to our minds. In other words "Has the link connection between home and work completely broken down?". To answer this question, we must know the dimensions of the problem between home and work. The vulnerability of the graph which represents the way between home and work should be searched. In graph theory some vulnerability parameters are defined to measure the vulnerability value of graphs such as connectivity [1], integrity [2], neighbor integrity [3], rupture degree [4], and neighbor rupture degree [5].

Terminology and notation not defined in this paper can be found in [5]. Let G be a simple graph and let u be any vertex of G. The set $N(u) = \{v \in V(G) \mid v \neq u; v$ and u are adjacent$\}$ is the open neighborhood of u, and $N[u] = \{u\} \cup N(u)$ is the closed neighborhood of u. A vertex u in G is said to be subverted if the closed neighborhood of u is removed from G. A set of vertices $S = \{u_1, u_2, \ldots, u_m\}$ is called a vertex subversion strategy of G if each of the vertices in S has been subverted from G. If S has been subverted from the graph G, then the remaining graph is called survival graph, denoted by G/S.

2. Basic Results

In this paper the new vulnerability parameter neighbor rupture degree was studied. The concept of neighbor rupture degree was introduced by Bacak-Turan and Kırlangıc in 2011 [5]. The definition of neighbor rupture degree and some results are given below.

Definition 1 (see [6]). The neighbor rupture degree of a non-complete connected graph G is defined to be

$$
\begin{aligned}
&\mathrm{Nr}\,(G) \\
&\quad = \max\left\{ w\left(\frac{G}{S}\right) - |S| - c\left(\frac{G}{S}\right) : S \subset V\,(G)\,, w\left(\frac{G}{S}\right) \geq 1 \right\},
\end{aligned}
\tag{1}
$$

where S is any vertex subversion strategy of G, $w(G/S)$ is the number of connected components in G/S, and $c(G/S)$ is the maximum order of the components of G/S.

In particular, the neighbor rupture degree of a complete graph K_n is defined to be $\mathrm{Nr}(K_n) = 1 - n$. A set $S \subset V(G)$ is said to be Nr-*set* of G if

$$
\mathrm{Nr}\,(G) = w\left(\frac{G}{S}\right) - |S| - c\left(\frac{G}{S}\right).
\tag{2}
$$

Some known results are listed below.

Theorem 2 (see [6]). (a) *Let P_n be a path graph with n vertices and $n \geq 2$,*

$$
\mathrm{Nr}\,(P_n) = \begin{cases} 0, & n \equiv 1 \pmod 4 \\ -1, & n \equiv 0, 2, 3 \pmod 4. \end{cases}
\tag{3}
$$

(b) *Let C_n be a cycle graph with n vertices and $n \geq 3$,*

$$
\mathrm{Nr}\,(C_n) = \begin{cases} -1, & n \equiv 0 \pmod 4 \\ -2, & n \equiv 1, 2, 3 \pmod 4. \end{cases}
\tag{4}
$$

(c) *Let $K_{n_1, n_2, n_3, \ldots, n_k}$ be a k-partite graph*

$$
\mathrm{Nr}\,(K_{n_1, n_2, \ldots, n_k}) = \max\{n_1, n_2, n_3, \ldots, n_k\} - 3.
\tag{5}
$$

(d) *Let W_n be a wheel graph with n vertices and $n \geq 5$,*

$$
\mathrm{Nr}\,(W_n) = \begin{cases} -1, & n \equiv 1 \pmod 4 \\ -2, & n \equiv 0, 2, 3 \pmod 4. \end{cases}
\tag{6}
$$

3. Graph Operations and Neighbor Rupture Degree

In this section some graph operations are operated on graphs and their neighbor rupture degrees are evaluated.

Definition 3 (see [7]). The union graph $G = G_1 \cup G_2 \cup \cdots \cup G_n$ has vertex set $V(G) = V(G_1) \cup V(G_2) \cup \cdots \cup V(G_n)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n)$. If a graph G consists of k $(k \geq 2)$ disjoint copies of a graph H, then we write $G = kH$.

Theorem 4. *Let $G_1, G_2, G_3, \ldots, G_n$ be connected graphs. Then*

$$
\begin{aligned}
&\mathrm{Nr}\,(G_1 \cup G_2 \cup \cdots \cup G_n) \\
&\quad \geq \mathrm{Nr}\,(G_1) + \mathrm{Nr}\,(G_2) + \cdots + \mathrm{Nr}\,(G_n).
\end{aligned}
\tag{7}
$$

Proof. Let $G = G_1 \cup G_2 \cup \cdots \cup G_n$ be the union of $G_1, G_2, \ldots, G_n$. Let $S_1, S_2, \ldots, S_n$ be Nr-sets of $G_1, G_2, \ldots, G_n$ respectively, and let $S = S_1 \cup S_2 \cup \cdots \cup S_n$ be a subversion strategy of G. Then we obtain

$$
\begin{aligned}
\mathrm{Nr}\,(G) &\geq w\left(\frac{G}{(S_1 \cup S_2 \cup \cdots \cup S_n)}\right) - |S_1 \cup S_2 \cup \cdots \cup S_n| \\
&\quad - c\left(\frac{G}{(S_1 \cup S_2 \cup \cdots \cup S_n)}\right) \\
&= w\left(\frac{G_1}{S_1}\right) + w\left(\frac{G_2}{S_2}\right) + \cdots + w\left(\frac{G_n}{S_n}\right) - |S_1| - |S_2| \\
&\quad - \cdots |S_n| - \max\left\{ c\left(\frac{G_1}{S_1}\right), c\left(\frac{G_2}{S_2}\right), \ldots, c\left(\frac{G_n}{S_n}\right) \right\} \\
&\geq w\left(\frac{G_1}{S_1}\right) + w\left(\frac{G_2}{S_2}\right) + \cdots + w\left(\frac{G_n}{S_n}\right) - |S_1| - |S_2| \\
&\quad - \cdots |S_n| - c\left(\frac{G_1}{S_1}\right) - c\left(\frac{G_2}{S_2}\right) - \cdots - c\left(\frac{G_n}{S_n}\right) \\
&= \mathrm{Nr}\,(G_1) + \mathrm{Nr}\,(G_2) + \mathrm{Nr}\,(G_3) + \cdots + \mathrm{Nr}\,(G_n).
\end{aligned}
\tag{8}
$$

Thus we have $\mathrm{Nr}(G_1 \cup G_2 \cup \cdots \cup G_n) \geq \mathrm{Nr}(G_1) + \mathrm{Nr}(G_2) + \cdots + \mathrm{Nr}(G_n)$. $\square$

Theorem 5. *Let $K_{n_1}, K_{n_2}, \ldots, K_{n_m}$ be complete graphs with $n_1 \leq n_2 \leq n_m$ where $n_{i+1} - n_i \geq 2$; $\forall i \in Z^+$. Then*

$$
\mathrm{Nr}\,(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}) = 2 - m - n_1.
\tag{9}
$$

Proof. Let S be a subversion strategy of $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}$. Since these are complete graphs, it is obvious that S contains at most one vertex from each K_{n_j}.

If $|S| = k$, then $w((K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m})/S) = m - k$ and $c((K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m})/S) \geq n_{m-k}$. Thus we have $w((K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \cdots \cup K_{n_m})/S) - |S| - c((K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \cdots \cup K_{n_m})/S) \leq m - 2k - n_{m-k} \leq m - 2k - n_1 - 2(m - k - 1)$ since

$$
n_{i+1} - n_i \geq 2,
$$

$$
n_{m-k} \geq n_1 + 2\,(m - k - 1)
\tag{10}
$$

$$
= 2 - m - n_1 \implies \mathrm{Nr} \leq 2 - m - n_1.
$$

There exist S^* such that $|S^*| = m - 1$, $w((K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m})/S^*) = 1$ and $c((K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m})/S^*) = n_1$. Then we have

$$
\begin{aligned}
&w\left(\frac{\left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}\right)}{S^*}\right) - |S^*| \\
&\quad - c\left(\frac{\left(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}\right)}{S^*}\right) \\
&= 2 - m - n_1 \\
&\implies \mathrm{Nr} \geq 2 - m - n_1.
\end{aligned}
\tag{11}
$$

From (10) and (11) we obtain $\mathrm{Nr}(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_m}) = 2 - m - n_1$. $\square$

The following theorem's proof is very similar to that of Theorem 5.

Theorem 6. *Let* $K_{n_1}, K_{n_2}, K_{n_3}, \ldots, K_{n_m}$ *be complete graphs with* $n_1 \leq n_2 \leq n_3 \cdots \leq n_m$ *where* $n_{i+1} - n_i \leq 2$; *for all* $i \in Z^+$. *Then*

$$\mathrm{Nr}\left(K_{n_1} \cup K_{n_2} \cup K_{n_3} \cup \cdots \cup K_{n_m}\right) = m - n_m. \quad (12)$$

Corollary 7. *Let*

$$\mathrm{Nr}\left(K_n \cup K_m\right) = \begin{cases} 2 - n, & 2 > n - m, \\ -m, & otherwise. \end{cases} \quad (13)$$

Definition 8 (see [7]). *The join graph* $G = G_1 + G_2$ *has vertex set* $V(G) = V(G_1) \cup V(G_2)$ *and edge set* $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}$.

In this part, neighbor rupture degree of join of some graphs is given.

Theorem 9. *Let* G_1 *and* G_2 *be two connected graphs. Then*

$$\mathrm{Nr}\left(G_1 + G_2\right) = \max\left\{\mathrm{Nr}\left(G_1\right), \mathrm{Nr}\left(G_2\right)\right\}. \quad (14)$$

Proof. Let S be a subversion strategy of $G_1 + G_2$. There are three cases according to the elements of S.

Case 1. Let $S = S_1 \subset V(G_1)$ be the Nr-set of G_1 such that $w(G_1/S_1) - |S_1| - c(G_1/S_1) = \mathrm{Nr}(G_1)$. Since any elements from G_1 are adjacent to every element of G_2 in $G_1 + G_2$, we have

$$w\left(\frac{G_1 + G_2}{S_1}\right) - |S_1| - c\left(\frac{G_1 + G_2}{S_1}\right)$$
$$= w\left(\frac{G_1}{S_1}\right) - |S_1| - c\left(\frac{G_1}{S_1}\right) = \mathrm{Nr}\left(G_1\right). \quad (15)$$

Case 2. Let $S = S_2 \subset V(G_2)$ be the Nr-set of G_2 such that $w(G_2/S_2) - |S_2| - c(G_2/S_2) = \mathrm{Nr}(G_2)$. Since any elements from G_2 are adjacent to every element of G_1 in $G_1 + G_2$, we have

$$w\left(\frac{G_1 + G_2}{S_2}\right) - |S_2| - c\left(\frac{G_1 + G_2}{S_2}\right)$$
$$= w\left(\frac{G_2}{S_2}\right) - |S_2| - c\left(\frac{G_2}{S_2}\right) = \mathrm{Nr}\left(G_2\right). \quad (16)$$

Case 3. Let $S \subset V(G_1) \cup V(G_2)$. Since S contains at least one vertex of $V(G_1)$ which is adjacent to all the vertices of $V(G_2)$ and S contains at least one vertex of $V(G_2)$ which is adjacent to all the vertices of $V(G_1)$ in $G_1 + G_2$, then $(G_1 + G_2)/S$ is empty. It contradicts to the definition of neighbor rupture degree.

By (15) and (16) $\mathrm{Nr}(G_1 + G_2) = \max\{\mathrm{Nr}(G_1), \mathrm{Nr}(G_2)\}$. □

For three or more disjoint graphs $G_1, G_2, G_3, \ldots, G_n$ sequential join $G_1 + G_2 + \cdots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$ [8].

The following theorem's proof is very similar to that of Theorem 9.

Theorem 10. *Let* G_1, G_2, *and* G_3 *be connected graphs. Then the neighbor rupture degree of sequential join of* G_1, G_2, *and* G_3 *is*

$$\mathrm{Nr}\left(G_1 + G_2 + G_3\right) = \max\left\{\mathrm{Nr}\left(G_1 \cup G_3\right), \mathrm{Nr}\left(G_2\right)\right\}. \quad (17)$$

Corollary 11. *If* $G_2 \cong K_n$, *then*

$$\mathrm{Nr}\left(G_1 + G_2 + G_3\right) \geq \mathrm{Nr}\left(G_1\right) + \mathrm{Nr}\left(G_3\right). \quad (18)$$

Corollary 12. *If* $n_1 \leq n_2 \leq n_3$, *then*

$$\mathrm{Nr}\left(K_{n_1} + K_{n_2} + K_{n_3}\right) = -n_3. \quad (19)$$

Definition 13 (see [9]). *The complement of a simple graph* G is obtained by taking the vertices of G and joining two of them whenever they are not joined in G and denoted by G^c.

Theorem 14. *Let* P_n *be a path graph of order* n. *Then*

$$\mathrm{Nr}\left(P_n^c\right) = -1. \quad (20)$$

Proof. Let S be a subversion strategy of P_n^c and let $S = \{u\}$ where $u \in V(P_n)$.

Case 1. If $\deg(u) = 1$ in P_n, then u is adjacent to all vertices in P_n^c except its neighbor in P_n. It means $|N[u]| = n - 1$ in P_n^c, then we have

$$w\left(\frac{P_n^c}{S}\right) - |S| - c\left(\frac{P_n^c}{S}\right) = 1 - 1 - 1 = -1. \quad (21)$$

Case 2. If $\deg(u) = 2$ in P_n, then u is adjacent to all vertices in P_n^c except its neighbors in P_n. It means $|N[u]| = n - 2$ in P_n^c where the remaining two vertices are adjacent. Therefore,

$$w\left(\frac{P_n^c}{S}\right) - |S| - c\left(\frac{P_n^c}{S}\right) = 1 - 1 - 2 = -2. \quad (22)$$

On the other hand, if we assume S is a subversion strategy with $|S| \geq 2$, then the remaining graph is empty. Therefore it contradicts to the definition of neighbor rupture degree.

From (21) and (22) we have $\mathrm{Nr}(P_n^c) = -1$. □

The following theorem's proof is very similar to that of Theorem 14.

Theorem 15. *Let* $W_{1,n}$ *be a wheel graph of order* $n + 1$. *Then*

$$\mathrm{Nr}\left(W_{1,n}^c\right) = -1. \quad (23)$$

Theorem 16. *Let* $K_{m,n}$ *be a complete bipartite graph. Then*

$$\mathrm{Nr}\left(K_{m,n}^c\right) = \begin{cases} 2 - n, & 2 > n - m, \\ -m, & otherwise. \end{cases} \quad (24)$$

Proof. It is obvious that $K_{m,n}^c = K_m \cup K_n$.
According to Corollary 7 we get the result. □

Corollary 17. *Let* $K_{1,n}$ *be a star graph of order* $n + 1$. *Then*

$$\mathrm{Nr}\left(K_{1,n}^c\right) = -1. \quad (25)$$

Definition 18 (see [7]). The cartesian product $G = G_1 \times G_2$ has $V(G) = V(G_1) \times V(G_2)$, and two vertices (u_1, u_2) and (v_1, v_2) of G are adjacent if and only if either

$$u_1 = v_1, \qquad u_2 v_2 \in E(G_2) \tag{26}$$

or

$$u_2 = v_2, \qquad u_1 v_1 \in E(G_1). \tag{27}$$

Theorem 19. *Let $P_2 \times P_{3a}$ be a cartesian product with $a \in Z^+$. Then*

$$\mathrm{Nr}\left(P_2 \times P_{3a}\right) = a - 1. \tag{28}$$

Proof. Let S be a subversion strategy of $P_2 \times P_{3a}$ and $|S| = r$. There are two cases according to the number of elements in S.

Case 1. Let $1 \le r \le a$. Then $N[S] \le 4r$ and $w \le 2r$.

Since $c((P_2 \times P_{3a})/S) \ge (|V(S)| - |N(S)|)/w \ge ((2.(3a) - 4r)/2r) = 3a/r - 2$ and

$$\begin{aligned} w\left(\frac{P_2 \times P_{3a}}{S}\right) &- |S| - c\left(\frac{P_2 \times P_{3a}}{S}\right) \\ &\le 2r - r - \left(\frac{3a}{r} - 2\right) \\ &= r + 2 - \frac{3a}{r}. \end{aligned} \tag{29}$$

Let $f(r) = r + 2 - 3a/r$. f is an increasing function since $f'(r) = 1 + 3a/r^2 > 0$. So it takes its maximum value at $r = a$. Then $f(a) = a - 3a/a + 2 = a - 1$. Hence,

$$\mathrm{Nr}\left(P_2 \times P_{3a}\right) \le a - 1. \tag{30}$$

Case 2. Let $a \le r \le |V(P_2 \times P_{3a})|$. Then $c((P_2 \times P_{3a})/S) \ge 1$ and $w((P_2 \times P_{3a})/S) \le 2(3a) - 4a - (r-a) = 3a - r$, thus we obtain $w((P_2 \times P_{3a})/S) - |S| - c((P_2 \times P_{3a})/S) \le 3a - r - r - 1 = 3a - 2r - 1$.

Let $f(r) = 3a - 2r - 1$. Since $f' < 0$, f is a decreasing function, so it takes its maximum value at $r = a$. Then $f(a) = 3a - 2a - 1 = a - 1$

$$\mathrm{Nr}\left(P_2 \times P_{3a}\right) \le a - 1. \tag{31}$$

From (30) and (31) we have

$$\mathrm{Nr}\left(P_2 \times P_{3a}\right) \le a - 1. \tag{32}$$

It is obvious that there exist S^* such that $|S^*| = a$, $w((P_2 \times P_{3a})/S^*) = 2a$ and $c((P_2 \times P_{3a})/S^*) = 1$ so

$$\mathrm{Nr}\left(P_2 \times P_{3a}\right) \ge a - 1. \tag{33}$$

From (32) and (33) we have $\mathrm{Nr}(P_2 \times P_{3a}) = a - 1$. □

The following theorems' proofs are very similar to that of Theorem 19.

Theorem 20. *Let $P_2 \times P_{3a+1}$ be a cartesian product with $a \in Z^+$. Then*

$$\mathrm{Nr}\left(P_2 \times P_{3a+1}\right) = a - 2. \tag{34}$$

Theorem 21. *Let $P_2 \times P_{3a+2}$ be a cartesian product with $a \in Z^+$. Then*

$$\mathrm{Nr}\left(P_2 \times P_{3a+2}\right) = a - 1. \tag{35}$$

Theorem 22. *Let K_m and K_n be two complete graphs with $(m \le n)$. Then*

$$\mathrm{Nr}\left(K_m \times K_n\right) = 1 - n. \tag{36}$$

Proof. Let S be a subversion strategy of $K_m \times K_n$ and let $|S| = r$. We have two cases according to the cardinality of S.

Case 1. If $0 \le r < m - 1$, then $w((K_m \times K_n)/S) = 1$ and $c((K_m \times K_n)/S) \ge (m-r)(n-r)$, so we have

$$\begin{aligned} w\left(\frac{K_m \times K_n}{S}\right) &- |S| - c\left(\frac{K_m \times K_n}{S}\right) \\ &\le 1 - r - (m-r)(n-r). \end{aligned} \tag{37}$$

Let $f(r) = 1 - r - mn + (m+n)r - r^2$. Since $f(r)$ is an increasing function in $(0, m-1)$, it takes its maximum value at $r = m - 1$ and $f(m-1) = 1 - n$. Thus we get

$$\mathrm{Nr}\left(K_m \times K_n\right) \le 1 - n. \tag{38}$$

Case 2. If $m - 1 \le r \le (m-1)(n-1)$, then $w((K_m \times K_n)/S) = 1$ and $c((K_m \times K_n)/S) \ge n - m + 1$. So we have $w((K_m \times K_n)/S) - |S| - c((K_m \times K_n)/S) \le 1 - r - (n - m + 1) = m - n - r$.

Let $f(r) = m - n - r$. Since $f(r)$ is a decreasing function, it takes its maximum value at $r = m - 1$ and $f(m-1) = 1 - n$. Thus we get

$$\mathrm{Nr}\left(K_m \times K_n\right) \le 1 - n. \tag{39}$$

From (38) and (39) we have

$$\mathrm{Nr} \le 1 - n. \tag{40}$$

There exist S^* such that $r = m - 1$, $w((K_m \times K_n)/S^*) = 1$ and $c((K_m \times K_n)/S^*) = n - m + 1$, thus we have

$$\mathrm{Nr} \ge 1 - n. \tag{41}$$

From (40) and (41) we get $\mathrm{Nr} = 1 - n$. □

Definition 23 (see [9]). The tensor product $G_1 \otimes G_2$ of two simple graphs G_1 and G_2 is the graph with $V(G_1 \otimes G_2) = V_1 \times V_2$ and where in (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \otimes G_2$ if, and only if, u_1 is adjacent to v_1 in G_1 and u_2 is adjacent to v_2 in G_2.

Theorem 24. *Let $P_3 \otimes P_n$ be a tensor product of P_3 and P_n and $n \equiv 0 \pmod 4$. Then*

$$\mathrm{Nr}\left(P_3 \otimes P_n\right) = n - 1. \tag{42}$$

Proof. Let S be a subversion strategy of $P_3 \otimes P_n$ and $|S| = r$ be the number of removing vertices from $P_3 \otimes P_n$. There are two cases according to the number of elements in S.

Case 1. If $0 \leq r \leq (n/2)$, then $w((P_3 \otimes P_n)/S) \leq n + r$ and $c((P_3 \otimes P_n)/S) \geq 1$. Thus we have

$$w\left(\frac{P_3 \otimes P_n}{S}\right) - |S| - c\left(\frac{P_3 \otimes P_n}{S}\right) \leq n - 1. \quad (43)$$

Case 2. If $(n/2) \leq r \leq 3n$, then $w((P_3 \otimes P_n)/S) \leq n + (n/2) - ((r - (n/2)) = 2n - r$ and $c((P_3 \otimes P_n)/S) \geq 1$. Thus we have

$$w\left(\frac{P_3 \otimes P_n}{S}\right) - |S| - c\left(\frac{P_3 \otimes P_n}{S}\right) \leq 2n - 2r - 1. \quad (44)$$

Let $f(r) = 2n - 2r - 1$ since $f'(r) < 0$ the function $f(r)$ is a decreasing function so it takes its maximum value at

$$r = \left(\frac{n}{2}\right), \qquad f\left(\frac{n}{2}\right) = n - 1. \quad (45)$$

From (43) and (45) we get

$$\mathrm{Nr}\left(P_3 \otimes P_n\right) \leq n - 1. \quad (46)$$

There exist S^* such that $|S^*| = (n/2), w((P_3 \otimes P_n)/S) = n + (n/2)$ and $c((P_3 \otimes P_n)/S) \geq 1$ thus we have

$$\mathrm{Nr}\left(P_3 \otimes P_n\right) \geq n - 1. \quad (47)$$

From (46) and (47) we get $\mathrm{Nr}(P_3 \otimes P_n) = n - 1$. $\qquad \square$

The following theorem's proof is very similar to that of Theorem 24.

Theorem 25. *Let $P_3 \otimes P_n$ be a tensor product of P_3 and P_n and $n \neq 0 \pmod 4$. Then*

$$\mathrm{Nr}\left(P_3 \otimes P_n\right) = 2n - 2\left\lceil\frac{(n+1)}{2}\right\rceil - 1. \quad (48)$$

Theorem 26. *Let the tensor product of K_m and K_n be $K_m \otimes K_n$. Then*

$$\mathrm{Nr}\left(K_m \otimes K_n\right) = \mathrm{Nr}\left(K_{m-1,n-1}\right) - 1 = \max\{m - 5, n - 5\}. \quad (49)$$

Proof. Let (a, b) be any vertex of $K_m \otimes K_n$ such that $a \in K_m$ and $b \in K_n$. The only vertices that are not adjacent to (a, b) in $(K_m \otimes K_n)$ are (a, c_j) with $(j = 1, 2, \ldots, n)$ and (d_i, b) with $(i = 1, 2, \ldots, m)$, where $c_j \in V(K_n)$ and $d_i \in V(K_m)$.

The vertices (a, c_j) are not adjacent to each other, neither do the vertices (d_i, b). But these are adjacent to each other, so

$$\frac{(K_m \otimes K_n)}{\{a, b\}} \cong K_{m-1, n-1},$$

$$\mathrm{Nr}\left(K_m \otimes K_n\right) = \mathrm{Nr}\left(K_{m-1, n-1}\right) - 1 \quad (50)$$

$$= \max\{m - 4, n - 4\} - 1$$

$$= \max\{m - 5, n - 5\}.$$

$\qquad \square$

Definition 27 (see [10]). The composition of simple graphs G and H is the simple graph $G[H]$ with vertex set $V(G) \times V(H)$, in which (u, v) is adjacent (u', v') if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$.

Theorem 28. *Let $P_3[P_n]$ be the composition of P_3 and P_n with $n \geq 5$. Then neighbor rupture degree of $P_3[P_n]$ is*

$$\mathrm{Nr}\left(P_3\left[P_n\right]\right) = \begin{cases} 1, & n \equiv 1 \pmod 4, \\ 0, & n \equiv 0, 2, 3 \pmod 4. \end{cases} \quad (51)$$

Proof. Let the vertex set of $P_3[P_n]$ be labeled as P_n^i, P_n^{ii}, and P_n^{iii}.

$$(52)$$
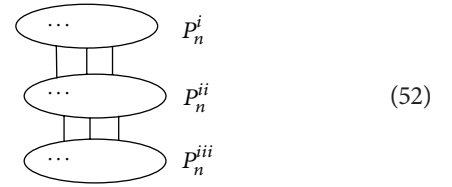
It can be easily seen that $P_3[P_n]$ is the sequential join of three disjoint path graphs, $P_3[P_n] \cong P_n + P_n + P_n$. Then, according to the Theorem 10 we get

$$\mathrm{Nr}\left(P_3\left[P_n\right]\right) = \mathrm{Nr}\left(P_n + P_n + P_n\right)$$
$$= \max\left\{\mathrm{Nr}\left(P_n \cup P_n\right), \mathrm{Nr}\left(P_n\right)\right\}. \quad (53)$$

By Theorem 2, we have

$$\mathrm{Nr}\left(P_n\right) = \begin{cases} 0, & n \equiv 1 \pmod 4, \\ -1, & n \equiv 0, 2, 3 \pmod 4. \end{cases} \quad (54)$$

To conclude the proof we need to find $\mathrm{Nr}(P_n \cup P_n)$. Let S be a subversion strategy of $P_n \cup P_n$. Since $P_n \cup P_n$ has two identical disjoint path graphs, let x denote the number of removing vertices of each P_n and let $|S| = 2x$.

Case 1. If $x \leq \lfloor(n-1)/4\rfloor$, $w((P_n \cup P_n) \setminus S) \leq 2x + 2$ and $c((P_n \cup P_n) \setminus S) \geq \lceil(2n - 6x)/(2x + 2)\rceil = \lceil(n - 3x)/(x + 1)\rceil$, then we get

$$w\left((P_n \cup P_n) \setminus S\right) - |S| - c\left((P_n \cup P_n) \setminus S\right)$$
$$\leq 2 - \left\lceil\frac{n - 3x}{x + 1}\right\rceil. \quad (55)$$

Let $f(x) = 2 - (n - 3x)/(x + 1)$, since $f(x)$ is an increasing function it takes its maximum value at $x = \lfloor(n-1)/4\rfloor$. $f(\lfloor(n-1)/4\rfloor) = 5 - (n+3)/(\lfloor(n - 1)/4\rfloor + 1)$. Since neighbor rupture degree has to be an integer, we have $\mathrm{Nr}(P_n \cup P_n) \leq \lceil 5 - (n + 3)/(\lfloor(n - 1)/4\rfloor + 1)\rceil$ and we get

$$\mathrm{Nr}\left(P_n \cup P_n\right) \leq \begin{cases} 1, & n \equiv 1 \pmod 4, \\ 0, & n \equiv 0, 2, 3 \pmod 4. \end{cases} \quad (56)$$

Case 2. If $x \geq \lfloor(n-1)/4\rfloor$, $w((P_n \cup P_n) \setminus S) \leq 2x$, and $c((P_n \cup P_n) \setminus S) \geq 1$, thus we get

$$\mathrm{Nr}\left(P_n \cup P_n\right) \leq -1. \quad (57)$$

From (56) and (57) we have

$$\text{Nr}\left(P_n \cup P_n\right) \le \begin{cases} 1, & n \equiv 1 \ (\text{mod } 4), \\ 0, & n \equiv 0, 2, 3 \ (\text{mod } 4). \end{cases} \quad (58)$$

There exist S^* such that $|S^*| = 2\lfloor(n-1)/4\rfloor$, $w((P_n \cup P_n) \setminus S) = (2\lfloor(n-1)/4\rfloor+1)$ and $c((P_n \cup P_n)\setminus S) = 2$ for $n \equiv 0, 2, 3(\text{ mod } 4)$, $c((P_n \cup P_n) \setminus S) = 1$ for $n \equiv 1(\text{mod } 4)$.

Thus we have

$$\text{Nr}\left(P_n \cup P_n\right) \ge \begin{cases} 1, & n \equiv 1 \ (\text{mod } 4), \\ 0, & n \equiv 0, 2, 3 \ (\text{mod } 4). \end{cases} \quad (59)$$

From (58) and (59) we get

$$\text{Nr}\left(P_n \cup P_n\right) = \begin{cases} 1, & n \equiv 1 \ (\text{mod } 4), \\ 0, & n \equiv 0, 2, 3 \ (\text{mod } 4). \end{cases} \quad (60)$$

By (54) and (60) we conclude the proof. $\qquad\square$

Theorem 29. *Neighbor rupture degree of composition of K_m and any graph G is*

$$\text{Nr}\left(K^m [G]\right) = \text{Nr}(G). \quad (61)$$

Proof. Let the vertex set of $K_m[G]$ be labeled as, $G^i, G^{ii}, \ldots, G^m$,

$$(62)$$
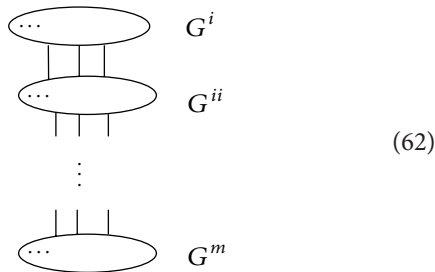
Let S be a subversion strategy of $K_m[G]$. We have two cases according to the elements of S.

Case 1. Let we choose one element from any vertex set G^i ($i = 1, 2, \ldots, m$); if $u \in V(G^i)$ and $S = \{u\}$, then it removes all of other vertex sets. So it depends on only G^i which we choose one element. Then we have

$$\text{Nr}\left(K_m [G]\right) = \text{Nr}\left(G^i\right) = \text{Nr}(G). \quad (63)$$

Case 2. Let we choose two elements from any vertex set G^i ($i = 1, 2, \ldots, m$) and G^j ($j = 1, 2, \ldots, m$) with $i \ne j$. Then $(K_m[G])/S$ is empty set. It contradicts to the definition of neighbor rupture degree. Thus we obtain

$$\text{Nr}\left(K_m [G]\right) = \text{Nr}(G). \quad (64)$$

$$\qquad\square$$

Definition 30 (see [11]). An nth power of a graph G is formed by adding an edge between all pairs of vertices of G with distance at most n. If $n = 2$ then it is called a *second power* of a graph also called a *square*.

Theorem 31. *Neighbor rupture degree of P_n^2 ($n > 6$) is*

$$\text{Nr}\left(P_n^2\right) = \begin{cases} 0, & n \equiv 1 \ (\text{mod } 6), \\ -1, & otherwise. \end{cases} \quad (65)$$

Proof. Let S be a subversion strategy of P_n^2 and let $|S| = r$. There are two cases according to the number of elements of S.

Case 1. If $0 \le r \le \lceil n/6\rceil - 1$, then $w(P_n^2/S) \le r + 1$, $c(P_n^2/S) \ge (n - 5r)/(r + 1)$; then we get

$$w\left(\frac{P_n^2}{S}\right) - |S| - c\left(P_n^2 \setminus S\right)$$

$$\le r + 1 - r - \frac{n - 5r}{r + 1} = 6 - \frac{n + 5}{r + 1}. \quad (66)$$

Let

$$f(r) = 6 - \frac{n + 5}{r + 1} = \begin{cases} 0, & n \equiv 1 \ (\text{mod } 6), \\ < 0, & otherwise. \end{cases} \quad (67)$$

Case 2. If $= \lceil n/6\rceil$, then $w(P_n^2/S) \le r$, $c(P_n^2/S) \ge 1$; then we get

$$w\left(\frac{P_n^2}{S}\right) - |S| - c\left(\frac{P_n^2}{S}\right)$$

$$\le r - r - 1$$

$$= -1 \text{ therefore Nr} \le -1. \quad (68)$$

Case 3. If $\lceil n/6\rceil + 1 \le r \le n$, then $w(P_n^2/S) \le r - 1$, $c(P_n^2/S) \ge 1$; then we get

$$w\left(\frac{P_n^2}{S}\right) - |S| - c\left(P_n^2 \setminus S\right)$$

$$\le r - 1 - r - 1$$

$$= -2 \text{ therefore Nr} \le -2. \quad (69)$$

According to (67), (68), and (69) we have

$$\text{Nr}\left(P_n^2\right) \le \begin{cases} 0, & n \equiv 1 \ (\text{mod } 6), \\ -1, & otherwise. \end{cases} \quad (70)$$

There exist S^* such that $|S^*| = \lceil(n-1)/6\rceil$, $w(P_n^2/S^*) = \lceil n/6\rceil$, $c(P_n^2/S^*) = 1$; then

$$w\left(\frac{P_n^2}{S^*}\right) - |S^*| - c\left(\frac{P_n^2}{S^*}\right) = \begin{cases} 0, & n \equiv 1 \ (\text{mod } 6), \\ 1, & otherwise. \end{cases} \quad (71)$$

By (70) and (71) we get the result,

$$\text{Nr}\left(\frac{P_n^2}{S}\right) = \max\left\{w\left(\frac{P_n^2}{S}\right) - |S| - c\left(\frac{P_n^2}{S}\right)\right\}$$

$$= \begin{cases} 0, & n \equiv 1 \ (\text{mod } 6), \\ 1, & otherwise. \end{cases} \quad (72)$$

$$\qquad\square$$

4. Conclusion

In this study, we investigate the neighbor rupture degree of graphs obtained by graph operations. The graph operations are used to obtain new graphs. Union, join, complement, composition, power, cartesian product, and tensor product are taken into consideration in this work. These operations are performed to various graphs and their neighbor rupture degrees were determined.

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