Journal of Applied Mathematics and Stochastic Analysis, **10**:4 (1997), 407-421.

THE MAP,M/G_1,G_2/1 QUEUE WITH PREEMPTIVE PRIORITY[1]

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(Received June, 1996; Revised December, 1996)

We consider the MAP, M/G_1,G_2/1 queue with preemptive resume priority, where low priority customers arrive to the system according to a Markovian arrival process (MAP) and high priority customers according to a Poisson process. The service time density function of low (respectively: high) priority customers is $g_1(x)$ (respectively: $g_2(x)$). We use the supplementary variable method with Extended Laplace Transforms to obtain the joint transform of the number of customers in each priority queue, as well as the remaining service time for the customer in service in the steady state. We also derive the probability generating function for the number of customers of low (respectively, high) priority in the system just after the service completion epochs for customers of low (respectively, high) priority.

Key words: The MAP/G/1 Queue, Preemptive Resume Priority, Supplementary Variable Method, Queue Length.
AMS subject classifications: 60K25, 90B22.

1. Introduction

The Markovian arrival process (MAP) is a good mathematical model for input traffics which have strong autocorrelations between cell arrivals and high burstiness in broadband-integrated services digital networks (B-ISDNs). Well-known processes such as the Poisson process, Interrupted Poisson process and Markov modulated Poisson process are special cases of the MAP [6].

The supplementary variable method, which is the main analytic tool in this paper, was first introduced by Cox [2] and has been applied by a number of authors. See Keilson and Kooharian [5], Hokstad [3], Sugahara et al. [11] and references therein. To apply the supplementary variable method to the MAP/G/1 type queues, Choi et al. [1] extended the notion of the Laplace Transform, which is suitable for dealing with matrix equations. Using this Extended Laplace Transform (ELT) they obtained

[1]This paper is in part supported by KOSEF(1995).

the joint transform of the number of customers and the remaining service time for the customer in service for the MAP/G/1 queue in the steady state.

There are several journal publications which have considered this kind of priority. Refer to [7, 13] and references therein. Takine and Hasegawa [13] studied the workload process in the MAP/G/1 queue with state-dependent service times. The results were applied to analyze the Laplace-Stieltjes Transform of the waiting time distribution in the preemptive resume priority MAP/G/1 queue. Machihara [7] studied the PH-MRP, M/G_1,G_2/1 queue with preemptive priority, where PH-MRP has high priority and Poisson process has low priority. With the help of the fundamental period of the PH-MRP/G/1 queue, he derived the distribution of the number of customers in the system at the service completion epochs for non-priority customers by the embedded Markov chain method. In addition, waiting times and interdeparture time distributions for non-priority customers were derived.

In this paper, we investigate the MAP, M/G_1,G_2/1 queue with preemptive resume priority by the supplementary variable method (with ELT) developed by Choi et al. [1]. From our supplementary variable analysis, we derive the joint transform of the number of customers in each priority queue, as well as the remaining service time for the customer in service in the steady state. We also derive the distribution for the number of customers of low (respectively, high) priority in the system just after the service completion epochs for customers of low (respectively, high) priority.

The overall organization of this paper is as follows: Section 2 reviews MAPs and the ELT; Section 3 derives the joint transform for the number of customers of each priority and the remaining service time in the steady state for our model; Section 4 derives the PGF (Probability Generating Function) for the number of customers of low (respectively, high) priority at the service completion epochs.

2. Preliminaries

A MAP is a process where arrivals are governed by an underlying m-state Markov chain [6]. Precisely, the MAP is characterized by two matrices C_1 and D_1. C_1 has negative diagonal elements and nonnegative off-diagonal elements, while D_1 has nonnegative elements. Here, $[C_1]_{ij}$, $i \neq j$ is the state transition rate from state i to state j in the underlying Markov chain without an arrival; $[D_1]_{ij}$ is the state transition rate from state i to state j in the underlying Markov chain with an arrival. We assume the underlying Markov chain is irreducible. Since $C_1 + D_1$ is the infinitesimal generator for the underlying Markov chain, we have:

$$(C_1 + D_1)e = 0,$$

where e is an $m \times 1$ column vector of which elements are all equal to 1. Since the finite state Markov chain is irreducible, there exists the stationary probability vector π such that

$$\pi(C_1 + D_1) = 0, \quad \pi e = 1.$$

Next, we introduce the ELT developed by Choi et al. [1]. Given $A = [A_{ij}]$ is the $m \times m$ matrix with $A_{ij} \geq 0$ for $i \neq j$, and $A_{ii} < 0$ for $1 \leq i \leq m$, we find a column vector A^0 with $Ae + A^0 = 0$ and construct a Markov process for the states $\{1, 2, \ldots, m, m+1\}$ with infinitesimal generator

$$\begin{pmatrix} A & A^0 \\ 0 & 0 \end{pmatrix}.$$

It is known that the (i, j)-component of e^{Ax} is the conditional probability that the Markov chain is in state j at time x, given that the Markov chain starts in state i at time 0[4, 9]. Further if A is irreducible, A^{-1} exists [8]. For $0 < z < 1$, let

$$\mathcal{T}(z) = \{S \mid S = [S_{ij}] \text{ is an irreducible } m \times m \text{ real matrix such that}$$

$$-S_{ij} \geq 0 \text{ for } i \neq j;\ \sum_j (-S_{ij}) \leq 0 \text{ with strictly inequality for some } i;$$

$$\text{and } S \text{ commutes with } C_1 + zD_1\}.$$

Note that the commutativity of S and $C_1 + zD_1$ is needed in taking the ELT for the matrix differential equations of the system (formulas (6), (7), and (8)).

We define the ELT with domain $\mathcal{T}(z)$. Let $f(x)$ and $h_i(x)$, $(i = 1, \ldots, m)$, be functions defined on $[0, \infty)$ such that

$$\int_0^\infty \mid f(x) \mid dx < \infty, \int_0^\infty \mid h_i(x) \mid dx < \infty.$$

Definition 1: Let S be an element in $\mathcal{T}(z)$. For a function $f(x)$, the Extended Laplace Transform $F^*(S)$ of $f(x)$ is the $m \times m$ matrix defined by

$$F^*(S) = \int_0^\infty f(x) e^{-Sx} dx.$$

For a vector of functions $H(x) = (h_1(x), \ldots, h_m(x))$, the Extended Laplace Transform $H^*(S)$ of $H(x)$ is the $1 \times m$ vector defined by

$$H^*(S) = \int_0^\infty H(x) e^{-Sx} dx.$$

Note that $F^*(S)$ and $H^*(S)$ exist because any component of e^{-Sx} is dominated by 1. If we identify s with sI (where I is the identity matrix of order m), then since sI commutes with any matrix, especially $C_1 + zD_1$, $\{s \mid s > 0\}$ can be considered as a subset of the domain $\mathcal{T}(z)$. For a positive real number s, we have:

$$H^*(sI) = H^*(s), \tag{1}$$

i.e., the ELT $H^*(sI)$ defined in Definition 1 is reduced to the vector $H^*(s)$, of which ith component is the ordinary Laplace Transform $H_i^*(s)$ of $h_i(x)$:

$$H_i^*(s) = \int_0^\infty h_i(x) e^{-sx} d.$$

Thus, Definition 1 is a natural generalization of the Laplace Transform. To determine the formula for the ordinary Laplace Transform version from the corresponding formula for the ELT, formula (1) is used. See formulas (25), (26) and (27) in Section 3 for more details.

3. Analysis of Our Model

We consider the MAP, $M/G_1,G_2/1$ queue with preemptive resume priority. The arrival process of low priority is an MAP with representation (C_1, D_1), and the arrival process of high priority is a Poisson process with rate γ. We assume that both arrival processes are independent, and that for each process there is an infinite capacity queue. The Poisson process with rate γ can be regarded as an MAP with representation $C_2 = -\gamma I$ and $D_2 = \gamma I$. Therefore, the superposed arrival process of an MAP and a Poisson process is considered as an MAP with representation $C = C_1 + C_2$ and $D = D_1 + D_2$. The service time density function of low (respectively: high) priority customers is $g_1(x)$ (respectively: $g_2(x)$). Also, it is assumed that the service times of customers are independent of each other. Considering the preemptive resume priority, a low priority customer who is interrupted during his service time will start his service again from where it was interrupted. We define

$$\lambda_1 = \pi D_1 e \quad \text{and } \lambda_2 = \pi D_2 e (= \gamma).$$

Let μ_1 (respectively: μ_2) be the mean service time for low (respectively: high) priority customers. Throughout this paper, we assume $\rho < 1$ to guarantee the stability of our system, where $\rho = \rho_1 + \rho_2$ and $\rho_1 = \lambda_1 \mu_1$, $\rho_2 = \lambda_2 \mu_2$.

We are now ready to analyze our system. Let $N_1(t)$ (respectively: $N_2(t)$) be the number of customers of low (respectively: high) priority in the low (respectively: high) priority queue and J_t the state of the underlying Markov chain of the MAP at time t. Let X_t (respectively: Y_t) be the remaining service time of the customer of low (respectively: high) priority in the process of service (if any) at time t. Let ξ_t be the state of the server at time t as

$$\xi_t = \begin{cases} 0, & \text{if server is idle,} \\ 1, & \text{if a customer of low priority is served,} \\ 2, & \text{if a customer of high priority is served and there} \\ & \text{exists an interrupted low priority customer,} \\ 3, & \text{if a customer of high priority is served and there} \\ & \text{does not exist an interrupted low priority customer.} \end{cases}$$

Note that the state of server enters the state 3 only when a customer of high priority arrives at idle period.

Define

$$p_{1i}(n_1, x; t)\Delta x = P\{N_1(t) = n_1, x < X_t < x + \Delta x, J_t = i, \xi_t = 1\},$$

$$p_{2i}(n_1,n_2,x,y;t)\Delta x\Delta y = P\{N_1(t)=n_1, N_2(t)=n_2, x<X_t<x+\Delta x,$$

$$y<Y_t<y+\Delta y, J_t=i, \xi_t=2\},$$

$$p_{3i}(n_1,n_2,y;t)\Delta y = P\{N_1(t)=n_1, N_2(t)=n_2, y<Y_t<y+\Delta y, J_t=i, \xi_t=3\},$$

$$q_i(t) = P\{J_t=i, \xi_t=0\}.$$

Let $p_1(n_1,x;t), p_2(n_1,n_2,x,y;t), p_3(n_1,n_2,y;t)$, and $q(t)$ be row vectors whose ith elements are $p_{1i}(n_1,x;t), p_{2i}(n_1,n_2,x,y;t)$, $p_{3i}(n_1,n_2,y;t)$, and $q_i(t)$, respectively. By Chapman-Kolmogorov's equations, we obtain the following for $n_1,n_2\geq 0$:

$$\begin{aligned}
p_1(n_1,x-\Delta t;t+\Delta t) &= p_1(n_1,x;t)[I+C\Delta t]\\
&\quad + p_1(n_1-1,x;t)D_1\Delta t\cdot 1\{n_1\geq 1\}\\
&\quad + p_1(n_1+1,0;t)g_1(x)\Delta t\\
&\quad + p_2(n_1,0,x,0;t)\Delta t\\
&\quad + q(t)D_1g_1(x)\Delta t\cdot 1\{n_1=0\}\\
&\quad + p_3(n_1+1,0,0;t)g_1(x)\Delta t + o(\Delta t)e',\\
p_2(n_1,n_2,x,y-\Delta t;t+\Delta t) &= p_2(n_1,n_2,x,y;t)[I+C\Delta t]\\
&\quad + p_2(n_1-1,n_2,x,y;t)D_1\Delta t\cdot 1\{n_1\geq 1\}\\
&\quad + p_2(n_1,n_2-1,x,y;t)D_2\Delta t\cdot 1\{n_2\geq 1\}\\
&\quad + p_1(n_1,x;t)D_2g_2(y)\Delta t\cdot 1\{n_2=0\}\\
&\quad + p_2(n_1,n_2+1,x,0;t)g_2(y)\Delta t + o(\Delta t)e',\\
p_3(n_1,n_2,y-\Delta t;t+\Delta t) &= p_3(n_1,n_2,y;t)[I+C\Delta t]\\
&\quad + p_3(n_1-1,n_2,y;t)D_1\Delta t\cdot 1\{n_1\geq 1\}\\
&\quad + p_3(n_1,n_2-1,y;t)D_2\Delta t\cdot 1\{n_2\geq 1\}\\
&\quad + q(t)D_2g_2(y)\Delta t\cdot 1\{n_1=n_2=0\}\\
&\quad + p_3(n_1,n_2+1,0;t)g_2(y)\Delta t + o(\Delta t)e',\\
q(t+\Delta t) &= q(t)[I+C\Delta t] + p_1(0,0,t)\Delta t\\
&\quad + p_3(0,0,0;t)\Delta t + o(\Delta t)e',
\end{aligned}$$

where e' is the transpose of the column vector e and $1\{\cdot\}$ denotes the indicator function.

The condition $\rho<1$ guarantees the existence of the stationary probability vectors defined as follows:

$$p_1(n_1,x) = \lim_{t\to\infty} p_1(n_1,x;t),$$

$$p_2(n_1,n_2,x,y) = \lim_{t\to\infty} p_2(n_1,n_2,x,y;t),$$

$$p_3(n_1,n_2,y) = \lim_{t\to\infty} p_3(n_1,n_2,y;t),$$

$$q = \lim_{t \to \infty} q(t).$$

From the above equations, we get Kolmogorov's forward differential equations as follows:

$$\begin{aligned}
-\frac{d}{dx}p_1(n_1,x) &= p_1(n_1,x)C + p_1(n_1-1,x)D_1 \cdot 1\{n \geq 1\} \\
&\quad + p_1(n_1+1,0)g_1(x) + p_2(n_1,0,x,0) \\
&\quad + p_3(n_1+1,0,0)g_1(x) \\
&\quad + qD_1g_1(x) \cdot 1\{n_1 = 0\}, \qquad (2) \\
-\frac{\partial}{\partial y}p_2(n_1,n_2,x,y) &= p_2(n_1,n_2,x,y)C + p_2(n_1-1,n_2,x,y)D_1 \cdot 1\{n_1 \geq 1\} \\
&\quad + p_2(n_1,n_2-1,x,y)D_2 \cdot \{n_2 \geq 1\} \\
&\quad + p_1(n_1,x)D_2g_2(y) \cdot 1\{n_2 = 0\} \\
&\quad + p_2(n_1,n_2+1,x,0)g_2(y) \qquad (3) \\
-\frac{d}{dy}p_3(n_1,n_2,y) &= p_3(n_1,n_2,y)C + p_3(n_1-1,n_2,y)D_1 \cdot 1\{n_1 \geq 1\} \\
&\quad + p_3(n_1,n_2-1,y)D_2 \cdot 1\{n_2 \geq 1\} \\
&\quad + qD_2g_2(y) \cdot 1\{n_1 = n_2 = 0\} \\
&\quad + p_3(n_1,n_2+1,0)g_2(y) \qquad (4) \\
0 &= qC + p_1(0,0) + p_3(0,0,0). \qquad (5)
\end{aligned}$$

We now define the following PGFs for $0 < z_1, z_2 < 1$:

$$P_1(z_1,x) = \sum_{n_1=0}^{\infty} p_1(n_1,x)z_1^{n_1},$$

$$P_2(z_1,z_2,x,y) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p_2(n_1,n_2,x,y)z_1^{n_1}z_2^{n_2},$$

$$P_2(z_1,0,x,y) = \sum_{n_1=0}^{\infty} p_2(n_1,0,x,y)z_1^{n_1},$$

$$P_3(z_1,z_2,y) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p_3(n_1,n_2,y)z_1^{n_1}z_2^{n_2},$$

$$P_3(z_1,0,y) = \sum_{n_1=0}^{\infty} p_3(n_1,0,y)z_1^{n_1}.$$

Multiplying $z_1^{n_1}$ in (2) and summing over n_1, we have

$$\begin{aligned}
-\frac{d}{dx}P_1(z_1,x) &= P_1(z_1,x)[C + z_1D_1] + \frac{1}{z_1}[P_1(z_1,0) - p_1(0,0]g_1(x) \\
&\quad + P_2(z_1,0,x,0) + \frac{1}{z_1}[P_3(z_1,0,0) - p_3(0,0,0)]g_1(x) + qD_1g_1(x). \qquad (6)
\end{aligned}$$

Multiplying $z_1^{n_1}$ and $z_2^{n_2}$ in (3) and (4) and summing over n_1 and n_2, we get

$$-\frac{\partial}{\partial y}P_2(z_1,z_2,x,y) = P_2(z_1,z_2,x,y)[C+z_1D_1+z_2D_2]$$
$$+P_1(z_1,x)D_2g_2(y)+\frac{1}{z_2}[P_2(z_1,z_2,x,0)-P_2(z_1,0,x,0)]g_2(y), \tag{7}$$
$$-\frac{d}{dy}P_3(z_1,z_2,y) = P_3(z_1,z_2,y)[C+z_1D_1+z_2D_2]+qD_2g_2(y)$$
$$+\frac{1}{z_2}[P_3(z_1,z_2,0)-P_3(z_1,0,0)]g_2(y). \tag{8}$$

Let, for each z_1 with $0 < z_1 < 1$,

$$\mathfrak{F}(z_1) = \{S \mid S = [S_{ij}] \text{ is an irreducible } m \times m \text{ real matrix such that}$$
$$-S_{ij} \geq 0 \text{ for } i \neq j;\ \sum_j(-S_{ij}) \leq 0 \text{ with strict inequality for some } i;$$
$$\text{and } S \text{ commutes with } C_1 + z_1D_1\}.$$

Now we introduce the Extended Laplace Transforms (ELTs) defined on $\mathfrak{F}(z_1)$ as follows for $S \in \mathfrak{F}(z_1)$ and $\zeta > 0$:

$$P_1^*(z_1,S) = \int_0^\infty P_1(z_1,x)e^{-Sx}dx,$$

$$P_2^*(z_1,z_2,x,S) = \int_0^\infty P_2(z_1,z_2,x,y)e^{-Sy}dy,$$

$$P_2^{**}(z_1,z_2,\zeta,S) = \int_0^\infty P_2^*(z_1,z_2,x,S)e^{-\zeta x}dx,$$

$$P_2^{0*}(z_1,z_2,S,0) = \int_0^\infty P_2(z_1,z_2,x,0)e^{-Sx}dx,$$

$$P_2^{0*}(z_1,0,S,0) = \int_0^\infty P_2(z_1,0,x,0)e^{-Sx}d,$$

$$P_3^*(z_1,z_2,S) = \int_0^\infty P_3(z_1,z_2,y)e^{-Sy}dy,$$

$$G_i^*(S) = \int_0^\infty g_i(x)e^{-Sx}dx, \qquad i=1,2.$$

By taking the ELT on both sides in formulas (6), (7) and (8), we get

$$- P_1^*(z_1,S)S + P_1(z_1,0) = P_1^*(z_1,S)[C + z_1D_1]$$
$$+ \frac{1}{z_1}[P_1(z_1,0) - p_1(0,0)]G_1^*(S) + P_2^{0*}(z_1,0,S,0)$$
$$+ \frac{1}{z_1}[P_3(z_1,0,0) - p_3(0,0,0)]G_1^*(S) + qD_1G_1^*(S),$$
$$- P_2^*(z_1,z_2,x,S)S + P_2(z_1,z_2,x,0) = P_2^*(z_1,z_2,x,S)[C + z_1D_1 + z_2D_2]$$
$$+ P_1(z_2,x)D_2G_2^*(S) + \frac{1}{z_2}[P_2(z_1,z_2,x,0) - P_2(z_1,0,x,0)]G_2^*(S),$$
$$- P_3^*(z_1,z_2,S)S + P_3(z_1,z_2,0) = P_3^*(z_1,z_2,S)[C + z_1D_1 + z_2D_2]$$
$$+ qD_2G_2^*(S) + \frac{1}{z_2}[P_3(z_1,z_2,0) - P_3(z_1,0,0)]G_2^*(S).$$

Thus,

$$P_1^*(z_1,S)[S + C + z_1D_1] = P_1(z_1,0)[I - \frac{1}{z_1}G_1^*(S)]$$
$$+ \frac{1}{z_1}p_1(0,0)G_1^*(S) - P_2^{0*}(z_1,0,S,0)$$
$$- \frac{1}{z_1}[P_3(z_1,0,0) - p_3(0,0,0)]G_1^*(S) - qD_1G_1^*(S), \tag{9}$$
$$P_2^*(z_1,z_2,x,S)[S + C + z_1D_1 + z_2D_2] = P_2(z_1,z_2,x,0)[I - \frac{1}{z_2}G_2^*(S)]$$
$$- P_1(z_1,x)D_2G_2^*(S) + \frac{1}{z_2}P_2(z_1,0,x,0)G_2^*(S), \tag{10}$$
$$P_3^*(z_1,z_2,S)[S + C + z_1D_1 + z_2D_2] = P_3(z_1,z_2,0)[I - \frac{1}{z_2}G_2^*(S)]$$
$$- qD_2G_2^*(S) + \frac{1}{z_2}P_3(z_1,0,0)G_2^*(S). \tag{11}$$

Let $S_2(z_1,z_2) = -C - z_1D_1 - z_2D_2$. Since $C_2 = -\gamma I$ and $D_2 = \gamma I$, we see that $S_2(z_1,z_2) \in \mathfrak{F}(z_1)$. Thus, we may take $S = S_2(z_1,z_2)$ to make the right hand sides of formulas (10) and (11) zero, so that

$$P_2(z_1,z_2,x,0)[I - \frac{1}{z_2}G_2^*(S_2(z_1,z_2))]$$
$$= P_1(z_1,x)D_2G_2^*(S_2(z_1,z_2)) - \frac{1}{z_2}P_2(z_1,0,x,0)G_2^*(S_2(z_1,z_2)), \tag{12}$$
$$P_3(z_1,z_2,0)[I - \frac{1}{z_2}G_2^*(S_2(z_1,z_2))]$$
$$= qD_2G_2^*(S_2(z_1,z_2)) - \frac{1}{z_2}P_3(z_1,0,0)G_2^*(S_1(z_1,z_2)). \tag{13}$$

Define a P_2 period by the duration of period from the epoch when a low priority customer is interrupted by an arriving customer of high priority to the epoch when the busy period for customers of high priority generated by the arriving customer of high priority ends. A P_2 period has the same distribution as the busy period for the $M/G_2/1$ queue. To solve $P_2(z_1,z_2,x,0)$ and $P_3(z_1,z_2,0)$ explicitly we need the following lemma.

Lemma 1: *Let $[\Theta(z_1)]_{ij}$ be the generating function of the customers of low priority arriving during a P_2 period which ends with underlying Markov chain in* *state j, given that the P_2 period starts with underlying Markov chain in state i. Let $\Theta(z_1)$ be the matrix of which the (i,j)-component is $[\Theta(z_1)]_{ij}$. Then $\Theta(z_1)$ is given by*

$$\Theta(z_1) = \int_0^\infty e^{(C_1 + z_1 D_1)x} dB(x),$$

where the Laplace-Stieltjes Transform of distribution $B(x)$ of the busy period for the $M/G_2/1$ queue is given by

$$B^*(s) = G_2^*(s + \gamma - \gamma B^*(s)),$$

where $G_2^*(s)$ is the ordinary Laplace Transform of $g_2(x)$.

Proof: Note that $B^*(s)$ is the Laplace-Stieltjes Transform of the length of a P_2 period. Since both arrival processes are independent of each other, we have

$$\Theta(z_1) = \int_0^\infty e^{(C_1 + z_1 D_1)x} dB(x). \qquad \square$$

The explicit formula for the distribution function $B(x)$ is given by equation (2.9*b*) in Takagi [12]. From Lemma 1, we have the following lemma:

Lemma 2: *$\Theta(z_1)$ commutes with $C_1 + z_1 D_1$, and $\Theta(z_1)$ is a substochastic matrix. Therefore, $-C - z_1 D_1 - D_2\Theta(z_1)$ is in $\mathfrak{F}(z_1)$.*

Proof: By irreducibility of $C_1 + z_1 D_1$ and Lemma 1, $\Theta(z_1)$ is a positive substochastic matrix. Thus, $C + z_1 D_1 + D_2\Theta(z_1)$ is irreducible. Further, for $0 < z_1 < 1$, $C + z_1 D_1 + D_2\Theta(z_1)$ has nonnegative off-diagonal elements with row sums less than 0. By Lemma 1, $\Theta(z_1)$ and $C_1 + z_1 D_1$ obviously commute. Therefore, $-C - z_1 D_1 - D_2\Theta(z_1)$ is in $\mathfrak{F}(z_1)$. $\square$

From Lemma 1 and Lemma 2, we can also obtain the matrix exponential form for $\Theta(z_1)$ as follows:

$$\Theta(z_1) = G_2^*(-C - z_1 D_1 - D_2\Theta(z_1)). \tag{14}$$

Lemma 3: *$P_2(z_1, 0, x, 0)$ and $P_3(z_1, 0, 0)$ are given by*

$$P_2(z_1, 0, x, 0) = P_1(z_1, x) D_2 \Theta(z_1),$$

$$P_3(z_1, 0, 0) = q D_2 \Theta(z_1).$$

Further $P_2^{0*}(z_1, 0, S, 0)$ is given by

$$P_2^{0*}(z_1, 0, S, 0) = P_1^*(z_1, S) D_2 \Theta(z_1).$$

Proof: Note that $P_1(n_1, x)\Delta t$ is the vector of the probability that the state of server is 1, the number of customers of low priority in the queue is n_1, and the remaining service time for the customer in service is in $(x, x + \Delta t)$ at an arbitrary time. By preemptive resume priority, the customers of high priority arriving during the services of low priority customers always generate P_2 periods. If we consider the epochs when P_2 periods start as embedded points, the vector of probability generating functions for the number of customers of low priority at the embedded points is given by

$$\frac{P_1(z_1,x)D_2}{P_1(1^-,x)D_2e}.$$

Hence, by the definition of $P_2(z_1,0,x,0)$, we have

$$\frac{P_2(z_1,0,x,0)}{P_2(1^-,0,x,0)e} = \frac{P_1(z_1,x)D_2\Theta(z_1)}{P_1(1^-,x)D_2e}. \tag{15}$$

Note that $\Theta(z_1)$ is the matrix of generating functions for the number of customers of low priority arriving during a single P_2 period. Further, note that $P_1(1^-,x)D_2e$ is the rate of starting P_2 periods with remaining service time x for the interrupted low priority customer, and $P_2(1^-,0,x,0)e$ is the rate of ending P_2 periods with remaining service time x for the interrupted low priority customer. So, in the steady state both rates must be the same; thus,

$$P_1(1^-,x)D_2e = P_2(1^-,0,x,0)e. \tag{16}$$

The above equation (16) can be also proved from formula (12) by letting $z_1 = z_2 = 1^-$ and multiplying e. Therefore, from formulas (15) and (16), we have

$$P_2(z_1,0,x,0) = P_1(z_1,x)D_2\Theta(z_1).$$

By taking the ELT on the above equation, we get

$$P_2^{0*}(z_1,0,S,0) = P_1^*(z_1,S)D_2\Theta(z_1).$$

By the similar argument, we can also prove that

$$P_3(z_1,0,0) = qD_2\Theta(z_1). \qquad \square$$

By Lemma 3 and taking the ordinary Laplace Transform on formula (12) with respect to x, for $\zeta > 0$, we get

$$\begin{aligned} &P_2^{0*}(z_1,z_2,\zeta,0)[I - \tfrac{1}{z_2}G_2^*(S_2(z_1,z_2))] \\ &= P_1^*(z_1,\zeta)D_2[I - \tfrac{1}{z_2}\Theta(z_1)]G_2^*(S_2(z_1,z_2)), \end{aligned} \tag{17}$$

where $P_2^{0*}(z_1,z_2,\zeta,0)$ and $P_1^*(z_1,\zeta)$ are the ordinary Laplace Transforms of $P_2(z_1,z_2,x,0)$ and $P_1(z_1,x)$. By Lemma 3 and formula (13), we get

$$\begin{aligned} &P_3(z_1,z_2,0)[I - \tfrac{1}{z_2}G_2^*(S_2(z_1,z_2))] \\ &= qD_2[I - \tfrac{1}{z_2}\Theta(z_1)]G_2^*(S_2(z_1,z_2)). \end{aligned} \tag{18}$$

Taking the ordinary Laplace Transform on formula (10) with respect to x, for $\zeta > 0$, we get

$$\begin{aligned} &P_2^{**}(z_1,z_2,\zeta,S)[S + C + z_1D_1 + z_2D_2] \\ &= P_2^{0*}(z_1,z_2,\zeta,0)[I - \tfrac{1}{z_2}G_2^*(S)] - P_1^*(z_1,\zeta)D_2G_2^*(S) \end{aligned}$$

$$+P_2^{0*}(z_1,0,\zeta,0)\frac{1}{z_2}G_2^*(S), \tag{19}$$

where $P_2^{0*}(z_1,0,\zeta,0)$ is the ordinary Laplace Transform of $P_2(z_1,0,x,0)$. By substituting (17) and (18) into (19) and (11), from Lemma 3 we get

$$P_2^{**}(z_1,z_2,\zeta,S)[S+C+z_1D_1+z_2D_2][I-\frac{1}{z_2}G_2^*(S_2(z_1,z_2))]$$

$$=P_1^*(z_1,\zeta)D_2[I-\frac{1}{z_2}\Theta(z_1)][G_2^*(S_2(z_1,z_2))-G_2^*(S)] \tag{20}$$

$$P_3^*(z_1,z_2,S)[S+C+z_1D_1+z_2D_2][I-\frac{1}{z_2}G_2^*(S_2(z_1,z_2))]$$

$$=qD_2[I-\frac{1}{z_2}\Theta(z_1)][G_2^*(S_2(z_1,z_2))-G_2^*(S)]. \tag{21}$$

From (5), (9) and Lemma 3, we have

$$P_1^*(z_1,S)[S+C+z_1D_1+D_2\Theta(z_1)]=P_1(z_1,0)[I-\frac{1}{z_1}G_1^*(S)]$$

$$-q[C+z_1D_1+D_2\Theta(z_1)]\frac{1}{z_1}G_1^*(S). \tag{22}$$

Let $S_1(z_1)=-C-z_1D_1-D_2\Theta(z_1)$. Since $S_1(z_1)\in\mathfrak{F}(z_1)$, by letting $S=S_1(z_1)$ in equation (22), we have

$$P_1(z_1,0)[I-\frac{1}{z_1}G_1^*(S_1(z_1))]=q[C+z_1D_1+D_2\Theta(z_1)]\frac{1}{z_1}G_1^*(S_1(z_1)). \tag{23}$$

By substituting equation (23) into equation (22), we get

$$P_1^*(z_1,S)[S+C+z_1D_1+D_2\Theta(z_1)][I-\frac{1}{z_1}G_1^*(S_1(z_1))]$$

$$=q[C+z_1D_1+D_2\Theta(z_1)][\frac{1}{z_1}G_1^*(S_1(z_1))-\frac{1}{z_1}G_1^*(S)]. \tag{24}$$

For an arbitrary $s>0$, since sI is an element of $\mathfrak{F}(z_1)$ for any z_1 with $0<z_1<1$, by taking $S=sI$ in equations (20), (21) and (24), we obtain the following formulas for the ordinary Laplace Transform version for any $0<z_1$, $z_2<1$, $\zeta>0$ and $s>0$:

$$P_2^{**}(z_1,z_2,\zeta,s)[sI+C+z_1D_1+z_2D_2][z_2I-G_2^*(S_2(z_1,z_2))]$$

$$=P_1^*(z_1,\zeta)D_2[z_2I-\Theta(z_1)][G_2^*(S_2(z_1,z_2))-G_2^*(s)I], \tag{25}$$

$$P_3^*(z_1,z_2,s)[sI+C+z_1D_1+z_2D_2][z_2I-G_2^*(S_2(z_1,z_2))]$$

$$=qD_2[z_2I-\Theta(z_1)][G_2^*(S_2(z_1,z_2))-G_2^*(s)I], \tag{26}$$

$$P_1^*(z_1,s)[sI+C+z_1D_1+D_2\Theta(z_1)][z_1I-G_1^*(S_1(z_1))]$$

$$=q[C+z_1D_1+D_2\Theta(z_1)][G_1^*(S_1(z_1))-G_1^*(s)I]. \tag{27}$$

The stationary invariant vector q of the underlying Markov chain during the server idle periods must still be calculated. From the same argument as in Takine and Hasegawa [13], q satisfies the following equation:

$$Q = C + \int_0^\infty dD(x)e^{Qx},$$

$$qQ = 0, \quad qe = 1 - \rho, \tag{28}$$

where $dD(x) = D_1 g_1(x)dx + D_2 g_2(x)dx$. Hence, we finally establish the main theorem.

Theorem 1: *For $0 < z_1, z_2 < 1, \zeta > 0$, and $s > 0$, the joint transforms $P_1^*(z_1, s)$, $P_2^*(z_1, z_2, \zeta, s)$, and $P_3^*(z_1, z_2, s)$ are given by*

$$P_1^*(z_1, s)[sI + C + z_1 D_1 + D_2 \Theta(z_1)][z_1 I - G_1^*(S_1(z_1))]$$

$$= q[C + z_1 D_1 + D_2 \Theta(z_1)][G_1^*(S_1(z_1)) - G_1^*(s)I],$$

$$P_2^{**}(z_1, z_2, \zeta, s)[sI + C + z_1 D_1 + z_2 D_2][z_2 I - G_2^*(S_2(z_1, z_2))]$$

$$= P_1^*(z_1, \zeta) D_2 [z_2 I - \Theta(z_1)][G_2^*(S_2(z_1, z_2)) - G_2^*(s)I],$$

$$P_3^*(z_1, \zeta) D_2 [z_2 I - C + z_1 D_1 + z_2 D_2][z_2 I - G_2^*(S_2(z_1, z_2))],$$

$$= qD_2[z_2 I - \Theta(z_1)][G_2^*(S_2(z_1, z_2)) - G_2^*(s)I],$$

where q is given by

$$Q = C + \int_0^\infty dD(x)e^{Qx},$$

$$qQ = 0, \quad qe = 1 - \rho,$$

$$dD(x) = D_1 g_1(x)dx + D_2 g_2(x)dx.$$

4. Marginal Queue Length Distributions

We find the distribution for the number of customers of low (respectively: high) priority just after the service completion epochs for customers of low (respectively: high) priority. Let $\Pi_1(x)$ (respectively: $\Pi_2(z)$) be the vector of probability generating functions for the number of customers of low (respectively: high) priority just after the service completion epochs of the low (respectively: high) priority customers. Then, $\Pi_1(z)$ and $\Pi_2(z)$ are given by

$$\Pi_1(z) = \frac{P_1(z, 0)}{P_1(1^-, 0)e},$$

$$\Pi_2(z) = \frac{P_2^{0*}(1^-, z, 0^+, 0) + P_3(1^-, z, 0)}{P_2^{0*}(1^-, 1^-, 0^+, 0)e + P_3(1^-, 1^-, 0)e}.$$

From formulas (23), (17) and (18) we have

$$\Pi_1(z)[zI - G_1^*(S_1(z))] = \frac{1}{P_1(1^-, 0)e} q[C + z_1 D_1 + D_2 \Theta(z)] G_1^*(S_1(z)), \tag{29}$$

$$\Pi_2(z)[zI - G_2^*(S_2(1^-,z))] = \frac{[q + P_1^*(1^-,0^+)]D_2[zI - \Theta(1^-)]}{P_2^{0*}(1^-,1^-,0^+,0)e + P_3(1^-,1^-,0)e}$$

$$\times G_2^*(1^-,z)). \tag{30}$$

Next we derive $P_1(1^-,0)e$ and $P_2^{0*}(1^-,1^-,0^+,0)e + P_3(1^-,1^-,0)e$.

Lemma 4:

$$P_1(1^-,0)$$

$$= q[C + D_1 + D_2\Theta(1^-)]G_1^*(S_1(1^-))[I - G_1^*(S_1(1^-)) + e\pi]^{-1} + P_1(1^-,0)e\pi,$$

$$P_1(1^-,0)e = \lambda_1,$$

$$P_1^*(1^-,0^+) = q[G_1^*(S_1(1^-)) - I][e\pi + I - G_1^*(S_1(1^-))]^{-1} + P_1^*(1^-,0^+)e\pi,$$

$$P_1^*(1^-,0^+)e = \rho_1.$$

Proof: The first equation is obtained from (29) by the same argument as in the proof of Corollary 3.2.7 in Ramaswami [10].

To obtain the second equation, recall that

$$\Theta(z) = G_2^*(-C - zD_1 - D_2\Theta(z)),$$

$$S_1(z) = -C - zD_1 - D_2\Theta(z).$$

By differentiating the above equations with respect to z and letting $z = 1^-$, we get

$$-\frac{d}{dz}S_1(z)\mid_{z=1^-}e = [e\pi - C_1 - D_1]^{-1}$$

$$\times [e\pi - C_1 - D_1 + D_2\frac{\mu_2 e\pi}{1-\rho_2} - D_2\Theta(1^-) + D_2]D_1e,$$

and by differentiating $G_1^*(S_1(z))$ and letting $z = 1^-$, we get

$$\frac{d}{dz}G_1^*(S_1(z))\mid_{z=1^-}e = [e\pi - C - D_1 - D_2\Theta(1^-)]^{-1}$$

$$\times [\mu_1 e\pi - G_1^*(S_1(1^-)) + I][-\frac{d}{dz}S_1(z)\mid_{z=1^-} - e].$$

By differentiating (29) with respect to z, letting $z = 1^-$ and multiplying e on both sides, we get, with the help of the above equations,

$$P_1(1^-,0)e = \lambda_1.$$

From (27), by letting $s = 0^+$, we get

$$P_1^*(z,0^+)[zI - G_1^*(S_1(z))] = q[G_1^*(S_1(z)) - I].$$

From the above equation, we compute the third equation by the same argument as in the proof of Corollary 3.2.7 in Ramaswami [10]. By differentiating the above equation with respect to z, letting $z = 1^-$ and multiplying e on both sides, we get

$$P_1^*(1^-, 0^+)e = \rho_1. \qquad \square$$

Note that $P_1(1^-, 0)e$ is the departure rate of low priority customers in the steady state. The second equation of Lemma 4 demonstrates that the input and output rates of low priority customers are the same, which is a natural property in the steady state.

From Lemma 4, we obtain the following lemma.

Lemma 5:

$$P_2^{**}(1^-, 1^-, 0^+, 0^+)e + P_3^*(1^-, 1^-, 0^+)e = \rho_2,$$

$$P_2^{0*}(1^-, 1^-, 0^+, 0)e + P_3(1^-, 1^-, 0)e = \lambda_2.$$

Proof: We know that

$$qe = 1 - \rho_1 - \rho_2,$$

$$P_1^*(1^-, 0^+)e = \rho_1.$$

So, the first equation follows from the fact that

$$qe + P_1^*(1^-, 0^+)e + P_2^{**}(1^-, 1^-, 0^+, 0^+)e + P_3^*(1^-, 1^-, 0^+)e = 1.$$

From Lemma 4 we derive the second equation by the same argument as in the proof of Lemma 4. $\square$

Note that $P_2^{0*}(1^-, 1^-, 0^+, 0)e + P_3(1^-, 1^-, 0)e$ is the departure rate of high priority customers in the steady sate. The second equation in Lemma 5 demonstrates that the input and output rates of high priority customers are the same, which is natural in the steady state. From Lemma 4 and Lemma 5 we get the following theorem.

Theorem 2: $\Pi_1(z)$ (*respectively:* $\Pi_2(z)$), *the vector of the probability generating functions for the number of customers of low (respectively: high) priority in the system just after the service completion epochs for customers of low (respectively: high) priority are given by for* $0 < z < 1$:

$$\Pi_1(z)[zI - G_1^*(S_1(z))] = \frac{1}{\lambda_1} q[C + z_1 D_1 + D_2 \Theta(z)] G_1^*(S_1(z)),$$

$$\Pi_2(z)[zI - G_2^*(S_2(1^-, z))] = \frac{1}{\lambda_2}[q + P_1^*(1^-, 0^+)] D_2 [zI - \Theta(1^-)] G_2^*(S_2(1^-, z)).$$

Remark: When $\gamma = 0$, our model is reduced to the MAP /G/1 queue. So we have, from Theorem 1,

$$P_1^*(z_1, s)[sI + C_1 + z_1 D_1][z_1 I - G_1^*(-C_1 - z_1 D_1))]$$

$$= q[C_1 + z_1 D_1][G_1^*(-C_1 - z_1 D_1)) - G_1^*(s)I]$$

where q is given by

$$Q = C + \int_0^\infty dD(x)e^{Qx},$$

$$qQ = 0, \ \ qe = 1 - \rho_1,$$

$$dD(x) = D_1 g_1(x)dx.$$

the above equations are in accordance with the results in Choi, et al. [1].

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