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Research Article

Dynamics of a Nonstandard Finite-Difference Scheme for a Limit Cycle Oscillator with Delayed Feedback

Yuanyuan Wang[1,2] and Xiaohua Ding[1]

[1] *Department of Mathematics, Harbin Institute of Technology (Weihai), Weihai 264209, China*
[2] *School of Science, China University of Petroleum (East China), Qingdao 266555, China*

Correspondence should be addressed to Xiaohua Ding; mathdxh@126.com

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We consider a complex autonomously driven single limit cycle oscillator with delayed feedback. The original model is translated to a two-dimensional system. Through a nonstandard finite-difference (NSFD) scheme we study the dynamics of this resulting system. The stability of the equilibrium of the model is investigated by analyzing the characteristic equation. In the two-dimensional discrete model, we find that there are stability switches on the time delay and Hopf bifurcation when the delay passes a sequence of critical values. Finally, computer simulations are performed to illustrate the theoretical results. And the results show that NSFD scheme is better than the Euler method.

1. Introduction

Reddy et al. [1] proposed the following model system of an autonomously driven single limit cycle oscillator:

$$\dot{z}(t) - \left(a + iw - |z(t)|^2\right) z(t) = -k_1 z(t-\tau) - k_2 z^2(t-\tau), \quad (1)$$

where $z = x + iy$ is a complex quantity, w is the frequency of oscillation, and a is a real constant. $\tau \geq 0$ is the time delay of the autonomous feedback term. k_1 and k_2 represent the strengths of the linear and nonlinear contributions of the feedback. Reddy et al. investigated the temporal dynamics of system (1) in various regimes characterized by the natural parameters of the oscillator, strengths of the feedback components (k_1, k_2), and the time delay parameter τ. Jiang and Wei [2] have studied the stability of system (1) and have drawn the bifurcation diagram in (a, k_1) plane in continuous-time model. Furthermore, it is found that there are stability switches on the time delay and Hopf bifurcation when the time delay crosses through some critical values.

But due to scientific computation and simulation, our interest focuses on the behavior of discrete dynamical system corresponding to (1). It is desired that the discrete-time model is "dynamically consistent" with the continuous-time model. In [3–12], the dynamics of numerical discrete difference equations can inherit those of the original differential equations. Wulf and Ford [13, 14] showed that the Euler forward method is "dynamically consistent" when applying it to solve the delay differential equation. It means that, for sufficiently small stepsize, the discrete model undergoes a Hopf bifurcation of the same type with the original model.

In this paper, we apply NSFD scheme [15–17] to discretize (1). We consider the autonomous delay differential equation

$$\begin{aligned} \dot{u} &= f(u(t), u(t-1)), \quad t \geq 0, \\ u(t) &= \psi(t), \quad -1 \leq t \leq 0. \end{aligned} \quad (2)$$

The first-order derivative is approximated by modified forward Euler expression

$$\frac{du(t)}{dt} \longrightarrow \frac{u_{k+1} - u_k}{\phi}, \quad (3)$$

with the denominator function ϕ such that

$$\phi(h) = h + O\left(h^2\right), \quad (4)$$

where $h = 1/m$ stands for step-size and u_k denotes the approximate value to $u(kh)$, so we get the method as follows:

$$u_{k+1} - u_k = \phi(h) f\left(u_k, u_{k-m}\right). \quad (5)$$

This method can seem as a modified forward Euler method [4]. NSFD scheme [15–17] tries to preserve the significant properties of their continuous analogues and consequently gives reliable numerical results. For small step-size we obtain the consistent dynamical results of the corresponding continuous-time model using Hopf bifurcation theory for discrete system [3, 6, 7, 13, 18]. Through the analysis, our results show that NSFD scheme is better than Euler method. To the best of our knowledge, to this day, by NSFD scheme, there are few results dealing with behavior of stability switches in the discrete model.

The paper is organized as follows. In Section 2, we discuss the distribution of the characteristic equation associated with the discrete limit cycle oscillator with delayed feedback and obtain the existence of the local Hopf bifurcation and stability switches. In Section 3, some computer simulations are performed to illustrate the theoretical results. In the final section, we summarize our results and give our future plans.

2. Stability Analysis

Let $z = x + iy$. Then (1) becomes

$$\begin{aligned}\dot{x}(t) &= \left(a - x^2(t) - y^2(t)\right)x(t) - wy(t) \\ &\quad - k_1 x(t-\tau) - k_2\left(x^2(t-\tau) - y^2(t-\tau)\right), \\ \dot{y}(t) &= wx(t) + \left(a - x^2(t) - y^2(t)\right)y(t) \\ &\quad - k_1 y(t-\tau) - 2k_2 x(t-\tau)\, y(t-\tau).\end{aligned} \tag{6}$$

Set $u_1(t) = x(\tau t), u_2(t) = y(\tau t)$. Then (6) can be rewritten as

$$\begin{aligned}\dot{u}_1(t) &= \tau\left[\left(a - u_1^2(t) - u_2^2(t)\right)u_1(t) - wu_2(t)\right. \\ &\quad \left. - k_1 u_1(t-1) - k_2\left(u_1^2(t-1) - u_2^2(t-1)\right)\right], \\ \dot{u}_2(t) &= \tau\left[wu_1(t) + \left(a - u_1^2(t) - u_2^2(t)\right)u_2(t)\right. \\ &\quad \left. - k_1 u_2(t-1) - 2k_2 u_1(t-1)\, u_2(t-1)\right].\end{aligned} \tag{7}$$

Let $U = (u_1, u_2)^T$. Given $h = 1/m$, where $m \in Z_+$, employ the NSFD scheme [15–17] to (7) and choose the "denominator function" ϕ as

$$\phi(h) = \frac{e^{a\tau h} - 1}{a\tau}. \tag{8}$$

It yields the difference equation

$$U_{n+1} = AU_n + BU_{n-m} + f\left(U_n, U_{n-m}\right), \tag{9}$$

where

$$A = \begin{pmatrix} e^{a\tau h} & -\dfrac{w\left(e^{a\tau h} - 1\right)}{a} \\ \dfrac{w\left(e^{a\tau h} - 1\right)}{a} & e^{a\tau h} \end{pmatrix}, \qquad B = \begin{pmatrix} -\dfrac{k_1\left(e^{a\tau h} - 1\right)}{a} & 0 \\ 0 & -\dfrac{k_1\left(e^{a\tau h} - 1\right)}{a} \end{pmatrix},$$

$$f = \begin{pmatrix} -\dfrac{e^{a\tau h} - 1}{a\tau}\tau\left(u_{1,n}^2 + u_{2,n}^2\right)u_{1,n} - \dfrac{e^{a\tau h} - 1}{a\tau}\tau k_2\left(u_{1,n-m}^2 - u_{2,n-m}^2\right) \\ -\dfrac{e^{a\tau h} - 1}{a\tau}\tau\left(u_{1,n}^2 + u_{2,n}^2\right)u_{2,n} - 2\dfrac{e^{a\tau h} - 1}{a\tau}\tau k_2\left(u_{1,n-m}u_{2,n-m}\right) \end{pmatrix}. \tag{10}$$

Introducing a new variable $Y_n = (U_n^T, U_{n-1}^T, \ldots, U_{n-m}^T)^T$, we can rewrite (9) as

$$Y_{n+1} = F\left(Y_n, \tau\right), \tag{11}$$

where $F = (F_0^T, F_1^T, \ldots, F_m^T)^T$, and

$$F_k = \begin{cases} AU_{n-k} + BU_{n-k-m} + f\left(U_{n-k}, U_{n-k-m}\right), & k = 0, \\ U_{n-k+1}, & 1 \le k \le m. \end{cases} \tag{12}$$

It is clear that the zero solution $(0, 0)$ is a fixed point of (11), and the linearization of (11) around $(0, 0)$ is

$$Y_{n+1} = \widehat{A}Y_n, \tag{13}$$

where

$$\widehat{\mathbf{A}} = \begin{bmatrix} A & 0 & \cdots & 0 & 0 & B \\ I & 0 & \cdots & 0 & 0 & 0 \\ 0 & I & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 & 0 \\ 0 & 0 & \cdots & 0 & I & 0 \end{bmatrix}, \tag{14}$$

where I is a 2×2 unit matrix. The characteristic equation of $\widehat{A}$ is given by

$$\det\left[\lambda^m(\lambda I - A) - B\right] = \left[\left(\lambda - e^{a\tau h}\right)\lambda^m + \frac{k_1\left(e^{a\tau h}-1\right)}{a}\right]^2 + \left(\frac{w\left(e^{a\tau h}-1\right)}{a}\lambda^m\right)^2 = 0. \tag{15}$$

Rewrite (15) in the more compact form

$$\Delta_{\pm} = \left[\left(\lambda - e^{a\tau h}\right)\lambda^m + \frac{k_1\left(e^{a\tau h}-1\right)}{a}\right] \pm \left(\frac{w\left(e^{a\tau h}-1\right)}{a}\lambda^m\right)i = 0. \tag{16}$$

Similar to [19], we only need to investigate

$$\Delta_{+} = \left(\lambda - e^{a\tau h}\right)\lambda^m + \frac{k_1\left(e^{a\tau h}-1\right)}{a} + \left(\frac{w\left(e^{a\tau h}-1\right)}{a}\lambda^m\right)i = 0. \tag{17}$$

Lemma 1. *If $k_1 > a$, then all roots of (17) have modulus less than one for sufficiently small $\tau > 0$; if $k_1 < a$, in all roots of (17), there is at least one root with modulus more than one.*

Proof. For $\tau = 0$, (17) becomes

$$\lambda^{m+1} - \lambda^m = 0. \tag{18}$$

The equation has an m-fold root and a simple root $\lambda = 1$.

Consider the root $\lambda(\tau)$ such that $\lambda(0) = 1$. This root is a C^1 function of τ. For (17), we have

$$\begin{aligned} \frac{d|\lambda|^2}{d\tau} &= \lambda\frac{d\bar{\lambda}}{d\tau} + \bar{\lambda}\frac{d\lambda}{d\tau}, \\ \left.\frac{d|\lambda|^2}{d\tau}\right|_{\lambda=1,\tau=0} &= 2\Re\left(\bar{\lambda}\frac{d\lambda}{d\tau}\right)\Big|_{\lambda=1,\tau=0} = 2\left(ha - hk_1\right), \\ \left.\frac{d|\lambda|^2}{d\tau}\right|_{\lambda=1,\tau=0} &< 0, \quad k_1 > a, \\ \left.\frac{d|\lambda|^2}{d\tau}\right|_{\lambda=1,\tau=0} &> 0, \quad k_1 < a. \end{aligned} \tag{19}$$

Consequently, when $k_1 > a$, all roots of (17) lie in $|\lambda| < 1$ for sufficiently small $\tau > 0$; when $k_1 < a$, in all roots of (17), there is at least one root with modulus more than one. □

A Hopf bifurcation occurs when two roots of the characteristic equation (17) cross the unit circle. We have to find values of τ such that there exist roots on the unit circle. The roots on the unit circle are given by $e^{i\omega_*}, \omega_* \in (-\pi, \pi]$. Since we are dealing with a real polynomial complex roots, we only need to look for $\omega_* \in (0, \pi]$. $e^{i\omega_*}$ is a root of (17) if and only if

$$e^{i(m+1)\omega_*} - e^{a\tau h}e^{im\omega_*} + \frac{k_1\left(e^{a\tau h}-1\right)}{a} + \frac{w\left(e^{a\tau h}-1\right)}{a}e^{im\omega_*}i = 0. \tag{20}$$

Hence

$$e^{i\omega_*} - e^{a\tau h} + \frac{k_1\left(e^{a\tau h}-1\right)}{a}e^{-im\omega_*} + \frac{w\left(e^{a\tau h}-1\right)}{a}i = 0. \tag{21}$$

Then

$$\begin{aligned} &\cos(m+1)\omega_* - e^{a\tau h}\cos m\omega_* + \frac{k_1\left(e^{a\tau h}-1\right)}{a} - \frac{w\left(e^{a\tau h}-1\right)}{a}\sin m\omega_* = 0, \\ &\sin(m+1)\omega_* - e^{a\tau h}\sin m\omega_* + \frac{w\left(e^{a\tau h}-1\right)}{a}\cos m\omega_* = 0. \end{aligned} \tag{22}$$

Or

$$\begin{aligned} &\cos\omega_* - e^{a\tau h} + \frac{k_1\left(e^{a\tau h}-1\right)}{a}\cos m\omega_* = 0, \\ &\sin\omega_* + \frac{w\left(e^{a\tau h}-1\right)}{a} - \frac{k_1\left(e^{a\tau h}-1\right)}{a}\sin m\omega_* = 0. \end{aligned} \tag{23}$$

So

$$\cos(m+1)\omega_* = \frac{a}{k_1} + \frac{\left(1 - e^{a\tau h}\right)\left(k_1^2 - a^2 - w^2\right)}{2ak_1}. \tag{24}$$

If the step-size h is sufficiently small, we obtain the following results.

Case I ($k_1 > a$). Consider the following:

(I1) if $k_1 > a > 0$, then $0 < a/k_1 < 1$, that is, $0 < \cos(m+1)\omega_* < 1$;

(I2) if $0 > k_1 > a$, then $a/k_1 > 1$, that is, $\cos(m+1)\omega_* > 1$, which yields a contradiction;

(I3) $k_1 > 0 > a$, if $a + k_1 < 0$, then $a/k_1 < -1$, that is, $\cos(m+1)\omega_* < -1$, which yields a contradiction;

(I4) $k_1 > 0 > a$, if $a + k_1 > 0$, then $-1 < a/k_1 < 0$, that is, $-1 < \cos(m+1)\omega_* < 0$.

Case II ($k_1 < a$). Consider the following:

(II1) if $0 < k_1 < a$, then $a/k_1 > 1$, that is, $\cos(m+1)\omega_* > 1$, which yields a contradiction;

(II2) if $k_1 < a < 0$, then $0 < a/k_1 < 1$, that is, $0 < \cos(m+1)\omega_* < 1$;

(II3) $k_1 < 0 < a$, if $a + k_1 > 0$, then $a/k_1 < -1$, that is, $\cos(m+1)\omega_* < -1$, which yields a contradiction;

(II4) $k_1 < 0 < a$, if $a + k_1 < 0$, then $-1 < a/k_1 < 0$, that is, $-1 < \cos(m+1)\omega_* < 0$.

Lemma 2. *If the step-size h is sufficiently small, and any one of (I2)(I3) and (III1)(II3) is satisfied, then* (17) *has no root with modulus one.*

If the step-size h is sufficiently small, any one of (I1)(I4) and (II2)(II4) (i.e., $k_1^2 > a^2$) is satisfied, then $|\cos(m+1)\omega_*| < 1$. From (23) we know that

$$\cos\omega_*^{\pm} = \left(-e^{a\tau h}\overline{M} \pm w \times \left(\frac{\left(e^{a\tau h}-1\right)^2}{a^2} \cdot \left[4\left[\left(e^{a\tau h}\right)^2 + \left(\frac{w\left(e^{a\tau h}-1\right)}{a} \right)^2 \right] - \overline{M}^2 \right] \right)^{1/2} \right) \times \left(2\left[\left(e^{a\tau h}\right)^2 + \left(\frac{w\left(e^{a\tau h}-1\right)}{a} \right)^2 \right] \right)^{-1}, \tag{25}$$

where $\overline{M} = ((k_1^2 - a^2 - w^2)/a^2)(e^{a\tau h}-1)^2 - 2e^{a\tau h}$.

It is clear that there exist values of the time delay parameters $\tau_j^{\pm}$ satisfying (23) according to $\omega_{*,j}^{\pm} \in (2j\pi/m, (2j+1)\pi/m)$, $j = 0, 1, 2, \ldots, [(m-1)/2]$.

Lemma 3. *Let $\lambda(\tau)$ be a root of* (17). *If the step-size h is sufficiently small and $k_1^2 > a^2$((I1)(I4) and (II2)(II4)), then*

$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^-,\omega_*=\omega_*^-} > 0,$$
$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^+,\omega_*=\omega_*^+} > 0, \quad \textit{when } w < \sqrt{k_1^2 - a^2}, \tag{26}$$
$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^+,\omega_*=\omega_*^+} < 0, \quad \textit{when } w > \sqrt{k_1^2 - a^2}.$$

Proof. From (17), we have

$$\lambda^m = \frac{k_1\left(1-e^{a\tau h}\right)}{a\lambda - ae^{a\tau h} + w\left(e^{a\tau h}-1\right)i}, \tag{27}$$

$$\begin{aligned}
&\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^{\pm},\omega_*=\omega_*^{\pm}} \\
&= 2\Re\left.\left(\overline{\lambda}\frac{d\lambda}{d\tau}\right)\right|_{\tau=\tau^{\pm},\omega_*=\omega_*^{\pm}} \\
&= \frac{2a^3he^{a\tau h}}{1-e^{a\tau h}} \cdot \left[(1+m)\left(\cos\omega_*^{\pm}-1\right) - \frac{m\left(e^{a\tau h}-1\right)^2\left(k_1^2-a^2-w^2\right)}{2a^2} \right] \\
&\quad \cdot \left((1) \times \left(\left[(1+m)a\cos\omega_*^{\pm} - mae^{a\tau h}\right]^2 + \left[(1+m)a\sin\omega_*^{\pm} + mw\left(e^{a\tau h}-1\right)\right]^2 \right)^{-1} \right).
\end{aligned} \tag{28}$$

Substituting (25) into (28), we get

$$\begin{aligned}
&\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^{\pm},\omega_*=\omega_*^{\pm}} \\
&= \left(-a^2h^2\tau^{\pm}\left[2(1+m)w\left(\pm\sqrt{k_1^2-a^2}-w\right) - (1+2m)\left(k_1^2-a^2-w^2\right) \right] + O\left(h^3\right) \right) \\
&\quad \times \left(\left[(1+m)a\cos\omega_*^{\pm} - mae^{a\tau h}\right]^2 + \left[(1+m)a\sin\omega_*^{\pm} + mw\left(e^{a\tau h}-1\right)\right]^2 \right)^{-1}.
\end{aligned} \tag{29}$$

Therefore

$$\begin{aligned}
&\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^-,\omega_*=\omega_*^-} \\
&= \left(-a^2h^2\tau^-\left[-2(1+m)w\left(w+\sqrt{k_1^2-a^2}\right) - (1+2m)\left(k_1^2-a^2-w^2\right) \right] + O\left(h^3\right) \right) \\
&\quad \times \left(\left[(1+m)a\cos\omega_*^- - mae^{a\tau h}\right]^2 + \left[(1+m)a\sin\omega_*^- + mw\left(e^{a\tau h}-1\right)\right]^2 \right)^{-1}
\end{aligned}$$

$$= \left(\left(w + \sqrt{k_1^2 - a^2}\right) \times \left(2a^2 h\tau^- \sqrt{k_1^2 - a^2}\right) + O\left(h^2\right)\right) \times \left(\left[(1+m)\, a \cos\omega_*^- - mae^{a\tau h}\right]^2 + \left[(1+m)\, a \sin\omega_*^- + mw\left(e^{a\tau h} - 1\right)\right]^2\right)^{-1}. \tag{30}$$

If the step-size h is sufficiently small and $k_1^2 > a^2$, we have

$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^-,\omega_*=\omega_*^-} > 0,$$

$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^+,\omega_*=\omega_*^+} = \left(-a^2h^2\tau^+\left[2(1+m)\, w\left(\sqrt{k_1^2-a^2} - w\right) - (1+2m)\left(k_1^2 - a^2 - w^2\right)\right] + O\left(h^3\right)\right) \times \left(\left[(1+m)\, a\cos\omega_*^+ - mae^{a\tau h}\right]^2 + \left[(1+m)\, a\sin\omega_*^+ + mw\left(e^{a\tau h}-1\right)\right]^2\right)^{-1}$$
$$= \left(\left(\sqrt{k_1^2-a^2} - w\right) \times \left(2a^2h\tau^+\sqrt{k_1^2-a^2}\right) + O\left(h^2\right)\right) \times \left(\left[(1+m)\, a\cos\omega_*^+ - mae^{a\tau h}\right]^2 + \left[(1+m)\, a\sin\omega_*^+ + mw\left(e^{a\tau h}-1\right)\right]^2\right)^{-1}, \tag{31}$$

If the step-size h is sufficiently small and $k_1^2 > a^2$, we have the following conclusions:

$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^+,\omega_*=\omega_*^+} > 0, \quad \text{when } w < \sqrt{k_1^2-a^2},$$
$$\left.\frac{d|\lambda(\tau)|^2}{d\tau}\right|_{\tau=\tau^+,\omega_*=\omega_*^+} < 0, \quad \text{when } w > \sqrt{k_1^2-a^2}. \tag{32}$$

The proof is complete. □

For the convenience, we denote (see the bifurcation diagram in [2])

$$D_1 : \sqrt{a^2+\omega^2} \le k_1, \qquad D_2 : |a| < k_1 < \sqrt{a^2+\omega^2},$$
$$D_3 : -\sqrt{a^2+\omega^2} < k_1 < -|a|, \qquad D_4 : k_1 \le -\sqrt{a^2+\omega^2},$$
$$D_5 : a < k_1 < -a, \qquad D_6 : -a < k_1 < a. \tag{33}$$

Theorem 4. *For (11), the following statements are true.*

(1) *If $(a, k_1) \in D_1$, then (11) undergoes a Hopf bifurcation at the origin $(0,0)$ when $\tau = \tau_j^\pm$; the zero solution is asymptotically stable when $\tau \in (0, \tau_0)$ and is unstable when $\tau > \tau_0$.*

(2) *If $(a, k_1) \in D_2$, then (11) undergoes a Hopf bifurcation at the origin $(0,0)$ when $\tau = \tau_j^\pm$. One controls (25) to make $\omega_*^\pm \in (0, \theta_1)$ (θ_1 is determined in the proof) then the zero solution is asymptotically stable when $\tau \in (0, \tau^-_{[(m-1)/2]}) \bigcup (\cup_{j=[(m-1)/2]}^{1} (\tau_j^+, \tau_{j-1}^-))$ and is unstable when $\tau \in \cup_{j=[(m-1)/2]}^{0} (\tau_j^-, \tau_j^+)$.*

(3) *If $(a, k_1) \in D_3$, then (11) undergoes a Hopf bifurcation at the origin $(0,0)$ when $\tau = \tau_j^\pm$. We control (25) to guarantee the following statements.*

(3i) *If $\omega_*^\pm \in (0, \theta_1)$, then the zero solution is unstable.*

(3ii) *If $\omega_*^\pm \in (\theta_1, \pi)$, then the zero solution is asymptotically stable when $\tau \in \cup_{j=0}^{[(m-1)/2]} (\tau_j^+, \tau_j^-)$ and is unstable when $\tau \in (0, \tau_0^+) \bigcup (\cup_{j=0}^{[(m-1)/2]-1} (\tau_j^-, \tau_{j+1}^+))$.*

(4) *If $(a, k_1) \in D_4$, then (11) undergoes a Hopf bifurcation at the origin $(0,0)$ when $\tau = \tau_j^\pm$, and the zero solution is unstable.*

(5) *If $(a, k_1) \in D_5$, then the zero solution of (11) is asymptotically stable.*

(6) *If $(a, k_1) \in D_6$, then the zero solution of (11) is unstable.*

Proof. (1) If $\sqrt{a^2+\omega^2} \le k_1$, applying Lemmas 1 and 3, we know that all roots of (17) have modulus less than one when $\tau \in (0, \tau_0)$, and (17) has at least a couple of roots with modulus greater than one when $\tau > \tau_0$. Due to Corollary 2.4 in Ruan and Wei [20], we get the conclusion.

(2) If $|a| < k_1 < \sqrt{a^2+\omega^2}$, the time delay satisfies

$$\frac{e^{a\tau h} - 1}{a} = \frac{\sin\omega_*^\pm}{k_1 \sin m\omega_*^\pm - w}, \tag{34}$$

so

$$k_1 \sin m\omega_*^\pm - w > 0. \tag{35}$$

From (25) for $\omega_*^\pm \in (0, \pi)$, we know that $\cos\omega_*^+ > \cos\omega_*^-$. Since $\cos\omega_*^\pm$ is a decreasing function for $\omega_*^\pm \in (0,\pi)$, we have $\omega_{*,j}^+ < \omega_{*,j}^-$.

Meanwhile, in view of (34), we have

$$he^{a\tau h}\frac{d\tau}{dw_*^\pm} = \frac{\left(k_1 \sin m\omega_*^\pm - w\right)\cos\omega_*^\pm - mk_1 \sin\omega_*^\pm \cos m\omega_*^\pm}{\left(k_1 \sin m\omega_*^\pm - w\right)^2}. \tag{36}$$

Set $g(\omega_*^\pm) = (k_1 \sin m\omega_*^\pm - w)\cos\omega_*^\pm - mk_1 \sin\omega_*^\pm \cos m\omega_*^\pm$. If $m > 1$, then

$$g'\left(\omega_*^\pm\right) = \left(w + \left(m^2 - 1\right) k_1 \sin m\omega_*^\pm\right) \sin\omega_*^\pm > 0. \tag{37}$$

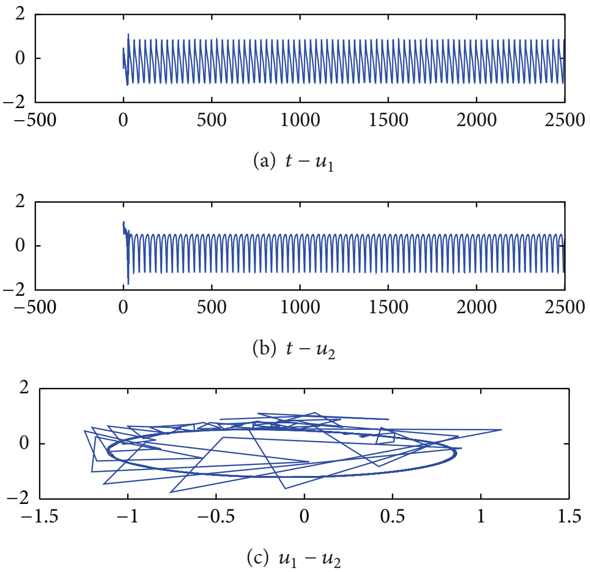


Figure 1: Waveform plot and phase (the Euler method) for system (39) with $a=-2$, $k_1=2.5\,((a,k_1)\in D_1)$, $h=1/2$, and $\tau=0.5\,(>0.4391=\tau_0)$. A periodic solution bifurcates from $(0,0)$ and is asymptotically stable.

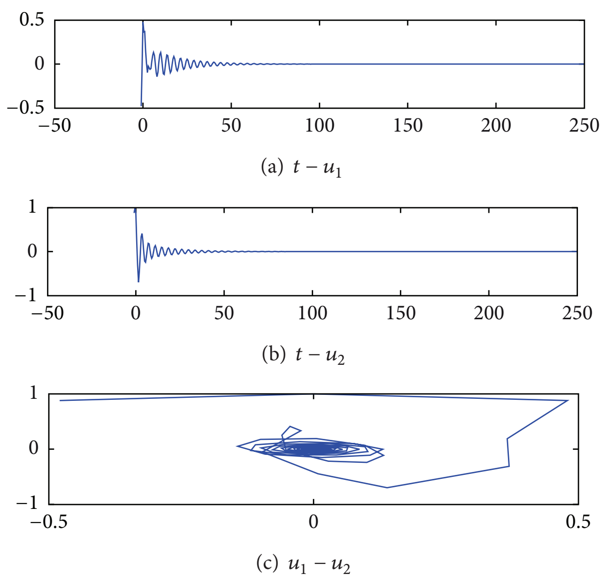


Figure 2: Waveform plot and phase for system (39) with $a=-2$, $k_1=2.5\,((a,k_1)\in D_1)$, $h=1/2$, and $\tau=0.5\,(<0.5781=\tau_0)$. The equilibrium $(0,0)$ is asymptotically stable.

There exists θ_1 such that when $\omega_*^{\pm}\in(0,\theta_1)$, $d\tau/d\omega_*^{\pm}<0$ and when $\omega_*^{\pm}\in(\theta_1,\pi)$, $d\tau/d\omega_*^{\pm}>0$. So we have $\tau_j^-<\tau_j^+$ when $\omega_*^{\pm}\in(0,\theta_1)$; that is, $0<\tau^-_{[(m-1)/2]}<\tau^+_{[(m-1)/2]}<\cdots<\tau_1^-<\tau_1^+<\tau_0^-<\tau_0^+$. And $\tau_j^->\tau_j^+$ when $\omega_*^{\pm}\in(\theta_1,\pi)$; that is, $0<\tau_0^+<\tau_0^-<\cdots<\tau^+_{[(m-1)/2]}<\tau^-_{[(m-1)/2]}$; this case is impossible. Applying Lemmas 1 and 3, we can arrive at the conclusion.

(3) If $-\sqrt{a^2+\omega^2}<k_1<-|a|$, applying Lemmas 1 and 3, in the same way as (2), we can get the conclusion.

(4) If $k_1\le-\sqrt{a^2+\omega^2}$, applying Lemmas 1 and 3, $((d|\lambda(\tau)|^2)/d\tau)|_{\tau=\tau_j,\omega=\omega_{*,j}}>0$, we know that the zero solution is unstable.

(5) (6) Applying Lemmas 1 and 2, we can get the conclusion. □

According to the conclusions of Theorem 4, we have the results that are consistent with those for the corresponding continuous-time model, for sufficiently small step-size.

3. Numerical Simulations

One of the purposes of this section is to test the results in Section 2; the second one is to show that NSFD scheme is better than the Euler method.

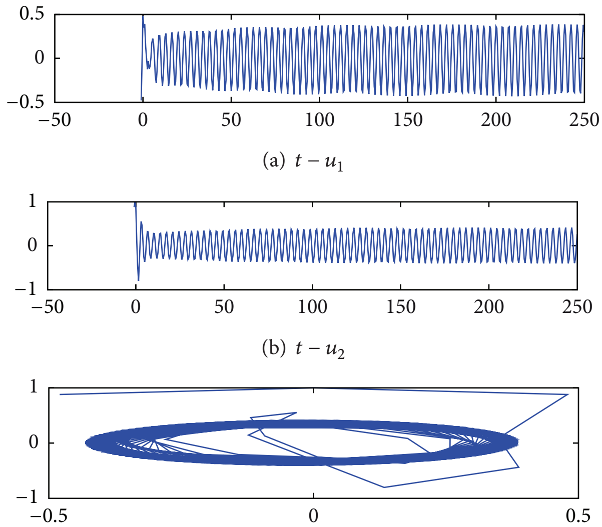
We present some numerical results to system (7) with different values of $a,\omega,k_i\ (i=1,2)$, and τ. We choose $k_2=1$, $\omega=1$; the system (7) is given by

$$\begin{aligned}\dot{u}_1(t)&=\tau\left[au_1(t)-u_2(t)-k_1u_1(t-1)-\left(u_1^2(t)+u_2^2(t)\right)u_1(t)\right.\\&\left.\quad-\left(u_1^2(t-1)-u_2^2(t-1)\right)\right],\\\dot{u}_2(t)&=\tau\left[u_1(t)+au_2(t)-k_1u_2(t-1)-\left(u_1^2(t)+u_2^2(t)\right)u_2(t)\right.\\&\left.\quad-2u_1(t-1)u_2(t-1)\right].\end{aligned}\tag{38}$$

Using NSFD scheme ($h=1/m$) to (38), we obtain

$$\begin{aligned}u_{1,n+1}&=u_{1,n}+\frac{e^{a\tau h}-1}{a}\left[au_{1,n}-u_{2,n}-k_1u_{1,n-m-(u_{1,n}^2+u_{2,n}^2)u_{1,n}}\right.\\&\left.\quad-\left(u_{1,n-m}^2-u_{2,n-m}^2\right)\right],\\u_{2,n+1}&=u_{2,n}+\frac{e^{a\tau h}-1}{a}\left[u_{1,n}+au_{2,n}-k_1u_{2,n-m}\right.\\&\left.\quad-\left(u_{1,n}^2+u_{2,n}^2\right)u_{2,n}-2u_{1,n-m}u_{2,n-m}\right].\end{aligned}\tag{39}$$

Choosing $a=-2$, $k_1=2.5\,((a,k_1)\in D_1)$), we obtaining Figures 2–5. We compute the bifurcation points of (39) for some step-size. We see that τ_0 is asymptotically convergent to 1.0033 with the increasing of m, which is the true value. Using NSFD scheme we have $h=1/2$, $\tau_0=0.5781$; $h=1/10$, $\tau_0=0.8687$; $h=1/20$, $\tau_0=0.9294$. Applying Euler method we obtain $h=1/2$, $\tau_0=0.4391$; $h=1/10$, $\tau_0=0.7975$; $h=1/20$, $\tau_0=0.8875$. From Figure 3 for $h=1/2$, $\tau=0.6$, we can obtain that the fixed point is not asymptotically stable. For $\tau=0.6$ and $h=1/10$ (Figure 4), the fixed point is asymptotically stable. With the Euler method (Figure 1) we

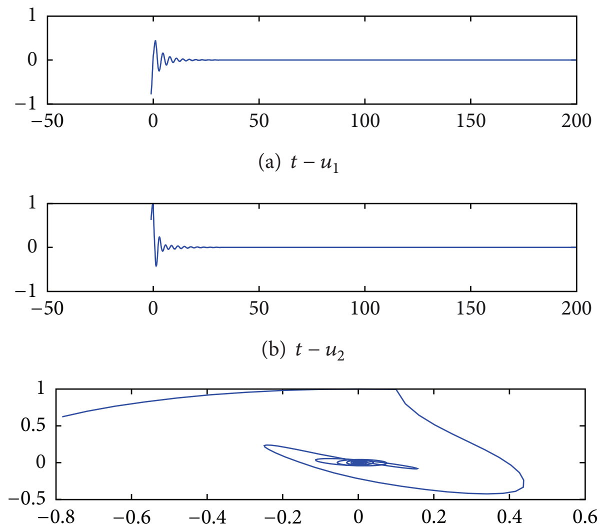


(a) $t-u_1$

(b) $t-u_2$

(c) u_1-u_2

Figure 3: Waveform plot and phase for system (39) with $a = -2$, $k_1 = 2.5\,((a,k_1)\in D_1)$, $h = 1/2$, and $\tau = 0.6\,(> 0.5781 = \tau_0)$. A periodic solution bifurcates from $(0,0)$ and is asymptotically stable.

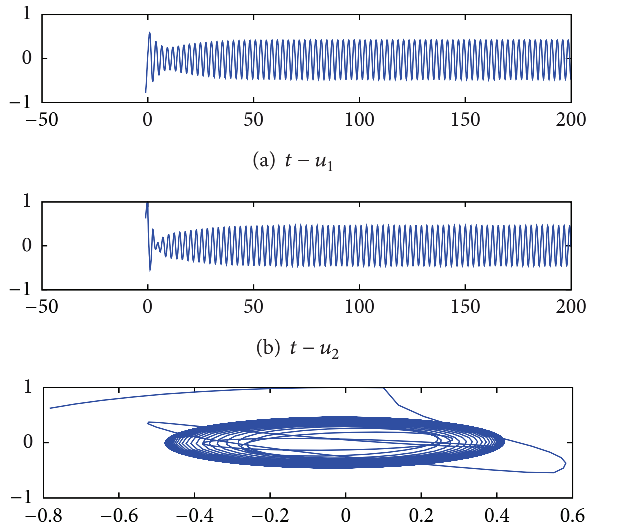


(a) $t-u_1$

(b) $t-u_2$

(c) u_1-u_2

Figure 4: Waveform plot and phase for system (39) with $a = -2$, $k_1 = 2.5\,((a,k_1)\in D_1)$, $h = 1/10$, and $\tau = 0.6\,(< 0.8687 = \tau_0)$. The equilibrium $(0,0)$ is asymptotically stable.

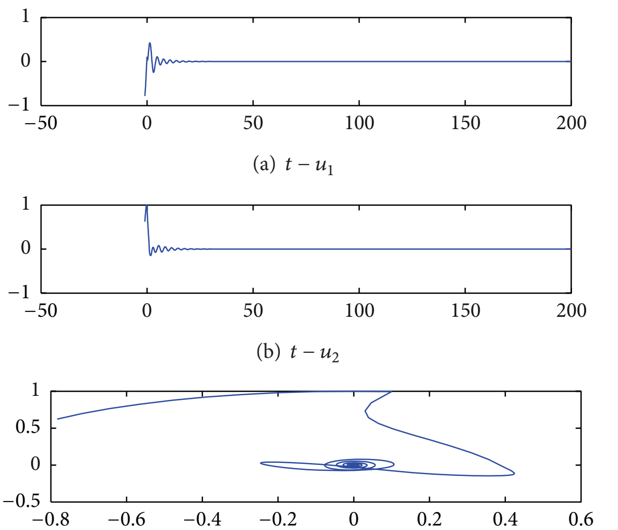


(a) $t-u_1$

(b) $t-u_2$

(c) u_1-u_2

Figure 5: Waveform plot and phase for system (39) with $a = -2$, $k_1 = 2.5\,((a,k_1)\in D_1)$, $h = 1/10$, and $\tau = 1\,(> 0.8687 = \tau_0)$. A periodic solution bifurcates from $(0,0)$ and is asymptotically stable.



(a) $t-u_1$

(b) $t-u_2$

(c) u_1-u_2

Figure 6: Waveform plot and phase for system (39) with $a = -1$, $k_1 = 1.09\,((a,k_1)\in D_2)$, $h = 1/10$, and $\tau = 1\,(< 1.4480 = \tau_0^-)$. The equilibrium $(0,0)$ is asymptotically stable.

can obtain that for $h = 1/2, \tau = 0.5$, the fixed point is not asymptotically stable. With NSFD scheme (Figure 2) for the same step-size and τ we can obtain that the fixed point is asymptotically stable.

Through the analysis, it demonstrates superiority of NSFD scheme over Euler method under the means of describing approximately the dynamics of the original system.

Analogous to the region D_1, now we give the results and figures of $h = 1/10$ in other five regions.

We choose $a = -1$, $k_1 = 1.09\,((a,k_1)\in D_2)$. We have $h = 1/10$, $\tau_0^- = 1.4480$, $\tau_0^+ = 4.2051$, and $\tau_1^- = 7.0458$. See Figures 6, 7, and 8.

We choose $a = -1$, $k_1 = -1.09$ $((a,k_1)\in D_3)$. We have $h = 1/10$, $\tau_0^+ = 0.8405$, $\tau_0^- = 2.8689$, and $\tau_1^+ = 5.5638$. A stability switch is found. See Figures 9, 10, and 11.

Choose $a = -0.5$, $k_1 = -1\,((a,k_1)\in D_3)$. The zero solution of (39) is unstable. See Figure 12.

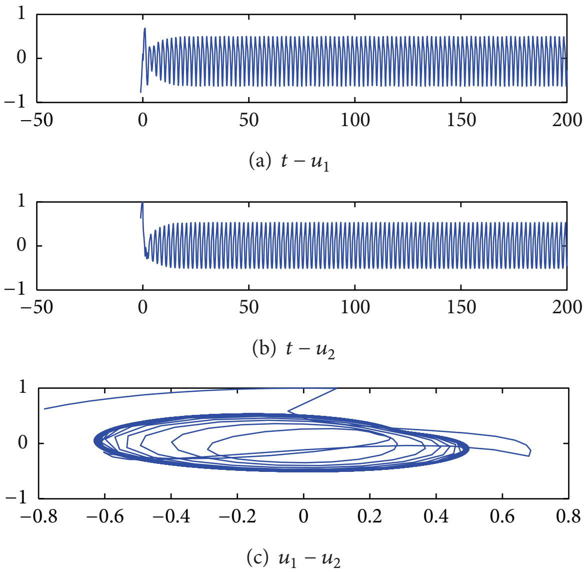


FIGURE 7: Waveform plot and phase for system (39) with $a = -1$, $k_1 = 1.09$ ($(a, k_1) \in D_2$), $h = 1/10$, and $\tau = 3$ ($\tau_0^- = 1.4480 < 3 < 4.2051 = \tau_0^+$). A periodic solution bifurcates from $(0, 0)$ and is asymptotically stable.

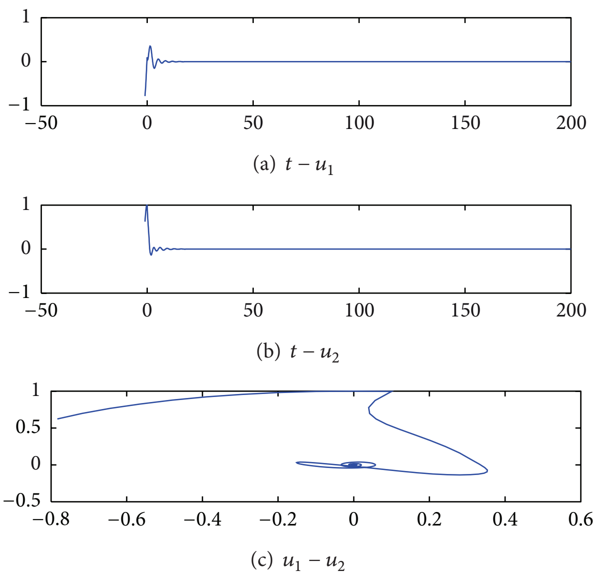


FIGURE 8: Waveform plot and phase for system (39) with $a = -1$, $k_1 = 1.09$ ($(a, k_1) \in D_2$), $h = 1/10$, and $\tau = 4.5$ ($\tau_0^+ = 4.2051 < 4.5 < 7.0458 = \tau_1^-$). The equilibrium $(0, 0)$ is asymptotically stable.

Choose $a = -2$, $k_1 = -2.5$ ($(a, k_1) \in D_4$). The zero solution of (39) is unstable. See Figure 13, where the delay is 1.

Take $a = -2$, $k_1 = -1$ ($(a, k_1) \in D_5$). The zero solution is asymptotically stable. See Figure 14.

Take $a = 2$, $k_1 = -1$ ($(a, k_1) \in D_6$). The zero solution is unstable. See Figure 15.

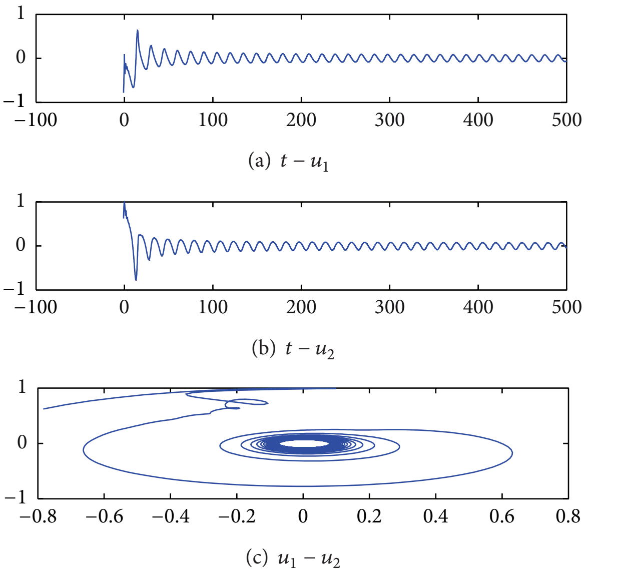


FIGURE 9: Waveform plot and phase for system (39) with $a = -1$, $k_1 = -1.09$ ($(a, k_1) \in D_3$), $h = 1/10$, and $\tau = 0.8$ ($< 0.8405 = \tau_0^+$). A periodic solution bifurcates from $(0, 0)$ and is asymptotically stable.

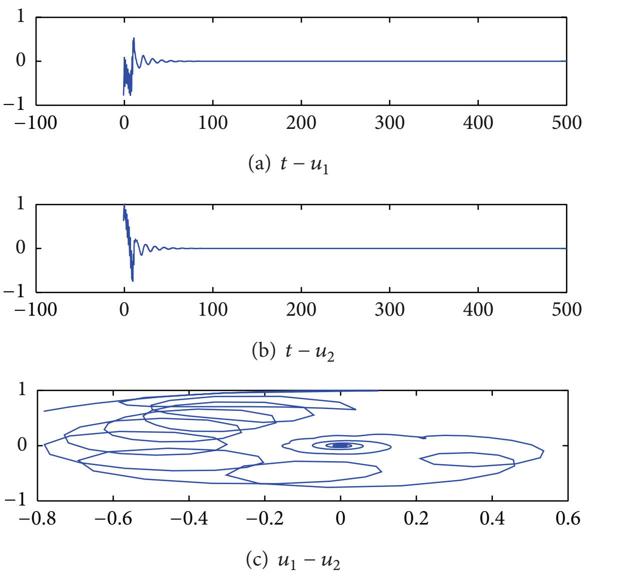
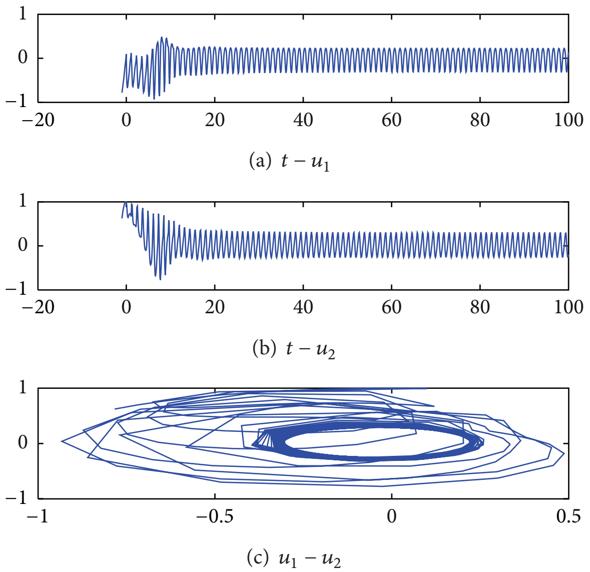


FIGURE 10: Waveform plot and phase for system (39) with $a = -1$, $k_1 = -1.09$ ($(a, k_1) \in D_3$), $h = 1/10$, and $\tau = 2$ ($\tau_0^+ = 0.8405 < 2 < 2.8689 = \tau_0^-$). The equilibrium $(0, 0)$ is asymptotically stable.

4. Conclusions and Future Plans

Reddy et al. [1] have studied the dynamics of a single Hopf bifurcation oscillator (the Stuart-Landau equation) in the presence of an autonomous time-delayed feedback. The feedback term has both a linear component and a simple quadratic nonlinear term. The model can also find more direct applications in simulation studies for feedback control

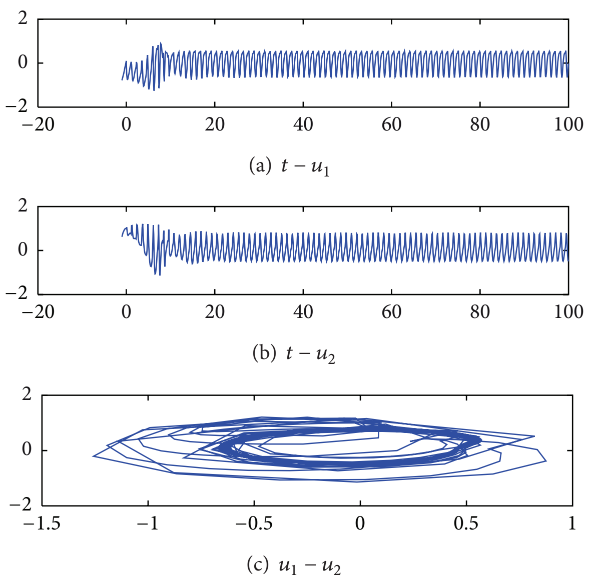


(a) $t - u_1$

(b) $t - u_2$

(c) $u_1 - u_2$

Figure 11: Waveform plot and phase for system (39) with $a = -1$, $k_1 = -1.09$ $((a, k_1) \in D_3)$, $h = 1/10$, and $\tau = 3$ $(\tau_0^- = 2.8689 < 3 < 5.5638 = \tau_1^+)$. A periodic solution bifurcates from $(0, 0)$ and is asymptotically stable.



(a) $t - u_1$

(b) $t - u_2$

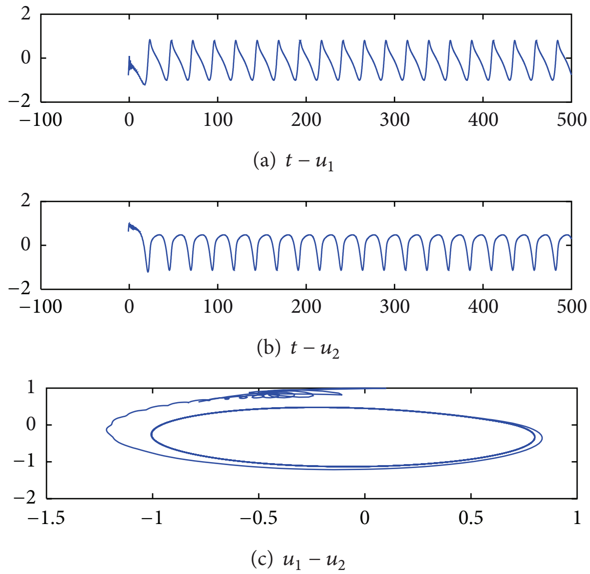
(c) $u_1 - u_2$

Figure 12: Waveform plot and phase for system (39) with $a = -0.5$, $k_1 = -1$ $((a, k_1) \in D_3)$, $h = 1/10$, and $\tau = 3$. A periodic solution bifurcates from $(0, 0)$ and is asymptotically stable.

of individual physical, chemical, or biological entities that have the basic nonlinear characteristics of Hopf oscillator.

In this article, we study a complex autonomously driven single limit cycle oscillator with delayed feedback. In Section 2, the original model is translated to a two-dimensional system. Using NSFD scheme we have investigated the dynamics of a discrete limit cycle oscillator with delayed feedback. Choose the "denominator function" ϕ as

$$\phi(h) = \frac{e^{a\tau h} - 1}{a\tau}. \tag{40}$$

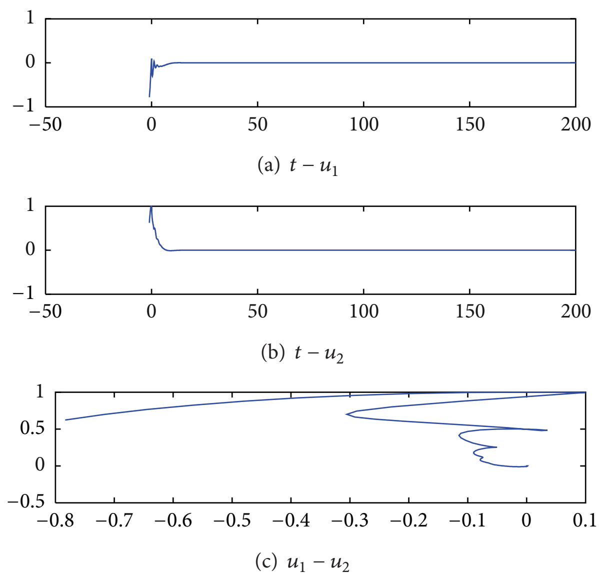


(a) $t - u_1$

(b) $t - u_2$

(c) $u_1 - u_2$

Figure 13: Waveform plot and phase for system (39) with $a = -2$, $k_1 = -2.5$ $((a, k_1) \in D_4)$, $h = 1/10$, and $\tau = 1$. A periodic solution bifurcates from $(0, 0)$ and is asymptotically stable.



(a) $t - u_1$

(b) $t - u_2$

(c) $u_1 - u_2$

Figure 14: Waveform plot and phase for system (39) with $a = -2$, $k_1 = -1$ $((a, k_1) \in D_5)$, $h = 1/10$, and $\tau = 3$. The equilibrium $(0, 0)$ is asymptotically stable.

Through analysis, we obtain Lemmas 1, 2, and 3 and Theorem 4. Equations (24) and (25) are important. For small step-size we obtain the consistent dynamical results of the corresponding continuous-time model. And we find stability switches in the two-dimensional discrete model. At the same time, it demonstrates superiority of NSFD scheme over the Euler method under the means of describing approximately the dynamics of the original system.

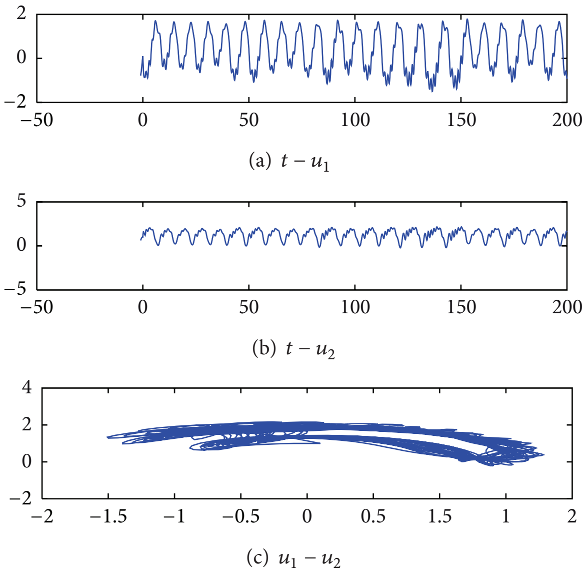


Figure 15: Waveform plot and phase for system (39) with $a = 2$, $k_1 = -1$ ($(a, k_1) \in D_6$), $h = 1/10$, and $\tau = 1$. A periodic solution bifurcates from $(0, 0)$ and is asymptotically stable.

Consider a first-order complex differential equations with delay

$$\dot{x}(t) = px(t) + qx(t-\tau), \tag{41}$$

where $\tau > 0$ is a constant, and p and q are both complex. Cahlon and Schmidt [21] pointed out that (41) plays an important role as a test equation for studying the numerical method applied to delay differential equation. By studying the asymptotic stability of the solutions of (41) for different values of p and q, we learn more about the effect of the delay on the solution. For instance, if $|q| > |p|$, the term with delay carries more weight, while the opposite holds for $|q| < |p|$ [21]. Wei and Zhang [19] have studied Hopf bifurcation and stability switches of (41) in the continuous-time model. In our future work, by many numerical methods, we will discuss those behaviors of (41) in the discrete-time model.

Acknowledgments

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