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QUASI-FELLER MARKOV CHAINS

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We consider the class of Markov kernels for which the weak or strong Feller property fails to hold at some discontinuity set. We provide a simple necessary and sufficient condition for existence of an invariant probability measure as well as a Foster-Lyapunov sufficient condition. We also characterize a subclass, the quasi (weak or strong) Feller kernels, for which the sequences of expected occupation measures share the same asymptotic properties as for (weak or strong) Feller kernels. In particular, it is shown that the sequences of expected occupation measures of strong and quasi strong-Feller kernels with an invariant probability measure converge setwise to an invariant measure.

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1. Introduction

We consider a Markov chain on a locally compact separable metric space. A common assumption when studying a Markov chain is that the stochastic kernel of transition probabilities is (weak) Feller, i.e., it maps the space of bounded continuous functions into itself (whereas the strong-Feller kernels map the bounded measurable functions into bounded continuous functions). Indeed, under such an (easy to check) assumption, various properties can be derived for the long-run behavior of the Markov chain. In particular, for the existence of invariant probability measures, simple necessary and sufficient conditions are available (for instance cf. [9, 7]). In addition, a nice property of (weak) Feller kernels is that every weak* limit point of the expected occupation measures is a (possibly trivial) invariant measure.

However, it may happen that the transition kernel fails to have the (weak or strong) Feller property at some points in an exceptional set. In some cases, this pathology is serious in that it prevents the kernel from having the above mentioned

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properties of (weak) Feller kernels. The example in Section 2 illustrates the dramatic consequences if the (weak) Feller property fails at a single point only, even on a compact metric space. In some other cases, the kernel behaves practically as a (weak) Feller kernel, i.e., the above pathology is not serious. Therefore, having a means to distinguish between those two types of kernels is highly desirable.

A practical example of interest, which motivated this work, is the important class of *Generalized Semi-Markov Processes* (GSMP), which permits modeling of the essential dynamical structure of a discrete-event system (cf. [6]). Indeed, a time-homogeneous GSMP can be studied via Markov chain techniques, particularly its long-run behavior via ergodic theorems (cf. [6]). However, the (discrete-time) associated Markov kernel is *not* (weak) Feller as discontinuities occur when (at least) two "clocks" run out of time simultaneously. See also the threshold models in Tong [10]. It is thus necessary to provide conditions of existence of an invariant probability measure for such pathological kernels.

In the present paper, we propose such conditions which are in fact a simple extension of the ones in [7] for (weak) Feller kernels. In addition, we characterize a class of kernels, the *quasi-Feller* kernels, i.e., those kernels with a discontinuity set but with the same properties as (weak) Feller kernels.

We also prove that strong-Feller kernels enjoy an additional nice property; namely, if the transition kernel has an invariant probability measure μ, then μ-a.e., the sequence of expected occupation measures converges *setwise* to an invariant probability measure, in contrast to the (only) weak convergence for (weak) Feller kernels. The corresponding class of quasi strong-Feller kernels also has this property. Finally, the necessary and sufficient condition for existence of a unique invariant probability measure proposed in [8] also applies to a quasi-Feller kernel.

2. Notation and Definitions

Let $(X, \mathfrak{B})$ be a measurable space with X a locally compact separable metric space and $\mathfrak{B}$ its usual Borel σ-field. Denote by

- $M(X)$, the Banach space of signed Borel measures on $(X, \mathfrak{B})$ endowed with the total variation form.
- $C_b(X)$, the Banach space of bounded continuous functions on X, endowed with the sup-norm.
- $C_0(X) \subset C_b(X)$, the Banach space of continuous functions on X that vanish at infinity, endowed with the sup-norm. By "vanish at infinity", we mean
  that $f \in C_0(X)$ if $\sup_{x \in K_n^c} |f(x)| \to 0$ whenever K_n compact and $K_n \uparrow X$.
- $B(X)$, the Banach space of bounded measurable functions on X, endowed with the sup-norm.

Let P be the transition probability kernel of a Markov chain with values in X, i.e., $P(x, \cdot)$ is a probability measure on $(X, \mathfrak{B})$ for every $x \in X$, and $P(\cdot, B)$ is a measurable function on X for every $B \in \mathfrak{B}$.

P is a (weak) Feller kernel if $Pf \in C_b(X)$ whenever $f \in C_b(X)$. Note that, as X is a locally compact separable metric space, this condition is in fact equivalent to the apparently weaker condition $Pf \in C_b(X)$ whenever $f \in C_0(X)$. P is (strong) Feller if $Pf \in C_b(X)$ whenever $f \in B(X)$.

A measure $\mu \in M(X)$ is invariant if and only if

$$\mu(B) = \int P(x, B)\mu(dx) \qquad B \in \mathfrak{B}. \tag{1}$$

3. Quasi-Feller Markov Kernels

Let P be a transition kernel for which the (weak) Feller property fails at some points $x \in D \subset X$. It is important to note that even if this property fails at only a single point, it can have dramatic consequences, as shown in the following elementary example on a compact metric space:

$$X: = [0,1], P(x, \{x/2\}) = 1 \forall x \neq 0, P(0, \{1\}) = 1.$$

It is trivial to check that the above kernel has *no* invariant probability measure, despite X being compact, and the kernel is "almost Feller." In addition to not being Feller, the necessary and sufficient condition of existence stated in [7], namely,

$$\lim_{n \to \infty} n^{-1} E_x \sum_{t=0}^{n-1} f_0(X_t) > 0 \tag{2}$$

for some $x \in X$ and some arbitrary $0 < f_0 \in C_0(X)$, is not valid (and similarly, equivalent conditions that use compact sets in Beneš [1] and Foguel [5]). Indeed, for a fixed arbitrary $0 < f_0 \in C_0(X)$, we have

$$\lim_{n \to \infty} n^{-1} E_x \sum_{t=0}^{n-1} f_0(X_t) = \lim_{n \to \infty} f_0(2^{-n}x) = f_0(0) > 0,$$

so that (2) holds, but there is no i.p.m.

This example contradicts the conjecture in [3] that if the property fails at a finite number of points, then the kernel is still well-behaved. Another example is the oscillating random walk (cf. [3])

$$x_{n+1} = x_n + \begin{cases} \xi_n & \text{if } X_n \geq 0 \\ \psi_n & \text{if } X_n < 0 \end{cases}$$

where $\{\xi_n\}$ and $\{\psi_n\}$ are two unrelated sequences of independent equally distributed random variables.

The corresponding transition kernel is (weakly) continuous except at $x = 0$. However, if for instance $\text{Prob}(\xi_1 = \sqrt{2}) = p_1 = 1 - \text{Prob}(\xi_1 = -1)$ and $\text{Prob}(\psi_1 = \sqrt{3}) = p_2 = 1 - \text{Prob}(\psi_1 = -1)$, the n-step probability distribution converges weakly to an invariant uniform probability distribution. In this case, the discontinuity at zero does not prevent the existence of an i.p.m.

Let D be the set of discontinuity of the transition kernel, i.e.,

$$x \notin D \Leftrightarrow Pf(x_n) \to Pf(x) \quad \text{whenever } f \in C_b(X) \text{ and } x_n \to x,$$

and let $Y: = X - D$ be the subspace of X with the usual induced topology and $\mathfrak{B}'$ its usual Borel σ-field. In many cases of interest, $D \in \mathfrak{B}$ and $P(x, D) = 0 \forall x \in X - D$ and $P(x, X - D) > 0 \forall x \in D$. This can be checked easily on the above two examples and also in the GSMP models in [6].

3.1 Existence of an Invariant Probability Measure

We now state a necessary and sufficient condition for existence of an invariant probability measure for Markov kernels with a discontinuity set D.

Theorem 3.1: *Assume that $D \in \mathfrak{B}$ is closed, $P(x, D) = 0$ $\forall x \in X - D$ and $P^n(x, D) < 1$ $\forall x \in D$ for some $n \geq 1$. Let $0 \leq f_0 \in C_0(X)$ be fixed arbitrary and such that f_0 vanishes on D and is strictly positive elsewhere. Then, P has an invariant probability measure (i.p.m) if and only if*

$$\limsup_{n \to \infty} E_x n^{-1} \sum_{t=0}^{n-1} f_0(X_t) > 0 \tag{3}$$

for some $x \in X - D$.

Proof: Let $Y: = X - D$. As a subset of X, $X - D$ is open and thus Y, with the topology induced by X, is a locally compact separable metric space and its usual Borel σ-field coincides with $\mathfrak{B}': = \{\mathfrak{B} \cap \mathfrak{Y} \mid \mathfrak{B} \in \mathfrak{B}\}$. The Banach space $C_0(Y)$ of continuous functions that vanish at "infinity" is the subset of functions in $C_0(X)$ that vanish on D. In addition, $M(Y)$, the Banach space of finite Borel sign measures on $(Y, \mathfrak{B}')$, is the topological dual of $C_0(Y)$.

Now, let P' be the restriction of P on Y, i.e.,

$$P'(x, B): = P(x, B) \text{ whenever } x \in Y,\ B \in \mathfrak{B}'.$$

It is trivial to check that P' is weak Feller. Moreover, from every initial state $x \in Y$, the Markov chain stays in Y with probability 1. Therefore, the Markov chain induced by P' coincides with the original chain for every initial state $x \in Y$. For every $x \in Y$, one may use indifferently $P'(x, \cdot)$ or $P(x, \cdot)$.

Let μ be an invariant probability measure for P. From $P(x, D) = 0$ $\forall x \in Y$, $P^n(x, Y) > 0$ $\forall x \in D$ for some $n \geq 1$, and the invariance of μ, we also have $\mu(D) = 0$, i.e., $\mu \in M(Y)$ and μ is also an invariant probability measure for P'. Conversely, if μ is an invariant probability measure for P', it is also invariant for P.

Therefore, one may apply directly to P' the necessary and sufficient condition for existence on an i.p.m. given in [7] which is (3) with "lim" instead of "lim sup" and the operator E'_x instead of E_x. However, since with initial state $x \in Y$ the Markov chain induced by P' coincides with the one induced by P, one may replace E'_x by E_x. Also a simple examination of the proof of Theorem 2.1 in [7] shows that one may use indifferently "lim", "lim inf" or "lim sup". □

The important thing to notice is that in (3), f_0 is in $C_0(Y)$, i.e., f_0 vanishes on D. In the first example, one may check that with $f_0 \in C_0(Y)$,

$$E_x n^{-1} \sum_{t=0}^{n-1} f_0(X_t) = n^{-1} \sum_{t=0}^{n-1} f_0(2^{-t}x) \to f_0(0) = 0 \text{ for every } x \in (0, 1],$$

since f_0 *vanishes* on $D{:}=\{0\}$. This confirms that P has no invariant probability measure. As already mentioned, (3) with $0<f_0\in C_0(X)$ (i.e. as in [7]) would yield $f_0(0)>0$.

We now give other properties, using limits of the expected occupation measures. Let

$$\mu_n^x(B)=E_x n^{-1}\sum_{t=0}^{n-1}1_B(x_t)\quad x\in X,\;\; B\in\mathfrak{B} \tag{4}$$

$$\nu_n^x(B)=\mu_n^x(B\cap Y),\;\; x\in Y. \tag{5}$$

For every $x\in X$, fixed arbitrary, $\{\mu_n^x\}$ is a sequence of probability measures on $(X,\mathfrak{B})$ and for every $x\in Y$, fixed arbitrary, $\{\nu_n^x\}$ is a sequence of probability measures on $(Y,\mathfrak{B}')$.

It is important to note that the weak* convergence in $M(Y)$ is not the same as the weak* convergence in $M(X)$.

Lemma 3.2: *Let D be closed with $P(x,D)=0$ $\forall x\in X-D$ and $P^n(x,D)<1$ $\forall x\in D$ for some $n\geq 1$. Then,*

(*a*) *For every $x\in Y$, fixed arbitrary, every weak* accumulation point of the sequence $\{\nu_x^n\}$ in $M(Y)$ is a (possibly trivial) invariant measure $\nu^x\in M(Y)$.*

(*b*) *If P has an invariant probability measure μ, then $\mu\in M(Y)$ and ν^x is an i.p.m. for P, μ-a.e. in Y. In addition, $\nu_n^x\Rightarrow\nu^x$, μ-a.e.*

(*c*) *Let $\mu^x\in M(X)$ be a weak* accumulation point in $M(X)$ of the sequence $\{\mu_n^x\}$, $x\in Y$. Then ν^x, the restriction of μ^x to $(Y,\mathfrak{B}')$, is a weak* accumulation point in $M(Y)$ of $\{\nu_n^x\}$, and therefore, an invariant measure for P. Hence, μ^x is an invariant measure if and only if $\mu^x(D)=0$.*

Proof: (*a*) As Y is a locally compact separable metric space, the unit ball in $M(Y)$ is weak* sequentially compact. Hence, consider an arbitrary (weak*) convergent subsequence $\nu_{n_k}^x\to\nu^x\in M(Y)$. From $\nu_n^xP'=\nu_n^x+n^{-1}((P')^n-\delta_x)$ (with δ_x the Dirac at x), we conclude that

$$\lim_{k\to\infty}\int fd(\nu_{n_k}^xP')=\int fd\nu^x \text{ for every } f\in C_0(Y). \tag{6}$$

On the other hand, as P' is weak Feller, $P'f\in C_b(Y)$ and for every $0\leq f\in C_0(Y)$

$$\lim_{k\to\infty}\int fd(\nu_{n_k}^xP')=\lim_{k\to\infty}\int(P'f)d\nu_{n_k}^x\geq\int(P'f)d\nu^x=\int fd(\nu^xP'). \tag{7}$$

Combining (6)-(7) yields

$$\int fd\nu^x\geq\int fd(\nu^xP') \text{ for every } 0\leq f\in C_0(Y), \tag{8}$$

i.e., $\nu^x\geq\nu^xP'$. As ν^x is a finite measure, this implies $\nu^xP'=\nu^x$, i.e., ν^x is an invariant (possibly trivial) measure. As the weak* accumulation point was arbitrary, the result follows.

(*b*) Assume that μ is an invariant probability measure for P, hence for P', i.e.,

$\mu \in M(Y)$. From the Birkhoff Individual and Mean Ergodic Theorem (cf. [11]), we have, for every $f \in L_1(Y, \mathfrak{B}', \mu)$,

$$\lim_{n\to\infty} E_x n^{-1} \sum_{t=0}^{n-1} f(X_t) = \lim_{n\to\infty} E_x n^{-1} \sum_{t=0}^{n-1} (P')^t f(X_t) = f^*(x),$$

μ-a.e. In addition, $\int f^* d\mu = \int f d\mu$. On the other hand,

$$f^*(x) = \lim_{n\to\infty} \int f d\nu_n^x$$

so that for every $f \in C_0(Y)$, and an arbitrary weak* accumulation point ν^x of $\{\nu_n^x\}$, we have μ-a.e.

$$f^*(x) = \int f d\nu^x. \tag{9}$$

Therefore,

$$\int f^* d\mu = \int f d\mu = \int \Big(\int f d\nu^x \Big) \mu(dx) \quad \forall f \in C_0(Y).$$

This in turn implies

$$\mu(B) = \int \nu^x(B)\mu(dx), \quad b \in \mathfrak{B}'.$$

As μ is a probability measure, this implies that μ-a.e. $\nu^x(Y) = 1$, i.e., ν^x is a probability measure μ-a.e. By the Portmanteau Theorem (cf. [2]), we also conclude that μ-a.e. every weak* accumulation point of $\{\nu_n^x\}$ is also a weak accumulation point.

As $C_0(Y)$ is separable, it contains a countable dense subset $F := \{f_1, \ldots,\} \subset C_0(Y)$. For each $f \in F$ there is a set N_f with $\mu(N_f) = 0$ such that, from (9),

$$\int f d\nu^x = f^*(x) \quad \forall x \notin N_f.$$

Hence, as $\mu(\bigcup_{f \in F} N_f) = 0$ and F is dense in $C_0(Y)$,

$$\int f d\nu^x = f^*(x) \quad \forall f \in C_0(Y), \ \forall x \notin \textstyle\bigcup_{f \in F} N_f. \tag{10}$$

As (10) holds for every weak* (hence weak) accumulation point and every $f \in C_0(Y)$, all the weak limit points ν^x are identical μ-a.e., i.e. μ-a.e., $\mu_n^x \Rightarrow \nu^x$.

(c) Let $\mu^x \in M(X)$ be a weak* accumulation point in $M(X)$ of $\{\mu_n^x\}$, i.e., there is some subsequence $\{\mu_{n_k}^x\}$ such that

$$\lim_{k\to\infty} \int f d\mu_{n_k}^x = \int f d\mu^x \quad \forall f \in C_0(X). \tag{11}$$

Now, for every $x \in Y$ and $f \in C_0(Y)$, $\int f d\mu_{n_k}^x$ is just $\int f d\nu_n^x$. Therefore, since $C_0(Y) \subset C_0(X)$, to every weak* accumulation point $\mu^x \in M(X)$ of $\{\mu_n^x\}$ corresponds to a weak* accumulation point $\nu^x \in M(Y)$ of $\{\nu_n^x\}$. From $\int f d\mu^x = \int f d\nu^x$ for every $f \in C_0(Y)$, we conclude that the restriction of μ^x to $(Y, \mathfrak{B}')$ is just ν^x, and, from (b), is an invariant measure. In addition, as $P(x, D) = 0$ $\forall x \in Y$ and

$P^n(x,D) < 1 \ \ \forall x \in D$ for some $n \geq 1$, μ^x is invariant only if $\mu^x(D) = \int_D P^n(y,D)\mu^x(dy) < \mu^x(D)$, i.e. only if $\mu^x(D) = 0$. On the other hand, if $\mu^x(D) = 0$ then as the restriction of μ^x to $X - D$ is an invariant measure, and $P(x,D) = 0 \ \forall x \in X - D$, then so is μ^x. □

The last statement suggests the following definition of a *quasi-Feller Markov kernel* with discontinuity set D.

Definition: If P has a closed (weak) discontinuity set $D \in \mathfrak{B}$ with $P(x,D) = 0$ $\forall x \in X - D$ and $P^n(x,D) < 1 \ \forall x \in D$ for some $n \geq 1$, then P is said to be *quasi-Feller* if every weak* accumulation point $\mu^x \in M(X)$ of the sequence of expected occupation measures $\{\mu_n^x\}$, $x \in X - D$, satisfies $\mu^x(D) = 0$.

For such kernels, every weak* accumulation point μ^x of $\{\mu_n^x\}$ $(x \in X)$, is a (possibly trivial) invariant measure (with $\mu^x(D) = 0$) as for (weak) Feller kernels, which justifies the label "quasi-Feller."

In the first example, one may easily check that P is not quasi-Feller since for every $x \in X$, $\mu^x = \delta_0$ and thus, $\mu^x(\{0\}) = 1$. On the other hand, in the second example, $\mu^x(\{0\}) = 0$ so that P is quasi-Feller. A sufficient condition for P to be quasi-Feller is as follows:

Corollary 3.3: *Let $D \in \mathfrak{B}$ be closed, with $P(x,D) = 0 \ \ \forall x \in X - D$ and $P^n(x, X - D) > 0 \ \ \forall x \in D$ for some $n \geq 1$. Let $D_\varepsilon \in \mathfrak{B}$ be open and $D_\varepsilon \downarrow D$ as $\varepsilon \to 0$. Then, P is quasi-Feller if for some scalar K and for every sufficiently small $\varepsilon > 0$,*

$$\liminf_{n \to \infty} \mu_n^x(D_\varepsilon) \leq K\varepsilon, \tag{12}$$

for every $x \in X - D$.

Proof: Assume that (12) holds and consider a subsequence $\{\mu_{n_k}^x\}$ that converges weakly* to some $\mu^x \in M(X)$. As D_ε is open, for sufficiently small ε, we have (e.g. cf. [4]).

$$K\varepsilon \geq \liminf_{k \to \infty} \mu_n^x(D_\varepsilon) \geq \mu^x(D_\varepsilon) \geq \mu^x(D).$$

Letting $\varepsilon \downarrow 0$ yields the desired result. □

Note that similar condition was given in [3] (cf. condition III(a), p. 546).

3.2 A Lyapunov Condition of Existence

A sufficient condition of existence, although stronger than necessary and sufficient condition, is also useful and sometimes easier to manipulate. The condition below is a Lyapunov-type condition for non-Feller kernels with a discontinuity set D.

Corollary 3.4: *Assume that $D \in \mathfrak{B}$ is closed, with $P(x,D) = 0 \ \forall x \in X - D$ and $P^n(x, X - D) > 0 \ \forall x \in D$ for some $n \geq 1$. Let $0 < f_0 \in C_0(Y)$ be fixed arbitrary.*

(a) *If there exists a nonnegative measurable finite function f and a scalar $0 < \lambda$ such that*

$$Pf(x) \leq f(x) - 1 + \lambda f_0(x,) \ \ x \in Y, \tag{13}$$

then there exists an i.p.m. $\mu \in M(Y)$.

(b) *If there exist a nonnegative measurable finite function f, a compact set $K \subset Y$ and a scalar $0 < \lambda$ such that*

$$Pf(x) \leq f(x) - 1 + \lambda 1_K(x), \quad x \in Y, \tag{14}$$

then there exists an i.p.m. $\mu \in M(Y)$.

Proof: (*a*) Iterating (13) n times and dividing by n yields

$$n^{-1}(P^n f(x) - f(x)) + 1 \leq \lambda E_x n^{-1} \sum_{t=0}^{n-1} f_0(X_t). \tag{15}$$

Therefore, as $0 \leq P^n f$ and $n^{-1} f(x) \to 0$

$$0 < \lambda^{-1} \leq \limsup_{n\to\infty} E_x n^{-1} \sum_{t=0}^{n-1} f_0(X_t). \tag{16}$$

In view of Theorem 3, this implies the existence of an i.p.m.

(*b*) As P' is (weak) Feller on $(Y, \mathfrak{B}')$, this follows from e.g. [9, p. 297]. □

The Lyapunov condition (14) is standard for (weak) Feller kernels [cf. [9, p. 296]). On the other hand, when using (13) one does not need to find an appropriate compact set $K \subset Y$. However, one must be careful and take a function f_0 that vanishes on the discontinuity set D.

3.3. Quasi Strong-Feller Kernels

We recall that a strong-Feller Markov kernel maps $B(X)$, the bounded measurable functions, into $C_b(X)$. Equivalently, $P(x, B)$ is a continuous function of x, for every $B \in \mathfrak{B}$. As we did for the (weak) Feller kernels, we also consider the class of kernels for which the strong-Feller property fails to hold at some (closed) discontinuity set $D \subset \mathfrak{B}$. We first need the following result.

Lemma 3.5: *Assume that* P *is strong-Feller with an i.p.m.* $\mu \in M(X)$. *Then* μ-*a.e.*

$$\mu_n^x \to \mu^x \text{ setwise} \tag{17}$$

and μ^x *is an i.p.m. with*

$$\mu(B) = \int \mu^x(B)\mu(dx) \quad B \in \mathfrak{B}. \tag{18}$$

Proof: From Lemma 3.2(*b*) $\mu_n^x \Rightarrow \mu^x$. In addition, μ^x is an i.p.m. μ-a.e. and

$$\mu(B) = \int \mu^x(B)\mu(dx) \quad B \in \mathfrak{B}.$$

It thus suffices to prove that the convergence is *setwise* instead of *weak*.

We know that

$$\mu_n^x P = \mu_n^x + n^{-1}(P^n(x, \cdot) - \delta_x(\cdot)).$$

Let $B \in \mathfrak{B}$ be fixed arbitrary, and $f: = 1_B$. As P is strong-Feller, $Pf \in C_b(X)$ so that from a weak convergence of μ_n^x of μ^x, we get

$$\lim_{n\to\infty} \mu^n(B) = \lim_{n\to\infty} \int f d(\mu_n^x P)$$

$$=\lim_{n\to\infty}\int (Pf)d\mu_n^x$$

$$=\int (Pf)d\mu^x = \int fd(\mu^x P)$$

$$=\int fd\mu^x = \mu^x(B),$$

the desired result. □

Hence, the strong-Feller property translates into the strong (setwise) convergence of the expected occupation measures.

Finally, consider a kernel P for which the (strong) Feller property fails at some (closed) discontinuity set D. In a manner similar to what we did for quasi-Feller kernels, we say that:

Definition: If P has a closed (strong) discontinuity set $D \subset \mathfrak{B}$ with $P(x,D)=0$ $\forall x \in X-D$ and $P^n(x,D)<1$ $\forall x \in D$ for some $n \geq 1$, then P is said to be *quasi (strong) Feller* if every weak* accumulation point $\mu^* \in M(X)$ of the sequence of expected occupation measures $\{\mu_n^x\}$, $x \in X-D$, satisfies $\mu^x(D)=0$.

Indeed, if μ is an i.p.m. for P, then $\mu \in M(Y)$ and μ-a.e., using Lemma 3.5, the sequence of expected occupation measures converge setwise to an i.p.m. (the proof is similar to the weak Feller case).

The sufficient condition in Corollary 3.3 is also sufficient for P to be quasi strong-Feller if D is a (strong) discontinuity set.

Finally, note that the necessary and sufficient condition for uniqueness of an i.p.m. proposed in [8] (in fact, for strong-Feller Markov chains) is also valid for quasi strong-Feller kernels.

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