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EXISTENCE OF MOMENTS OF INCREASING PREDICTABLE PROCESSES ASSOCIATED WITH ONE- AND TWO-PARAMETER POTENTIALS

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The criterion and sufficient condition for the existence of moments of one-parameter increasing predictable processes is presented in terms of an associated potential. The estimates of moments of special functional connected with two-parameter increasing predictable processes are given in the case when the associated potential is bounded. The application of these estimates to the local time for purely discontinuous strong martingales in the plane is also presented.

Key words: Increasing Predictable Process, Associated Potential, Moments of all Orders, Local Time, Discontinuous Strong Martingale.

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1. Introduction, Definitions and Notations

We estimate the moments of one- and two-parameter increasing processes in terms of associated potentials. It is well known that a one-parameter increasing predictable process has moments of all orders if the associated potential is bounded. In Section 2 we give the criterion and sufficient conditions for the existence of moments of all orders for an increasing process in the case when the associated potential is unbounded, with examples that demonstrate the optimality of the criterion. In Section 3 we give estimates of moments for special functionals connected with increasing predictable two-parameter processes in the case where the associated potential is bounded. The problem of estimating moments of increasing predictable processes associated with two-parameter potentials arises from a question whether the local time for two-parameter purely discontinuous strong martingales has all finite moments. Note that in a one-parameter case Bass [1] gave a positive answer for this question. In Section 4 we apply the results of Section 3 to the local time for purely discontinuous strong martingales on the plane. We estimate moments of special functionals connected with the local time, and formulate the conditions sufficient for the local time to have all finite moments.

The following are necessary definitions and notations: For two points $s=(s_1,s_2)$ and $t=(t_1,t_2)$ in R^2_+, $s\le t$ means that $s_1\le t_1$ and $s_2\le t_2$, and $s<t$ means that $s_1<t_1$ and $s_2<t_2$. If $s<t$, then $(s,t]$ is the rectangle $(s_1,t_1]\times(s_2,t_2]$. Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probability space with the filtration $\{\mathcal{F}_t, t\in R^2_+\}$ which satisfies the following properties [2]:

(F1) if $t<t'$ then $\mathcal{F}_t\subset\mathcal{F}_{t'}$;

(F2) $\mathcal{F}_0$ contains all null sets of $\mathcal{F}$;

(F3) for each $t, \mathcal{F}_t=\bigcup\limits_{t<t'}\mathcal{F}_{t'}$;

(F4) for each t, $\mathcal{F}^1_t$ and $\mathcal{F}^2_t$ are conditionally independent given $\mathcal{F}_t$,

where $\mathcal{F}^1_t=\bigvee\limits_{t_2\ge 0}\mathcal{F}_t$, $\mathcal{F}^2_t=\bigvee\limits_{t_1\ge 0}$. Let $\mathcal{F}^*_t=\mathcal{F}^1_t\vee\mathcal{F}^2_t$, $\mathcal{F}_{t+-}=\bigvee\limits_{s_2<t_2}\mathcal{F}_{t_1s_2}$, $\mathcal{F}_{t-+}=\bigvee\limits_{s_1<t_1}\mathcal{F}_{s_1t_2}$, $\mathcal{F}_{t-}=\bigvee\limits_{s<t}\mathcal{F}_s$, $\mathcal{F}^1_{t-}=\bigvee\limits_{s_1<t_1,t_2\ge 0}\mathcal{F}_{s_1,t_2}$, $\mathcal{F}^2_{t-}=\bigvee\limits_{s_2<t_2,t_1\ge 0}\mathcal{F}_{t_1s_2}$. The definitions of strong, weak 1- and 2-martingale, and increasing process in the plane will follow those of [2]. The definitions and notation of two-parameter predictable, 1- and 2-predictable processes, predictable and dual predictable projection will follow those of [7]. The definition of one-parameter potential will follow that of [3]. If $[0,t]\subset R^2_+$, we denote $\lambda^n_i=\{0=t^0_i<t^1_i<\cdots<t^n_i=t_i\}$, $i=1,2$, $\lambda^n_t=\lambda^n_1\times\lambda^n_2$, $\lambda^m_t\supset\lambda^n_t$ if $m>n$, the partition of $[0,t]$. We also denote $t_{ik}=(t^i_1,t^k_2)$, $X_{ik}=X_{t_{ik}}$, $\square_{ik}=(t_{ik},t_{i+1k+1}]$, $\Delta^1_{ik}X=X_{i+1k}-X_{ik}$, $\Delta^2_{ik}X=X_{ik+1}-X_{ik}$, $\square_{ik}X=\Delta^1_{ik+1}X-\Delta^1_{ik}X$. The increment of process X on the rectangle $(s,t]$ is $X(s,t]=X_t-X_{s_1t_2}-X_{t_1s_2}+X_s$. Let $\Delta^1X_t=X_t-X_{t-+}$, $\Delta^2X_t=X_t-X_{t+-}$, $\square X_t=X_t-X_{t-+}-X_{t+-}+X_{t-}$, where $X_{t+}=\lim\limits_{s\to t, s\in Q^1_t}X_s$, $X_{t-}=\lim\limits_{s\to t, s\in Q^3_t}X_s$, $X_{t+-}=\lim\limits_{s\to t, s\in Q^4_t}X_s$ with $Q^1_t=\{s: t<s\}$, $Q^2_t=\{s: s_1<t_1, t_2\le s_2\}$, $Q^3_t=\{s: s<t\}$, $Q^4_t=\{s: t_1\le s_1, s_2<t_2\}$. All processes are assumed to have these limits, to be constant on $\Gamma=(\{0\}\times[0,\infty))\cup([0,\infty)\times\{0\})$, and to be continuous in Q^1_t.

A functional

$$\int\limits_{[0,t]}X_t dY_t=P-\lim_{|\lambda^n_t|\to 0}\sum_{i,k=0}^{n-1}X_{ik}\square_{ik}Y$$

is called a *stochastic integral of the first kind* if this limit exists for any sequence of partitions λ^n_t. A functional

$$\int\limits_{[0,t]}X_t d_1Y_t d_2Z_t=P-\lim_{|\lambda^n_t|\to 0}\sum_{i,k=0}^{n-1}X_{ik}\Delta^1_{ik}Y\,\Delta^2_{ik}Z$$

is called a *stochastic integral of the second kind* if this limit exists for any sequence of partitions λ^n_t.

A process $X=\{X_t, t\in R^2_+\}$ is called a *weak submartingale* if X_t is integrable for each t, X is adapted and $E(X(s,t]/\mathcal{F}_s)\ge 0$ for each $s\le t$.

If $\{A_t, t\in R^2_+\}$ is an increasing process, then define $A_{t_1\infty}=\lim\limits_{t_2\to\infty}A_t$, $A_{\infty t_2}=\lim\limits_{t_1\to\infty}A_t$, $A_\infty=\lim\limits_{t\to\infty}A_t$.

A process $\{X_t, t \in R^2_+\}$ is called a *two-parameter potential* if X_t is a nonnegative weak submartingale such that $X_{t_1}(t_2)$ and $X_{t_2}(t_1)$ are one-parameter potentials. If a potential X_t is bounded, then it is associated with the increasing predictable process A_t in the following way [5]

$$X_t = A_t + E(A_\infty - A_{t_1\infty} - A_{\infty t_2}/\mathfrak{F}_t). \tag{1}$$

2. Some Properties of One-Parameter Potentials

Let $\{X_t, \mathfrak{F}_t, t \in R_+\}$ be a one-parameter potential associated with predictable increasing process A_t. It means that $X_t = E(A_\infty - A_t/\mathfrak{F}_t)$ [3]. All increasing processes are assumed to be integrable.

Theorem 1: *The following statements are equivalent:*

(*a*) *for every* $k \geq 1$, $E\sup_t X_t^k \leq c_k < \infty$,

(*b*) *for every* $k \geq 1$, $EA_\infty^k \leq d_k < \infty$.

Proof: Suppose condition (a) holds. Let $A_t^{(n)} = A_t \wedge n$, $X_t^{(n)}$ be a potential associated with $A_t^{(n)}$. Then for any $k \geq 2$

$$E[A_\infty^{(n)}]^k = EA_\infty^{(n)}[A_\infty^{(n)}]^{k-1} = E\int_0^\infty (X_t^{(n)} + A_t^{(n)})d[A_t^{(n)}]^{k-1}$$
$$= E\int_0^\infty (X_{t-}^{(n)} + A_{t-}^{(n)})d[A_t^{(n)}]^{k-1},$$

i.e.,

$$2E[A_\infty^{(n)}]^k = E\int_0^\infty (X_t^{(n)} + X_{t-}^{(n)} + A_t^{(n)} + A_{t-}^{(n)})d[A_t^{(n)}]^{k-1}.$$

On the other hand, we have from integration by parts that

$$2E[A_\infty^{(n)}]^k = E\int_0^\infty (A_t^{(n)} + A_{t-}^{(n)})d[A_t^{(n)}]^{k-1} + E\int_0^\infty ([A_t^{(n)}]^{k-1} + [A_{t-}^{(n)}]^{k-1})dA_t^{(n)}, \tag{2}$$

hence,

$$E\int_0^\infty (X_t^{(n)} + X_{t-}^{(n)})d[A_t^{(n)}]^{k-1} = E\int_0^\infty ([A_t^{(n)}]^{k-1} + [A_{t-}^{(n)}]^{k-1})dA_t^{(n)}. \tag{3}$$

We can estimate the integral in the left-hand side of (3)

$$E\int_0^\infty (X_t^{(n)} + X_{t-}^{(n)})d[A_t^{(n)}]^{k-1}$$

$$= E\int_0^\infty (X_t^{(n)} + X_{t-}^{(n)})d[A_t^{(n)}]^{k-1} I\{\sup_t X_t^{(n)} > \frac{A_\infty^{(n)}}{4k}\}$$

$$+ E\int_0^\infty (X_t^{(n)} + X_{t-}^{(n)})d[A_t^{(n)}]^{k-1} I\{\sup_t X_t^{(n)} < \frac{A_\infty^{(n)}}{4k}\} \tag{4}$$

$$\leq 8kE(\sup_t X_t^{(n)})^k + \frac{1}{2k}E[A_\infty^{(n)}]^k.$$

Note that

$$\int_0^\infty A_t^{(n)} d[A_t^{(n)}]^{k-1} = \lim_{|\lambda_t^n| \to 0} \sum_{k=0}^{n-1} A_{t_i}^{(n)}([A_{t_i}^{(n)}]^{k-1} - [A_{t_{i-1}}^{(n)}]^{k-1})$$

$$\leq (k-1) \lim_{|\lambda_t^n| \to 0} \sum_{k=0}^{n-1} [A_{t_i}^{(n)}]^{k-1} \Delta A_{t_i}^{(n)} = (k-1)\int_0^\infty [A_{t_i}^{(n)}]^{k-1} dA_t^{(n)}.$$

From (2)-(4) we have

$$E[A_\infty^{(n)}]^k \leq kE\int_0^\infty [A_{t_i}^{(n)}]^{k-1} dA_t^{(n)} \leq 8k^2 E(\sup_t X_t^{(n)})^k + \frac{1}{2}E[A_\infty^{(n)}]^k$$

or

$$E[A_\infty^{(n)}]^k \leq 16k^2 E(\sup_t X_t^{(n)})^k.$$

Since $X_t^{(n)} = E(A_\infty \wedge n - A_t \wedge n / \mathfrak{F}_t) \leq X_t$, we have that

$$E[A_\infty^{(n)}]^k \leq 16k^2 E\sup_t X_t^k.$$

While $n \to \infty$, we obtain

$$EA_\infty^k \leq 16k^2 E\sup_t X_t^k.$$

Suppose condition (b) holds. Then obviously $X_t \leq E(A_\infty / \mathfrak{F}_t)$, and by Doob's inequality

$$E\sup_t X_t^k \leq E\sup_t (E(A_\infty / \mathfrak{F}_t))^k \leq \left(\frac{k}{k-1}\right)^k EA_\infty^k.$$

The theorem is proved. □

Remark 1: Theorem 1 generalizes the well known result that if an associated potential is bounded, then the increasing process has all bounded moments. We give examples to demonstrate that the existence of moments of X_t does not supply the existence of moments of A_t. This means that the conditions of Theorem 1 are quite optimal.

Example 1: Suppose $\{\xi_n, n \geq 1\}$ is a sequence of independent random variables, and each of them has gamma-distribution with parameters $\alpha_n = \frac{1}{n^2}$ and $\beta_n = \frac{1}{\sqrt{n}}$. Then $E\xi_n^k = \frac{1}{\beta_n^k}(\alpha_n + k - 1) \cdot \ldots \cdot (\alpha_n + 1)\alpha_n < \infty$. Consider the filtration $\mathfrak{F}_n = \sigma\{\xi_k, k \leq n\}$ and an increasing predictable process $A_n = \sum_{k=1}^{n-1} \xi_k$. Hence, a potential associated with A_n is

$$X_n = E(A_\infty - A_n/\mathfrak{F}_n) = E\sum_{k=n+1}^{\infty} \xi_k + \xi_n.$$

Note that X_n is unbounded, but it has all bounded moments. It is easy to see that $EA_\infty = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} < \infty$, but $EA_\infty^2 \geq \sum_{n=1}^{\infty} \frac{n^2+1}{n^3} = \infty$.

Let $\alpha_n = \frac{1}{n^2}$ and $\beta_n = 1$. One can check that a random variable $\eta = \sum_{n=1}^{\infty} \xi_n$ has gamma-distribution with parameters $\alpha = \frac{\pi^2}{6}$ and $\beta = 1$. Therefore, for any $k \geq 1$ $EA_\infty^k = (\frac{\pi^2}{6} + k - 1) \cdot \ldots \cdot \frac{\pi^2}{6} < \infty$, i.e., the increasing process has all bounded moments.

Example 2: Consider a standard Wiener process $\{w_t, \mathfrak{F}_t, t \in R_+\}$. Let $\zeta_t = e^{aw_t - bt}$, where $a, b \in R$, $b > 0$. Then $E\zeta_t^k = e^{(\frac{k^2a^2}{2} - kb)t} < \infty$, $t > 0$, $k \geq 1$, i.e., ζ_t has all moments. By Itô's formula

$$\zeta_t^k = 1 + ka\int_0^t \zeta_s^k dw_s + \left(\frac{k^2a^2}{2} - kb\right)\int_0^t \zeta_s^k ds. \tag{5}$$

If $k \leq \frac{2b}{a^2}$, then ζ_t^k is a potential; if $k = \frac{2b}{a^2}$, then ζ_t^k is a martingale; and if $k > \frac{2b}{a^2}$, then ζ_t^k is a submartingale. Suppose $\frac{2b}{a^2} > 1$. Then ζ_t is a potential with $A_\infty = (-\frac{a^2}{2} + b)\int_0^\infty \zeta_s ds$. Suppose there exists such $n \in N$ that $2^{n-1} < \frac{b}{a^2} < 2^n$. Consider the following cases:

(i) Let $k = 2^n$. Then $k < \frac{2b}{a^2}$, ζ_t^i, $i \leq k$ is a potential, $\zeta_\infty^i = 0$, $i \leq k$, and from (5) (the constants depend on i and r)

$$E\left(\int_0^\infty \zeta_s^i ds\right)^r \leq c_1 + c_2 E\left(\int_0^\infty \zeta_s^i dw_s\right)^r \tag{6}$$

for every $r \geq 1$. Moreover, by Burkholder's inequalities

$$c_3 E\left(\int_0^\infty \zeta_s^i ds\right)^r \leq E\left(\int_0^\infty \zeta_s^{i/2} dw_s\right)^{2r} \leq c_4 E\left(\int_0^\infty \zeta_s^i ds\right)^r. \tag{7}$$

From (6) and the second part of (7) we obtain

$$E\left(\int_0^\infty \zeta_s ds\right)^k \leq c_5 + c_6 E\left(\int_0^\infty \zeta_s dw_s\right)^k \leq c_5 + c_7 E\left(\int_0^\infty \zeta_s^2 ds\right)^{k/2} \leq \ldots$$

$$\leq c_8 + c_9 E\left(\int_0^\infty \zeta_s^{2^n} ds\right) = c_8 + c_9 E\int_0^\infty e^{kaw_s - kbs} ds$$

$$= c_8 + c_9 \int_0^{\infty} e^{(\frac{k^2a^2}{2} - kb)s} ds < \infty.$$

Thus, $EA_{\infty}^k < \infty$.

(*ii*) Let $k = 2^{n+1}$. Then ζ_t^i, $i \geq k$ is a submartingale,

$$E\zeta_t^i = e^{(\frac{i^2a^2}{2} - ib)t} \to \infty,\ t \to \infty;\ E\left(\int_0^{\infty} \zeta_s^i ds\right) = \infty.$$

Hence, from the first part of (7) $E(\int_0^{\infty} \zeta_s^{k/2} dw_s)^2 = \infty$. Further, from the potential property and from (5) $E(\int_0^{\infty} \zeta_s^{k/2} ds)^2 = \infty$. Again, from the first part of (7) $E(\int_0^{\infty} \zeta_s^{k/4} dw_s)^4 = \infty$, and from (5) $E(\int_0^{\infty} \zeta_s^{k/4} ds)^4 = \infty$. Continuing in the same way, we obtain that $E(\int_0^{\infty} \zeta_s ds)^k = \infty$; i.e., $EA_{\infty}^k = \infty$.

From (*i*) and (*ii*) we have that $EA_{\infty}^i < \infty$ for $i < 2^{n-1} < \frac{b}{a^2}$ and $EA_{\infty}^i = \infty$ for $i \geq 2^n \geq \frac{b}{a^2}$.

Now we will prove a sufficient condition for the existence of moments EA_{∞}^k, $k \geq 1$ in the case when a potential $X_t = E(A_{\infty} - A_t / \mathfrak{F}_t)$ and associated increasing process A_t are continuous. Denote a martingale $M_t = E(A_{\infty} / \mathfrak{F}_t)$, $\langle M \rangle_t$ is its quadratic characteristic.

Theorem 2: *Let X_t be a continuous one-parameter potential, associated with continuous increasing process A_t. Assume that for every $k \geq 1$, $E\int_0^{\infty} X_s^k d\langle M \rangle_s < \infty$. Then*

$$EA_{\infty}^k < \infty,\ \ k \geq 1.$$

Proof: By Itô's formula for every $k \geq 2$

$$X_t^k = k\int_0^t X_s^{k-1} dX_s + \frac{k(k-1)}{2}\int_0^t X_s^{k-2} d\langle M \rangle_s.$$

Therefore,

$$\sup_t X_t^k \leq k \sup_t \left| \int_0^t X_s^{k-1} dM_s \right| + \frac{k(k-1)}{2}\int_0^{\infty} X_s^{k-2} d\langle M \rangle_s.$$

If the condition of the theorem holds, then by Gundy's inequality

$$E\sup_t X_t^k \leq kE\left(\int_0^{\infty} X_s^{2k-2} d\langle M \rangle_s\right)^{1/2} + \frac{k(k-1)}{2} E\int_0^{\infty} X_s^{k-2} d\langle M \rangle_s < \infty,$$

hence, by Theorem 1 we have the proof. □

Remark 2: Suppose $\langle M \rangle_t = \int_0^t a_s^2 ds$, where a_s is a nonrandom function with $\int_0^{\infty} a_s^2 ds < \infty$. Then $E\int_0^{\infty} X_s^k d\langle M \rangle_s = \int_0^{\infty} EX_s^k a_s^2 ds$. Since $EX_s^k \to 0$ when $s \to \infty$, then a continuous function EX_s^k is bounded and, therefore, $E\int_0^\infty X_s^k d\langle M\rangle_s < \infty$.

3. Two-Parameter Potentials Associated with Increasing Processes

We assume in this section that $\{X_t, \mathcal{F}_t, t \in R_+^2\}$ is a two-parameter, nonnegative, bounded potential associated with increasing predictable integrable process A_t; i.e., (1) holds.

Theorem 3: *If $X_t \leq c$, then for every $p \geq 1$*

$$E(E(A_\infty - A_t/\mathcal{F}_t))^p \leq 2^{p+1}c^p p!.$$

Proof: Fix $t_2 \geq 0$. Consider a process

$$Z_{t_1}(t_2) = E(A_{t_1\infty} - A_t/\mathcal{F}_t), \quad t_1 \geq 0. \tag{8}$$

It is obvious that $Z_{t_1}(t_2)$ is 1-predictable with respect to the filtration $\{\mathcal{F}_t^1, t_1 \geq 0\}$. Since $A_{t_1\infty} - A_t$ is F_t^1-measurable and from the condition (F4) [2] we have

$$Z_{t_1}(t_2) = E(E(A_{t_1\infty} - A_t/\mathcal{F}_t^1)/\mathcal{F}_t^2) = E(A_{t_1\infty} - A_t/\mathcal{F}_t^2).$$

Therefore, $Z_{t_1}(t_2)$ is predictable in t_1. Moreover, if $s_1 < t_1$, then

$$Z_{t_1}(t_2) - Z_{s_1}(t_2) = E(A_{t_1\infty} - A_t - A_{s_1\infty} + A_{s_1t_2}/\mathcal{F}_t^2) \geq 0.$$

Thus, $Z_{t_1}(t_2)$ is a predictable increasing integrable process as a one-parameter process with a parameter t_1. Denote $X_{t_1}(t_2)$ a potential associated with $Z_{t_1}(t_2)$. Then from (8)

$$\begin{aligned} X_{t_1}(t_2) &= E(Z_\infty(t_2) - Z_{t_1}(t_2)/\mathcal{F}_t) \\ &= E(A_\infty - A_{\infty t_2} - A_{t_1\infty} + A_t/\mathcal{F}_t) = X_t. \end{aligned} \tag{9}$$

Consequently, $X_{t_1}(t_2)$ is bounded; and we can apply the Garsia inequality [3] that leads to the estimate $E(Z_\infty(t_2))^p \leq c^p p!$ or

$$E(E(A_\infty - A_{\infty t_2}/\mathcal{F}_t^2))^p \leq c^p p!, \quad p \geq 1.$$

It follows easily from the last inequality and from (F4) [2] that for every $t \in R_+^2$

$$E(E(A_{t_1\infty} - A_t/\mathcal{F}_t))^p \leq c^p p!,$$

and by a symmetric argument

$$E(E(A_{\infty t_2} - A_t/\mathcal{F}_t))^p \leq c^p p!.$$

Finally, we obtain

$$E(E(A_\infty - A_t/\mathcal{F}_t))^p \leq 2^p E(E(A_\infty - A_{\infty t_2}/\mathcal{F}_t))^p$$

$$+2^pE(E(A_{\infty t_2}-A_t/\mathfrak{F}_t))^p \le 2^{p+1}c^pp!.$$

The theorem is proved. □

The next statement follows directly from Theorem 1, (8) and (9).

Corollary 1: *The following conditions are equivalent:*

(a) $EA^k_\infty \le c_k$ *for every* $k \ge 1$,

(b) $E\sup\limits_{t_2}(E(A_{\infty t_2}-A_t/\mathfrak{F}_t))^k \le d_k$ *for every* $k \ge 1$.

Let A_t be an integrable predictable increasing process. Denote

$$\tau_n = \inf\{t_1 \in R_+ : A_{t_1t_1} \ge n\}$$

and

$$A^n_t = \int\limits_{[0,t]} \text{III}_{[0,\tau_n[^2}(s)dA_s. \tag{10}$$

Note that A^n_t is a predictable process and has all bounded moments. In fact, according to [7], $A^n_t = (\int\limits_{[0,t]} I_{[0,\tau_n[^2}(s)dA_s)^\pi$, where we denote $(\,)^\pi$ the dual predictable projection [7] of the corresponding increasing predictable process. Further, again according to [7], for every $p \ge 1$ there exist constants c_p such that

$$E(A^n_t)^p \le c_pE\left(\int\limits_{[0,t]} I_{[0,\tau_n[^2}(s)dA_s\right)^p \le c_pn^p.$$

Suppose now that A_t and B_t are two different increasing predictable processes; A^n_t is defined by (10). Denote $\sigma_n = \inf\{t_1 \in R_+ : B_{t_1t_1} \ge n\}$ and

$$B^n_t = \int\limits_{[0,t]} \text{III}_{[0,\sigma_n[^2}(s)dB_s. \tag{11}$$

The next auxiliary result will be used for the proof of the main theorem.

Lemma 1: *Let* A_t, B_t *be two predictable increasing processes,* A^n_t *is defined by* (10), B^n_t *is defined by* (11). *Then*

$$\lim_{n\to\infty} E\int\limits_{[0,t]} B^n_s dA^n_s = E\int\limits_{[0,t]} B_s dA_s.$$

Proof: Using (10), (11), and the definition of predictable projection [7], we can write

$$E\int\limits_{[0,t]} B^n_s dA^n_s = E\int\limits_{[0,t]}\left(\int\limits_{[0,s]} \text{III}_{[0,\sigma_n[^2}(u)dB_u\right)\text{III}_{[0,\tau_n[^2}(s)dA_s$$

$$= E\int\limits_{[0,t]}\left(\int\limits_{[0,s]} \text{III}_{[0,\sigma_n[^2}(u)dB_u\right)I_{[0,\tau_n[^2}(s)dA_s.$$

Note that $\inf\limits_{s \le t} I_{[0,\tau_n]^2}(s) \to 1$ a.s. Thus, it is sufficient to prove that

$$\sup_{s \leq t} | B_s - \int_{[0,s]} \text{III}_{[0,\sigma_n[^2}(u) dB_u | = \int_{[0,t]} \text{III}_{R^2_+ \setminus [0,\sigma_n[^2}(s) dB_s \to 0 \text{ a.s.}$$

The sequence $\{I_{R^2_+ \setminus [0,\sigma_n[^2}(s), n \geq 1\}$ is decreasing; hence, it is sufficient to prove that the last integral converges to zero in probability. But

$$P\left\{ \int_{[0,t]} \text{III}_{R^2_+ \setminus [0,\sigma_n[^2}(s) dB_s > \varepsilon \right\}$$

$$\leq \frac{1}{\varepsilon} E \int_{[0,t]} I_{R^2_+ \setminus [0,\sigma_n[^2}(s) dB_s \to 0, \quad n \to \infty,$$

and we obtain the proof. □

Further, we assume that σ-fields $\mathfrak{F}^1_t$ and $\mathfrak{F}^2_t$ are continuous on the right.

Lemma 2: *Suppose X_t is a bounded potential associated with increasing predictable process A_t, that has all finite moments, and B_t is an increasing predictable process that has all moments. Then the following formula holds*

$$E \int_{[0,\infty)} X_t dB_t + E \int_{[0,\infty)} (B_{t+-} + B_{t-+} - B_{t-}) dA_t$$

$$+ E \int_{[0,\infty)} d_1 X_t d_2 B_t + E \int_{[0,\infty)} d_1 B_t d_2 X_t = 0.$$

Proof: Denote $M^1_t = -E(A_{\infty t_2} / \mathfrak{F}_t)$, $M^2_t = -E(A_{t_1 \infty} / \mathfrak{F}_t)$, $M_t = E(A_\infty / \mathfrak{F}_t)$. Then M^1_t is 1-martingale, M^2_t is 2-martingale, M_t is martingale, and from (1) $X_t = M^1_t + M^2_t + M_t + A_t$. The process B_t is 1-predictable, hence,

$$EM^1_t B_t = E \int_0^{t_1} M_{s_1^- t_2} d_1 B_{s_1 t_2} = E \lim_{|\lambda^n_t| \to 0} \sum_{i=0}^{n-1} M^1_{t^i_1 t_2} \Delta^1_{t^i_1 t_2} B \text{ a.s.,}$$

where $M_{s_1^- t_2} = M_{(s_1, t_2)-+}$, and $0 = t^0_1 < t^1_1 < \ldots < t^n_1 = t_1$ is the partition of $[0, t_1]$. We can write

$$\sum_{i=0}^{n-1} M^1_{t^i_1 t_2} \Delta^1_{t^i_1 t_2} B = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} M^1_{ij} \Box_{ij-1} B + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Delta^2_{ij-1} M^1 \Delta^1_{ij-1} B.$$

(Here and further we replace index $t^i_1 t^j_2$ by index ij.) Under the assumption of continuity the σ-fields $\mathfrak{F}^1_t$ and $\mathfrak{F}^2_t$ on the right, the process M^1_t satisfies the conditions of Theorems 3.4 and 4.2 [8]. Therefore, M^1_t has a modification with limits in Q^i_t, $i = 2, 3, 4$ which is continuous in Q^1_t. For such modification

$$\lim_{|\lambda^n_t| \to 0} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} M^1_{ij} \Box_{ij-1} B = \int_{[0,t]} M^1_{s-+} dB_s.$$

Consequently, there exists

$$\lim_{|\lambda_t^n| \to 0} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Delta^2_{ij-1} M^1 \Delta^1_{ij-1} B = \int_{[0,t]} d_1 B_s d_2 M^1_s.$$

Furthermore, $\sum_{i=0}^{n-1} M^1_{t_1^i t_2} \Delta^1_{t_1^i t_2} B \leq B_t \sup_t E(A_\infty / \mathfrak{F}_t)$; and the right-hand side of this inequality has all moments. Hence $\sum_{i=0}^{n-1} M^1_{t_1^i t_2} \Delta^1_{t_1^i t_2} B$ is uniformly integrable. Similarly, we can consider the sums $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} M^1_{ij} \square_{ij-1} B$ and obtain their uniform integrability. Thus, $\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Delta^2_{ij-1} M^1 \Delta^1_{ij-1} B$ is uniformly integrable and

$$EM^1_t B_t = E \int_{[0,t]} M^1_{s-+} dB_s + E \int_{[0,t]} d_1 B_s d_2 M^1_s. \tag{12}$$

In the same way we obtain that

$$EM^2_t B_t = E \int_{[0,t]} M^2_{s+-} dB_s + E \int_{[0,t]} d_1 M^2_s d_2 B_s. \tag{13}$$

Also,

$$EM_t B_t = E \int_{[0,t]} M_s dB_s. \tag{14}$$

It is easy to check that the formula of integration by parts for two increasing processes A_t and B_t holds

$$EA_t B_t = E \int_{[0,t]} A_s dB_s + E \int_{[0,t]} (B_{s+-} + B_{s-+} - B_{s-}) dA_s$$

$$+ E \int_{[0,t]} d_1 A_s d_2 B_s + E \int_{[0,t]} d_1 B_s d_2 A_s. \tag{15}$$

Note that

$$E \int_{[0,t]} (M^1_s - M^1_{s-+}) dB_s = E \int_{[0,t]} \Pi^1 (M^1_s - M^1_{s-+}) dB_s$$

$$= E \int_{[0,t]} (M^1_{s-+} - M^1_{s-+}) dB_s = 0.$$

Therefore,

$$E \int_{[0,t]} M^1_s dB_s = E \int_{[0,t]} M^1_{s-+} dB_s. \tag{16}$$

Similarly,

$$E\int_{[0,t]} M_s^2 dB_s = E\int_{[0,t]} M_{s+-}^2 dB_s. \tag{17}$$

Adding (12)-(15) and using (16)-(17), we have

$$EX_tB_t = E\int_{[0,t]} X_s dB_s + E\int_{[0,t]} (B_{s+-} + B_{s-+} - B_{s-})dA_s$$

$$+E\int_{[0,t]} d_1X_s d_2B_s + E\int_{[0,t]} d_1B_s d_2X_s$$

Letting $t \to \infty$ in the last equality and using the definition of the potential, we obtain the proof. □

Consider the following sequence of predictable increasing processes

$$B_t^{(0)} = 1,\ B_t^{(1)} = A_t, B_t^{(2)} = \int_{[0,t]} B_s^{(1)} dA_s, \ldots, B_t^{(k)} = \int_{[0,t]} B_s^{(k-1)} dA_s, \ldots$$

The next theorem is the main result of this section.

Theorem 4: *Let $X_t \le c$. Then for any $k \ge 1$ $EB_\infty^{(k)} < \infty$.*

Proof: We use the induction in k.

(1) It is obvious that $EB_\infty^{(1)} = EA_\infty < \infty$.

(2) Suppose that $EB_\infty^{(k)} < \infty$ for some k, and prove that $EB_\infty^{(k+1)} < \infty$. Denote $\sigma_{n,k} = \inf\{t_1 \in R_+ : B_{t_1 t_1}^{(k)} \ge n\}$ and

$$B_t^{(k,n)} = \int_{[0,t]} \Pi I_{[0,\sigma_{n,k}[^2}(s) dB_s^{(k)},\quad k \ge 1,\quad n \ge 1.$$

Let $A_t^{(n)}$ be defined by (10), $X_t^{(n)}$ be a potential associated with $A_t^{(n)}$. If we prove that under inductive hypothesis $EB_\infty^{(k+1,n)} \le C < \infty$, then, letting $n \to \infty$ and using Lemma 1, we obtain the proof.

Thus, prove that $EB_\infty^{(k+1,n)} < \infty$. For any process B_t use the representation $B_t = B_{t-} + \Delta^1 B_t + \Delta^2 B_t - \Box B_t \le C$. Since

$$EB_\infty^{(k+1,n)} = E\int_{[0,\infty)} B_t^{(k,n)} dA_t^{(n)} = E\int_{[0,\infty)} B_{t-}^{(k,n)} dA_t^{(n)}$$

$$+E\int_{[0,\infty)} \Delta^1 B_t^{(k,n)} dA_t^{(n)} + E\int_{[0,\infty)} \Delta^2 B_t^{(k,n)} dA_t^{(n)} - \int_{[0,\infty)} \Box B_t^{(k,n)} dA_t^{(n)},$$

it is sufficient to prove finiteness of each term in the right-hand side of the last equality.

(a) By Lemma 2 we can write

$$EX_\infty^{(n)} B_\infty^{(k,n)} = E\int_{[0,\infty)} X_t^{(n)} dB_t^{(k,n)}$$

$$+E\int_{[0,\infty)}(B^{(,n)}_{t+-}+B^{(k,n)}_{t-+}-B^{(k,n)}_{t-})dA^{(n)}_t$$

$$+E\int_{[0,\infty)}d_1X^{(n)}_t d_2B^{(k,n)}_t+E\int_{[0,\infty)}d_1B^{(k,n)}_t d_2X^{(n)}_t, \tag{18}$$

where $A^{(n)}_t$ is defined by (10), and $X^{(n)}_t$ is a potential associated with $A^{(n)}_t$. Taking (1) into account, we obtain that

$$E\int_{[0,\infty)}d_1X^{(n)}_t d_2B^{(k,n)}_t=E\int_{[0,\infty)}d_1(A^{(n)}_t-A^{(n)}_{t_1\infty})d_2B^{(k,n)}_t.$$

By the definition of the stochastic integral of the second kind

$$\int_{[0,\infty)}d_1(A^{(n)}_{t_1\infty}-A^{(n)}_t)d_2B^{(k,n)}_t=\lim_{|\lambda^n_t|\to 0}\sum_{i,j=0}^{n-1}\Delta^1(A^{(n)}_{i\infty}-A^{(n)}_{ij})\Delta^2_{ij}B^{(k,n)}$$

$$=\lim_{|\lambda^n_t|\to 0}\sum_{i,j=0}^{n-1}\sum_{l=j}^{n-1}\square_{il}A^{(n)}\Delta^2_{ij}B^{(k,n)}=\lim_{|\lambda^n_t|\to 0}\sum_{i,l=0}^{n-1}B^{(k,n)}_{il+1}\square_{il}A^{(n)}$$

$$=\int_{[0,\infty)}B^{(k,n)}_{t-+}dA^{(n)}_t.$$

Hence,

$$E\int_{[0,\infty)}d_1X^{(n)}_t d_2B^{(k,n)}_t=-E\int_{[0,\infty)}B^{(k,n)}_{t-+}dA^{(n)}_t. \tag{19}$$

Similarly,

$$E\int_{[0,\infty)}d_1B^{(k,n)}_t d_2X^{(n)}_t=-E\int_{[0,\infty)}B^{(k,n)}_{t+-}dA^{(n)}_t. \tag{20}$$

Substituting (19) and (20) into (18), we obtain

$$EX^{(n)}_\infty B^{(k,n)}_\infty=E\int_{[0,\infty)}X^{(n)}_t dB^{(k,n)}_t$$

$$+E\int_{[0,\infty)}(B^{(k,n)}_{t+-}+B^{(k,n)}_{t-+}-B^{(k,n)}_{t-})dA^{(n)}_t-E\int_{[0,\infty)}B^{(k,n)}_{t+-}dA^{(n)}_t$$

$$-E\int_{[0,\infty)}B^{(k,n)}_{t-+}dA^{(n)}_t=E\int_{[0,\infty)}X^{(n)}_t dB^{(k,n)}_t-E\int_{[0,\infty)}B^{(k,n)}_{t-}dA^{(n)}_t. \tag{21}$$

Since the left-hand side and the first term of the right-hand side of (21) are finite, we have by virtue of boundedness of X and inductive hypothesis, that

$$E\int_{[0,\infty)}B^{(k,n)}_{t-}dA^{(n)}_t<\infty. \tag{22}$$

(b) First, prove that

$$E\int_{[0,\infty)} \square X_t^{(n)} dB_t^{(k,n)} = E\int_{[0,\infty)} \square A_t^{(n)} dB_t^{(k,n)}.$$

Consider the following martingales

$$M_t = X_t - A_t + E(A_{t_1\infty} - A_{\infty t_2}/\mathcal{F}_t) = E(A_\infty/\mathcal{F}_t)$$

and

$$M_t^{(n)} = X_t^{(n)} - A_t^{(n)} + E(A_{t_1\infty}^{(n)} - A_{\infty t_2}^{(n)}/\mathcal{F}_t) = E(A_\infty^{(n)}/\mathcal{F}_t),$$

where $M_\infty = A_\infty$ and $M_\infty^{(n)} = A_\infty^{(n)}$. Therefore, from [4]

$$EA_\infty^{(n)} B_\infty^{(k,n)} = E\int_{[0,\infty)} E(A_\infty^{(n)}/\mathcal{F}_t) dB_t^{(k,n)}$$

$$= E\int_{[0,\infty)} (X_t^{(n)} - A_t^{(n)} + E(A_{t_1\infty}^{(n)} + A_{\infty t_2}^{(n)}/\mathcal{F}_t)) dB_t^{(k,n)}.$$

Note that $M_t^{(n)}$ is a nonnegative measurable process, hence, by [7]

$$\Pi^1 M_t^{(n)} = E(M_t^{(n)}/\mathcal{F}_{t-}^1) = E(E(A_\infty^{(n)}/\mathcal{F}_t)/\mathcal{F}_{t-}^1)$$

$$= E(A_\infty^{(n)}/\mathcal{F}_{t-+}) = M_{t-+}^{(n)}.$$

Similarly, $\Pi^2 M_t^{(n)} = M_{t+-}^{(n)}$ and $\Pi M_t^{(n)} = M_{t-}^{(n)}$. From the definition of the predictable rejection we have

$$E\int_{[0,\infty)} M_t^{(n)} dB_t^{(k,n)} = E\int_{[0,\infty)} M_{t-+}^{(n)} dB_t^{(k,n)}$$

$$= E\int_{[0,\infty)} M_{t+-}^{(n)} dB_t^{(k,n)} = E\int_{[0,\infty)} M_{t-}^{(n)} dB_t^{(k,n)}.$$

Thus, we can write $EA_\infty^{(n)} B_\infty^{(k,n)}$ in the four following ways:

$$EA_\infty^{(n)} B_\infty^{(k,n)} = E\int_{[0,\infty)} (X_t^{(n)} - A_t^{(n)} + E(A_{t_1\infty}^{(n)} + A_{\infty t_2}^{(n)}/\mathcal{F}_t)) dB_t^{(k,n)}, \tag{23}$$

$$EA_\infty^{(n)} B_\infty^{(k,n)} = E\int_{[0,\infty)} (X_{t+-}^{(n)} - A_{t+-}^{(n)} + E(A_{t_1\infty}^{(n)} + A_{\infty t_2^-}^{(n)}/\mathcal{F}_{t+-})) dB_t^{(k,n)}, \tag{24}$$

$$EA_\infty^{(n)} B_\infty^{(k,n)} = E\int_{[0,\infty)} (X_{t-+}^{(n)} - A_{t-+}^{(n)} + E(A_{t_1^-\infty}^{(n)} + A_{\infty t_2}^{(n)}/\mathcal{F}_{t-+})) dB_t^{(k,n)}, \tag{25}$$

$$EA_\infty^{(n)}B_\infty^{(k,n)} = E\int\limits_{[0,\infty)} (X_{t-}^{(n)} - A_{t-}^{(n)} + E(A_{t_1^-\infty}^{(n)} + A_{\infty t_2^-}^{(n)} / \mathfrak{F}_{t-}))dB_t^{(k,n)}. \qquad (26)$$

Subtracting (24) and (25) from (23) and adding (26), we obtain, using predictability of $B_t^{(k,n)}$ and properties of conditional expectation, that

$$E\int\limits_{[0,\infty)} (\square X_t^{(n)} - \square A_t^{(n)})dB_t^{(k,n)} = 0$$

or

$$E\int\limits_{[0,\infty)} \square X_t^{(n)} dB_t^{(k,n)} = E\int\limits_{[0,\infty)} \square A_t^{(n)} dB_t^{(k,n)}. \qquad (27)$$

The process $A_t^{(n)}$ admits the representation $A_t^{(n)} = A_t^{(n)^c} + A_t^{(n)^{cd}} + A_t^{(n)^{dc}} + A_t^{(n)^d}$, where $A_t^{(n)^c}$ is continuous, $A_t^{(n)^{cd}}$ is continuous in t_1 and purely discontinuous in t_2, $A_t^{(n)^{dc}}$ is purely discontinuous in t_1 and continuous in t_2, and, finally, $A_t^{(n)^d}$ is purely discontinuous. Therefore, only the integral $E\int\limits_{[0,\infty)} \square B_t^{(k,n)} dA_t^{(n)^d}$ is nonzero, and

$$E\int\limits_{[0,\infty)} \square B_t^{(k,n)} dA_t^{(n)} = \lim_{|\lambda_t^n| \to 0} E\sum_{i,j=0}^{n-1} \square_{ij} B^{(k,n)} \square_{ij} A^{(n)}$$

$$= E\int\limits_{[0,\infty)} \square A_t^{(n)} dB_t^{(k,n)}. \qquad (28)$$

From (27) and (28) we have

$$E\int\limits_{[0,\infty)} \square B_t^{(k,n)} dA_t^{(n)} = E\int\limits_{[0,\infty)} \square X_t^{(n)} dB_t^{(k,n)} < 4cEB_\infty^{(k)} < \infty \qquad (29)$$

by the inductive hypothesis.

(c) Subtracting (25) from (23), we obtain

$$E\int\limits_{[0,\infty)} (\Delta^1 X_t^{(n)} - \Delta^1 A_t^{(n)} + \Delta^1 A_{t_1\infty}^{(n)})dB_t^{(k,n)} = 0$$

or

$$E\int\limits_{[0,\infty)} \Delta^1 X_t^{(n)} dB_t^{(k,n)} = E\int\limits_{[0,\infty)} (\Delta^1 A_t^{(n)} - \Delta^1 A_{t_1\infty}^{(n)})dB_t^{(k,n)}. \qquad (30)$$

Consider the integral in the right-hand side of (30)

$$\int\limits_{[0,\infty)} (\Delta^1 A_t^{(n)} - \Delta^1 A_{t_1\infty}^{(n)})dB_t^{(k,n)}$$

$$= \lim_{|\lambda_t^n| \to 0} \sum_{i,j=0}^{n} \Delta^1 (A_{ij}^{(n)} - A_{i\infty}^{(n)}) \square_{ij} B^{(k,n)}$$

$$= -\lim_{|\lambda_t^n|\to 0} \sum_{i,j=0}^{n-1} \sum_{l=j}^{n-1} \square_{il} A^{(n)} \square_{ij} B^{(k,n)}$$

$$= -\lim_{|\lambda_t^n|\to 0} \sum_{i,l=0}^{n-1} \square_{il} A^{(n)} \Delta^1_{il+1} B^{(k,n)} = -\int_{[0,\infty)} \Delta^1 B_t^{(k,n)} dA_t^{(n)}. \tag{31}$$

From (30) and (31) we have

$$E\int_{[0,\infty)} \Delta^1 B_t^{(k,n)} dA_t^{(n)} = -E\int_{[0,\infty)} \Delta^1 X_t^{(n)} dB_t^{(k,n)}.$$

An absolute value of the right-hand side of the last equality does not exceed $2cEB_\infty^{(k)} < \infty$, therefore,

$$E\int_{[0,\infty)} \Delta^1 B_t^{(k,n)} dA_t^{(n)} < \infty. \tag{32}$$

(*d*) Similarly, using (24) instead of (25), one can prove that

$$E\int_{[0,\infty)} \Delta^2 B_t^{(k,n)} dA_t^{(n)} < \infty. \tag{33}$$

Now, from (*a*)-(*d*) we get

$$EB_\infty^{(k+1,n)} = E\int_{[0,\infty)} B_t^{(k,n)} dA_t^{(n)} < 8cEB_\infty^{(k)} < \infty.$$

Letting $n\to\infty$ and using Lemma 1, we obtain the proof of the theorem. □

In Theorem 4 we have proved that the boundedness of the potential implies the integrability of the process $B_\infty^{(k)} \le A_\infty^k$.

4. Application to the Local Time

The process $\{Y_t, t \in R^2_+\}$ is said to have the *local time* $L_t(x)$ if $L_t(x)$ is an increasing process such that for all $t \in R^2_+$ and for all Borel locally integrable functions f on R

$$\int_R L_t(x) f(x) dx = \int_{[0,t]} f(Y_s) ds \quad \text{a.s.} \tag{34}$$

A weak predictable set [4] $D(\omega) \subset R^2_+$ is called a *stopping set* if $[0,t] \subset D(\omega)$ when $t \in D(\omega)$, and the event $\{t \in D(\omega)\} \in \mathcal{F}_t$. We say that the process Y belongs to a class $\mathcal{S}$ *locally* if there exists a sequence $\{D^n(\omega), n \ge 1\}$ of stopping sets and a sequence of processes $\{Y^n, n \ge 1\} \subset \mathcal{S}$ such that $D^n \subset D^{n+1}$ for every $n \ge 1$, $\bigcup_{n\ge 1} D^n = R^2_+$, and $(Y_t - Y_t^n) I_{D^n}(t) = 0$. Let X_t be a local purely discontinuous strong martingale that has local characteristics (a_s, ν_s). This means that

- a_s and ν_s are $\mathcal{F}_s$-adapted,
- for each s and ω ν_s is a σ-finite measure on $R^2 \backslash \Gamma$,
- for all Borel B such that $\bar{B} \in R^2 \backslash \Gamma$ is compact, $\sum_{s \le t} I_B(\square X_s) - \int_{[0,t]} \nu_s(B) ds$ is

a local strong martingale,

- $X_t - \sum_{s \leq t} \square X_s I\{|\square X_s| > 1\} - \int_{[0,t]} a_s ds$ is a local strong martingale,

where the process $\int_{[0,t]} \nu_s(B)ds$ is a weak predictable projection [4] of the process $\sum_{s \leq t} I_B(\square X_s)$. X_t purely discontinuous means that X_t is a uniform limit of

$$X_t^{\varepsilon} = \int_{[0,t]} a_s ds + \sum_{s \leq t} \square X_s I\{|\square X_s| > \varepsilon\}$$

$$- \int_{[0,t]} \int_{\varepsilon \leq |h| \leq 1} h\nu_s(dh)ds \quad \text{as } \varepsilon \to 0.$$

If $1 < \alpha < 2$, let $\theta_\alpha(dh) = \xi_\alpha |h|^{-(1+\alpha)}dh$ be the Levy measure for a stable symmetric process $\{Z_t, t \in R_+^2\}$ of index α, where $\xi_\alpha > 0$ is the constant such that $Eexp(isZ_t) = exp(-t_1 t_2 |s|^\alpha)$.

We assume that the following conditions hold:

(A) *X_t is a local purely discontinuous strong martingale such that*

(a) *for some K_1 $\sup_s |\int_R h^2 \wedge 1 \nu_s(dh)| \leq K_1$ a.s.,*

(b) *for some $1 < \alpha < 2$ $\nu_s(dh) = \theta_\alpha(dh)$, if $|h| > 1$,*

(c) *for some $1 < \alpha < 2$, $K_2 > 1$, any $0 < \gamma < \frac{\alpha - 1}{2}$*

$$\sup_s \int_{-1}^{1} |h|^{\alpha - \gamma} |\nu_s - \theta_\alpha|(dh) \leq K_2 \text{ a.s.},$$

(d) *X_t does not have more than one jump along any line parallel to coordinate axes.*

Example 3: Let N_t, $t \in R_+^2$ be a Poisson field with parameters $EN_t = EN_t^2 = \lambda t_1 t_2$ (see, for example, [6]). Then $N_t - t_1 t_2$ is a purely discontinuous local strong martingale with local characteristics $\nu_s(B) = I_B(1)$ and $a_s = 1$. Consider local purely discontinuous strong martingale $\widetilde{N}_t$ with local characteristics $\widetilde{\nu}_s(B) = I_B(1) + \theta_\alpha(B)$ for all Borel B such that $\widetilde{B} \in R^2 \backslash \Gamma$ is compact, where $\theta_\alpha(dh)$ is the Levy measure described above. Then $\widetilde{N}_t$ satisfies conditions (A).

Let $Q_t(\omega, \cdot)$ be a regular conditional probability distribution for $\mathfrak{F}_t$. That is for every $A \in \mathfrak{F}$ $Q_t(\cdot, A)$ is $\mathfrak{F}_t$-measurable, for every ω $Q_t(\omega, \cdot)$ is a probability measure on $\mathfrak{F}$, and $Q_t(\cdot, A) = P(A/\mathfrak{F}_t)$ a.s. for every $A \in \mathfrak{F}$. Q_t exists since $\mathfrak{F}_t$ is the completion of a countably generated σ-fields and X_t is real-valued. Denote $Q_t Y(\omega) = \int Y(\omega')Q_t(\omega, d\omega') = E(Y(\omega)/\mathfrak{F}_t)$.

Fix $t_0 \in R_+^2$, $\lambda > 0$. It was proved in [9] that there exists a nonnegative Borel in x bounded function $V_{t_0}(\lambda, x)(\omega)$ such that for every Borel $A \in R$

$$Q_{t_0}\left(\int_{[0,\infty)} e^{-\lambda t_1 - \lambda t_2} I_A(X_{t+t_0})dt \right)(\omega) = \int_A V_{t_0}(\lambda, x)(\omega)dx.$$

Further, the process $\{U_t(\lambda, x) = e^{-\lambda t_1 - \lambda t_2} V_t(\lambda, x), \mathfrak{F}_t, t \in Q_+^2\}$ for a.e. x is a weak submartingale, supermartingale, and a.s. for any $s \in Q_+^2$ and for any sequence $\{t_n, n \geq 1\} \subset Q_+^2$ such that $t_n \downarrow s$, $n \to \infty$, for a.e. x $EU_{t_n}(\lambda, x) \to EU_s(\lambda, x)$, $n \to \infty$. Here we denote $Q_+^2 = Q_+ \times Q_+$, $Q_+ Q \cap [0, \infty)$.

Under the assumption

(*B*) *for a.e.* x *the process* $(U_t(\lambda, x), \mathfrak{F}_t, t \in R^2_+)$ *is a weak submartingale, supermartingale and for every* $s \in R^2_+$ $EU_t(\lambda, x) \to EU_s(\lambda, x)$ *if* $t \to s$, $t > s$, $t \in R^2_+$.

The process $U_t(\lambda, x)$ for a.e. x allows the representation [9]

$$U_t(\lambda, x) = M_t(\lambda, x) + L_t(\lambda, x),$$

where $M_t(\lambda, x)$ is a weak martingale, and $L_t(\lambda, x)$ is an increasing predictable process. Note that $U_t(\lambda, x)$ is a bounded potential associated with $L_t(\lambda, x)$.

Fix $\lambda > 0$ and let

$$L_t(x) = \int_{[0,t]} e^{\lambda s_1 + \lambda s_2} dL_s(\lambda, x). \tag{35}$$

Then from [9] $L_t(x)$ is a local time for X_t, and it does not depend on λ. Consider the sequences of increasing predictable processes

$$C_t^{(0)}(\lambda, x) = 1,\ C_t^{(1)}(\lambda, x) = L_t(\lambda, x),\ C_t^{(2)}(\lambda, x) = \int_{[0,t]} C_s^{(1)}(\lambda, x) dL_s(\lambda, x),$$

$$\ldots, C_t^{(k)}(\lambda, x) = \int_{[0,t]} C_s^{(k-1)}(\lambda, x) dL_s(\lambda, x), \ldots$$

and

$$C_t^{(0)}(x) = 1,\ C_t^{(1)}(x) = L_t(x),\ C_t^{(2)}(x) = \int_{[0,t]} C_s^{(1)}(x) dL_s(x), \ldots,$$

$$C_t^{(k)}(x) = \int_{[0,t]} C_s^{(k-1)}(x) dL_s(x), \ldots.$$

Since potential $U_t(\lambda, x)$ is bounded, from Theorem 4 for any $k \geq 1$ $EC_\infty^{(k)}(\lambda, x) < \infty$. Therefore, using (35), we have that $EC_t^{(k)}(x) < \infty$.

Suppose that one more condition holds:

(*C*) $E \sup_{t_2} (E(L_{\infty t_2}(\lambda, x) - L_t(\lambda, x) / \mathfrak{F}_t))^k \leq d_k$ *for every* $k \geq 1$.

Then, from Corollary 1 $E[L_\infty(\lambda, x)]^k < \infty$ for every $k \geq 1$. Hence, $E[L_t(x)]^k \leq e^{k\lambda(t_1 + t_2)} E[L_t(\lambda, x)]^k < \infty$, $k \geq 1$.

Now we can formulate the result:

Theorem 5: *Suppose* X_t *is a purely discontinuous local strong martingale and* X_t *satisfies conditions* (*A*)-(*B*). *Then*

(*a*) *there exists jointly measurable continuous in* Q_t^1 *increasing process* $L_t(x)$ *such that* (34) *holds,*

(*b*) *for any* $k \geq 1$ $EC_t^{(k)}(x) < \infty$,

(*c*) *if, in addition, condition* (*C*) *holds, then for every* $t \in R^2_+$ *and for every* x $L_t(x)$ *has moments of all orders.*

Remark 3: The existence and estimations of moments of local time for one-parameter purely discontinuous local martingales were obtained in [1].

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