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SMOOTHNESS OF INVARIANT DENSITY FOR EXPANDING TRANSFORMATIONS IN HIGHER DIMENSIONS

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Let Ω be a bounded region in $\mathbb{R}^n$ and let $\mathcal{P} = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of closed subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Let $\tau: \Omega \rightarrow \Omega$ be an expanding transformation on $\mathcal{P}$ where, $\tau_i = \tau \mid_{P_i}$ and $\tau_i \in C^M$, $m \geq 2$. We show that the τ-invariant density $h \in C^{M-2}$.

Key words: Absolutely Continuous Invariant Measure, Ergodic, Expanding Transformation, Perron-Frobenius Operator.

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1. Introduction

There has been a recent surge of interest in the study of existence and properties of absolutely continuous invariant measures (**acim**) of higher dimensional transformations. Let Ω be a bounded region in $\mathbb{R}^n$ and let $\mathcal{P} = \{P_i\}_{i=1}^m$ be a partition of Ω into a finite number of subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Let $\tau: \Omega \rightarrow \Omega$ be piecewise C^2 on $\mathcal{P}$ where $\tau_i = \tau \mid_{P_i}$ is a C^2 diffeomorphism onto its image and expanding in the sense that there exists $\alpha > 1$ such that for any $i = 1, 2, \ldots, m$, $\| D\tau_i^{-1} \|_E < \alpha^{-1}$, where $D\tau_i^{-1}$ is the derivative matrix of τ_i^{-1} and $\| \cdot \|_E$ is the Euclidean matrix norm. Then, under general conditions [1], it has been shown that τ has an **acim**, which is a generalization of the results proved in [11, 12, 5] and [8]. We are then interested in properties associated with the **acim**. The properties of interest include: number of ergodic **acim** [9, 2], uniqueness [4, 21], stability [13, 6] and the smoothness of their invariant densities.

It is important to know which properties the invariant density, if one exists, inherits from its underlying transformation. For Lasota-Yorke maps [15] on $[0,1]$, Szewc [22] proved that the densities of invariant measures for Lasota-Yorke maps of class C^M are of class C^{M-1}. It should be noted, however, that these smoothness properties are assumed to hold only piecewise - that is, relative to a partition of Ω. The smoothness of the invariant density in Szewc's result is actually piecewise-smoothness relative to another partition that is obtained from the given one through refinement with all of its forward images. Thus in most cases (with the exception of some simple classes of maps, such as Markov maps) the underlying partition for the piecewise smoothness of the invariant density has (possibly infinitely) many more elements than the original one.

In this paper, we investigate the smoothness of invariant densities of **acim** in higher dimensions for a subset of Lasota-Yorke maps. We prove that if a transformation is expanding in the sense of the maps considered by Mañé [16] and of class C^M, then its invariant density is of class C^{M-2}. It should be noted, however, that this somewhat weaker smoothness result is valid on the original partition. We conjecture that this is the sharpest result possible for the given partition. In dimension one, the smoothness of the invariant density of an **acim** for transformations on an interval considered by Rényi [18] was established by Halfant [10]. An alternate proof of this has been presented in [17]. Some applications of expanding maps in a number theoretical context can be found in [20].

2. Definitions and Conditions

We denote by λ, the Lebesgue measure on $\mathbb{R}^n$. For a $n \times n \times \ldots \times n$ (k-times) array M_k we define its norm by $\| M_k \| = \max_{\mathcal{N}_k} |(m)_{i_1 \ldots i_k}|$, where

$$\mathcal{N}_k = \{i_1 i_2 \ldots i_k : 1 \leq i_j \leq n \text{ for } 1 \leq j \leq k\}.$$

For a real-valued function $f: \mathbb{R}^n \to \mathbb{R}$ we denote by $\mathbf{D}f$ its derivative, and by $\mathbf{D}^{(M)}f$ the M-th derivative of f. If $f(x) = f(x_1, x_2, \ldots, x_n)$ then $(\mathbf{D}f)_x$ is a linear map: $\mathbb{R}^n \to \mathbb{R}$ and $(\mathbf{D}f)_x(v_1, v_2, \ldots, v_n) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}$.

Let $\mathcal{P} = \{P_i\}_{i=1}^{m}$ be a partition of Ω into a finite number of subsets having piecewise C^2 boundaries of finite $(n-1)$-dimensional measure. Henceforth we work mainly with the open domain $\Omega_0 = \Omega \setminus \bigcup_{i \in \mathcal{I}_0} \partial P_i$. We use the notation τ_t^{-k} to mean a specific composition of inverses of the form $\tau_{i_k}^{-1} \circ \tau_{i_{k-1}}^{-1} \circ \ldots \circ \tau_{i_1}^{-1}$ where each $i_j \in \{1, 2, \ldots, m\}$ for $1 \leq j \leq k$, and denote by $\mathcal{I}_k$ the set of subscripts t for which τ_t^{-k} is defined (we use letters $t, u, \ldots$ as subscript here to distinguish between $\tau_i = \tau \mid_{P_i}$ and the composition of inverses of such maps as described.) Thus for any open set A, each τ_t^{-k} is a C^M diffeomorphism of $A \cap \tau_{i_1}(P_{i_1})$ onto its image. Of course, $\tau^{-k}(A) = \bigcup_{t \in \mathcal{I}_k} \tau_t^{-k}(A)$.

Definition 1: For an invariant measure μ, absolutely continuous with respect to λ, the *invariant density* h is given by $\mu = \int h d\lambda$.

Mañé [16] defined a class of *almost* Markov expanding maps satisfying five conditions.

Definition 2: Let $(\Omega, \mathcal{B}, \mu)$ be a probability space, where Ω is a separable metric space and $\mathcal{B}$ its Borel σ-algebra. We say a map $\tau: \Omega \to \Omega$ is *Mañé -expanding* if there exists a sequence of partitions $(\mathcal{P}_i)_{i \geq 0}$ of open sets such that:

(*a*) $\bigcup_{P \in \mathcal{P}_0} = \Omega$ (mod 0).

(*b*) For every $k \geq 0$ and $P \in \mathcal{P}_{k+1}, \tau(P)$ is a union (mod 0) of atoms of $\mathcal{P}_k$, and $\tau \mid_P$ is injective.

(*c*) There exists $0 < \eta < 1$ and $K > 0$ such that,

$$\| \tau_t^{-k}(x) - \tau_t^{-k}(y) \|_E \leq K\eta^k \| x - y \|_E$$

for every $k \geq 0$, and for every x, y in the domain of τ_t^{-k}.

(*d*) There exists $l > 0$ such that, for every pair of atoms $P, Q \in \mathcal{P}_0$, we have $\mu(\tau^{-l}(P) \cap Q) \neq 0$.

(*e*) There exists $C > 0$ such that, for every $k \geq 0$ and $0 < \zeta \leq 1$ whenever x, y are contained in the same atom of $\mathcal{P}_k$ we have

$$\left| \frac{\mathcal{T}_\tau(y)}{\mathcal{T}_\tau(x)} - 1 \right| \leq C \| \tau(x) - \tau(y) \|_E^\zeta.$$

($\mathcal{T}_t$ is the absolute value of the determinant of the Jacobian matrix of τ).

Mañé proved that these conditions are sufficient for the existence of an **acim** (see Theorem 1, Section 3).

Remarks: Such maps need not be Markov, but all Markov maps satisfy condition (*b*) (see [16, pp. 170-171]). The class of maps considered by Mañé is, by condition (*e*), $C^{1+\zeta}$. This class contains C^M; $M \geq 2$.

For our considerations (assuming that an **acim** exists) only one of these conditions is needed:

Expanding condition: There exists $0 < \eta < 1$ and $K > 0$ such that,

$$\| \tau_t^{-k}(x) - \tau_t^{-k}(y) \|_E \leq K\eta^k \| x - y \|_E$$

for every $k \geq 0$, for every t (corresponding to the maps τ_t^{-k} as described above) and for every x, y in the domain of τ_t^{-k}.

We will also need Mañé 's condition (*e*), but, as remarked above, this condition is satisfied whenever τ is C^M, $M \geq 2$. The expanding condition implies that for a given $\varepsilon > 0$, for N large enough and for any $u \in \mathcal{I}_N$ and any x, y in the domain of τ_u^{-N} we have:

$$\frac{\| \tau_u^{-N}(x) - \tau_u^{-N}(y) \|_E}{\| x - y \|_E} < \varepsilon < 1. \tag{1}$$

Set $\tau_u^{-N} = (\phi_1, \phi_2, \ldots, \phi_n)$. Thus for a fixed x, if y approaches x in the direction of any of the n coordinate axis, it follows that $\left| \frac{\partial \phi_j(x)}{\partial x_k} \right| < \varepsilon < 1$ where $1 \leq j,\ k \leq n$. Thus there exists an N such that for all $u \in \mathcal{I}_N$ we have

$$\| \mathbf{D}\tau_u^{-N}(x) \| < \varepsilon < 1 \text{ for all } x \text{ in the domain of } \tau_u^{-N}. \tag{2}$$

For most of the results that follow we need only expanding in the sense of (2). In other words, if we assume that an **acim** exists, the results of this paper follow from the weaker expanding condition (2) and the distortion condition (see Lemma 1).

Definition 3: We define the measures λ_k by $\lambda_k(A) = \lambda(\tau^{-k}(A))$.

Definition 4: *The Perron-Frobenius operator* $P_\tau: \mathcal{L}^1(\Omega) \to \mathcal{L}^1(\Omega)$ is

$$P_\tau f(x) = \sum_{i=1}^{m} \frac{f(\tau_i^{-1}(x))}{\mathcal{T}_\tau(\tau_i^{-1}(x))}, \; f \in \mathcal{L}^1(\Omega) \text{ and } m = \#(\mathcal{P}).$$

Definition 5: We define *iterated densities* $S_k(x)$ for almost all $x \in \Omega$ by the finite sum $S_k(x) = P_\tau^k 1(x) = \sum_{t \in \mathcal{I}_k} \mathcal{T}_{\tau_t^{-k}}(k)$. We note that, $\int_A (P_\tau^k 1) d\lambda = \int_{\tau^{-k}(A)} 1 d\lambda = \int_A d\lambda_k$, and therefore $S_k = \frac{d\lambda_k}{d\lambda}$.

3. Main Results

The following result was proved in [16, Chapter III, Theorem 1.3].

Theorem 1: (Mañé). *If τ is Mañé-expanding then it admits an invariant probability measure μ, absolutely continuous with respect to Lebesgue measure, and* $\lim_{k \to \infty} \lambda_k(A) = \mu(A)$ *for all A in $\mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra of Ω.*

The next result does not require the existence of an **acim**.

Theorem 2: *Let $\tau: \Omega \to \Omega$ satisfy the expanding condition and $\tau_i \in C^M$, $M \geq 2$. Then the sequence* $\{\sup_{x \in \Omega_0} \| \mathbf{D}^{(M-1)} S_k(x) \| \}$ *is uniformly bounded.*

Our final result, Theorem 3, states that if an **acim** exists, and if τ is of class C^M and satisfies our expanding condition, then the density function must be of class C^{M-2}. In this context, the existence of the **acim** is a separate problem, and, as stated in the introduction, there are many existence proofs in the literature. Of course, they all place some additional condition(s) on τ. For example, τ could satisfy all of Mañé's conditions, and then Theorem 1 applies.

Theorem 3: *Let $\tau: \Omega \to \Omega$ admit an absolutely continuous invariant measure and satisfy the expanding condition, and let $\tau_i \in C^M$, $M \geq 2$. Then the invariant density $h(x) \in C^{M-2}$ for all $x \in \Omega_0$.*

4. Proofs and Lemmas

The following lemma establishes what is historically called the distortion condition.

Lemma 1: *There exists a constant $B^{(0)}$ such that for any $k \geq 1$ and all $t \in \mathcal{I}_k$, we have*

$$\frac{\sup_{x \in \Omega_0} \mathcal{T}_{\tau_t^{-k}}(x)}{\inf_{x \in \Omega_0} \mathcal{T}_{\tau_t^{-k}}(x)} \leq B^{(0)}.$$

Proof: We adapt the proof of [16] for dimension 1. From the definition of $\mathcal{T}_\tau$ we have

$$\frac{\mathcal{T}_{\tau_t^{-k}}(x)}{\mathcal{T}_{\tau_t^{-k}}(y)} = \frac{\mathcal{T}_{\tau^k}(\tau_t^{-k}(y))}{\mathcal{T}_{\tau^k}(\tau_t^{-k}(x))} = \prod_{j=0}^{k-1} \frac{\mathcal{T}_\tau(\tau_v^{-j-1}(y))}{\mathcal{T}_\tau(\tau_v^{-j-1}(x))}$$

$$\leq \prod_{j=0}^{k-1}(1+C\,\|\,\tau_w^{-j}(x)-\tau_w^{-j}(y)\,\|\,_E^{\zeta}),$$

where $v \in \mathfrak{I}_{j+1}$ (respectively $w \in \mathfrak{I}_j$) corresponds to $\tau^{k-(j+1)} \circ \tau_t^{-k}$ (respectively $\tau^{k-j} \circ \tau_t^{-k}$). Thus

$$\frac{\mathfrak{T}_{\tau_t^{-k}}(x)}{\mathfrak{T}_{\tau_t^{-k}}(y)} \leq \prod_{j=0}^{k-1}(1+CK^{\zeta}\eta^{j\zeta}\,\|\,x-y\,\|\,_E^{\zeta}) \leq \prod_{j=0}^{\infty}(1+CK^{\zeta}\eta^{j\zeta}\,\|\,x-y\,\|\,_E^{\zeta})$$

$$\leq \prod_{j=0}^{\infty}(1+CK^{\zeta}\eta^{j\zeta}(\text{diam}\Omega)^{\zeta}).$$

This converges if and only if the series $\sum_{j=0}^{\infty}\eta^{j\zeta}$ converges. Thus since $\eta < 1$,

$$\frac{\sup_{x\in\Omega_0}\mathfrak{T}_{\tau_t^{-k}}(x)}{\inf_{x\in\Omega_0}\mathfrak{T}_{\tau_t^{-k}}(x)} \leq B^{(0)}. \qquad \square$$

Lemma 2: *$S_k(x) \leq B^{(0)}$ for every $x \in \Omega_0$ and $k \geq 1$ (where $B^{(0)}$ is the bound in Lemma 1).*

Proof: Starting with Definition 5, the following relations are valid:

$$\frac{\sup_{x\in\Omega_0} S_k(x)}{\inf_{x\in\Omega_0} S_k(x)} = \frac{\sup_{x\in\Omega_0}\sum_{t\in\mathfrak{I}_k}\mathfrak{T}_{\tau_t^{-k}}(x)}{\inf_{x\in\Omega_0}\sum_{t\in\mathfrak{I}_k}\mathfrak{T}_{\tau_t^{-k}}(x)} \leq \frac{\sum_{t\in\mathfrak{I}_k}\sup_{x\in\Omega_0}\mathfrak{T}_{\tau_t^{-k}}(x)}{\sum_{t\in\mathfrak{I}_k}\inf_{x\in\Omega_0}\mathfrak{T}_{\tau_t^{-k}}(x)}$$

$$\frac{\sum_{t\in\mathfrak{I}_k}\inf_{x\in\Omega_0}\mathfrak{T}_{\tau_t^{-k}}(x)}{\sum_{t\in\mathfrak{I}_k}\inf_{x\in\Omega_0}\mathfrak{T}_{\tau_t^{-k}}(x)} \text{(by Lemma 1)} = B^{(0)}.$$

Since $\int_{\Omega} S_k d\lambda = 1$, it follows that $\frac{1}{B^{(0)}} \leq \inf_{x\in\Omega_0} S_k(x) \leq \sup_{x\in\Omega_0} S_k(x) \leq B^{(0)}$. $\square$

The following lemma was proved in [7] in dimension 1.

Lemma 3: *Let $F(x) = f(\phi(x))\mathfrak{T}_{\phi}(x)$ for all $x \in \Omega$. Then*

$$\mathbf{D}^M F = \mathbf{D}^M f(\phi)(\mathbf{D}\phi)^M \mathfrak{T}_{\phi} + \sum_{j=0}^{M-1}\left(\mathbf{D}^j f(\phi)\left(\sum_{i=0}^{M}\mathbf{D}^i\mathfrak{T}_{\phi}P_{i,j,M}(\mathbf{D}\phi,\ldots,\mathbf{D}^M\phi)\right)\right),$$

where $P_{i,j,M}(\mathbf{D}\phi,\ldots,\mathbf{D}^M\phi)$ is a polynomial.

Proof: Straightforward proof by induction. $\square$

We now prove that the sequence $\{\,\|\,\mathbf{D}^{M-1}S_k(x)\,\|\,\}$ is uniformly bounded, if τ is piecewise C^M. This will guarantee that for every $0 \leq j \leq M-2$, the sequence $\{\,\|\,\mathbf{D}^j S_k(x)\,\|\,\}$ is also uniformly bounded and equicontinuous.

Proof of Theorem 2: We prove the theorem by induction. First we note that

$$S_{K+N}(x) = \sum_{i \in \mathfrak{I}_N} S_K(\tau_u^{-N}(x))\mathfrak{T}_{\tau_u^{-N}}(x).$$

Let $M = 2$. By differentiation we obtain

$$\mathbf{D}S_{K+N}(x) = \sum_{u \in \mathfrak{I}_N} \{\mathbf{D}S_K(\tau_u^{-N}(x))\mathbf{D}\tau_u^{-N}(x)\mathfrak{T}_{\tau_u^{-N}}(x) + S_K(\tau_u^{-N}(x))\mathbf{D}\mathfrak{T}_{\tau_u^{-N}}(x)\}.$$

We take N so that (2) is satisfied (for $\varepsilon = \frac{1}{B^{(0)}}$). We put

$$\gamma = \sup_{\substack{x \in \Omega_0 \\ u \in \mathfrak{I}_N}} \| \mathbf{D}\tau_u^{-N}(x) \| \leq \frac{1}{B^{(0)}} < 1.$$

Since τ is piecewise C^2 on Ω, we also have

$$\beta_1^{(2)} = \sup_{x \in \Omega_0} \sum_{u \in \mathfrak{I}_N} \| \mathbf{D}\mathfrak{T}_{\tau_u^{-N}}(x) \| < \infty \text{ and } B_k \sup_{x \in \Omega_0} \| \mathbf{D}S_k(x) \| < \infty.$$

Using Lemma 2 and Definition 5, we obtain $B_{K+N} \leq B_K\gamma B^{(0)} + B^{(0)}\beta_1^{(2)}$. This implies that the sequence $\{B_{K+iN}\}_{i=1}^{\infty}$, is uniformly bounded by some number $\widehat{B}_K$ and hence the entire sequence $\{B_k\}$ is bounded by $B^{(1)} = \max_{0 \leq i \leq N-1}\{\widehat{B}_i\}$. Thus $\{\sup_{x \in \Omega_0} \| \mathbf{D}S_k(x) \| \}$ is uniformly bounded by $B^{(1)}$.

Now we assume the theorem is true for M and prove it for $M+1$ (i.e., we assume that$\tau \in C^{M+1}$ and prove that $\{\sup_{x \in \Omega_0}\mathbf{D}^{(M)}S_k(x)\}$ is uniformly bounded).

By Lemma 3 we have

$$\mathbf{D}^{(M)}S_{K+N}(x) = \sum_{u \in \mathfrak{I}_N} \Bigg\{\mathbf{D}^{(M)}S_K(\tau_u^{-N}(x))(\mathbf{D}\tau_u^{-N}(x))^M\mathfrak{T}_{\tau_u^{-N}}(x)$$
$$+\sum_{j=0}^{M-1}\left(\mathbf{D}^{(j)}S_K(\tau_u^{-N}(x))\sum_{i=0}^{M}\left(\mathbf{D}^i\mathfrak{T}_{\tau_u^{-N}}(x)P_{i,j,M}(\mathbf{D}\tau_u^{-N}(x),\ldots,\mathbf{D}^M\tau_u^{-N}(x))\right)\right)\Bigg\}.$$

Now by induction we have, for $j = 0, 1, \ldots, M-1$, constants $B^{(j)}$ which are bounds for the sequences $\{\sup_{x \in \Omega_0}\mathbf{D}^{(j)}S_k(x)\}$ respectively.

Since τ is piecewise C^{M+1} on Ω, for each j there exists a constant $\beta_j^{(M)}$ such that

$$\beta_j^{(M)} = \sum_{u \in \mathfrak{I}_N} \sum_{i=0}^{M}\left(\mathbf{D}^i\mathfrak{T}_{\tau_u^{-N}}(x)P_{i,j,M}(\mathbf{D}\tau_u^{-N}(x),\ldots,\mathbf{D}^M\tau_u^{-N}(x))\right) < \infty.$$

Let $B_k^{(M)} = \sup_{x \in \Omega_0} \| \mathbf{D}^{(M)}S_k \| (x) < \infty$. Then

$$B_{K+N}^{(M)} \leq B_K^{(M)}\gamma^M B^{(0)} + \sum_{j=0}^{M-1}\beta^{(j)}\beta_j^{(M)}.$$

Since $\gamma^M B^{(0)} < 1$, the sequence $\{B_{K+iN}^{(M)}\}_{i=1}^{\infty}$ is uniformly bounded by some number $\widehat{B}_K^{(M)}$. Thus the entire sequence $\{B_k^{(M)}\}$ is bounded by $B^{(M)} = \max_{0 \le i \le N-1}\{\widehat{B}_i^{(M)}\}$. □

Lemma 4: *Let $f:\mathbb{R}^n \to \mathbb{R}$ and $\{S_k\}$ be a sequence of functions $S_k:\mathbb{R}^n \to \mathbb{R}$ such that $S_k \to f$. If $\mathbf{D}S_k \rightrightarrows g$ (uniformly) then $\mathbf{D}f = g$.*

Proof: This is a standard result, e.g., Theorem 7.17 of [19]. □

Proof of Theorem 3: Using Theorem 2, we conclude that the sequence $\{\mathbf{D}^{(K)}S_k\}$ is uniformly bounded and equicontinuous for $0 \le K \le M-2$ on Ω_0. By the Ascoli-Arzela Theorem, there is a subsequence $\{\mathbf{D}^{(K)}S_{k_l}\}$ with a continuous limit, $f_K(x)$. In particular, we have $\lim_{l\to\infty} S_{k_l}(x) = f_0(x) = f(x)$ $\forall x \in \Omega_0$. From Lemma 4, it follows that $f_K(x) = \mathbf{D}^{(K)}f(x)$ for $K = 0,1,2,\ldots,M-2$ and for all $x \in \Omega_0$. To complete the proof, we note that since $\lim_{l\to\infty}\int S_{k_l}d\lambda = \lim_{l\to\infty}\lambda_{k_l} = \int f d\lambda$, and since we know $\{\lambda_k\}$ converges (Theorem 1) it follows that $(\lim_{k\to\infty}\lambda_k)(A) = \int_A f d\lambda = \mu(A)$.

This means f is equal to the invariant density h a.e. Hence, h can be chosen to be $M-2$ times differentiable and we have $f^{(j)} = h^{(j)}$, for $1 \le j \le M-2$. □

Corollary 3.1: *Let $\tau:\Omega \to \Omega$ be expanding and $\tau_i \in C^M$, $M \ge 2$. Then for $0 \le j \le M-2$, the sequence $\{\mathbf{D}^{(j)}S_k\}$ converges uniformly to $h^{(j)}$.*

Proof: This follows immediately from Theorem 3 and the fact that $\{\mathbf{D}^{(j)}S_{k_l}\}$ is uniformly bounded and equicontinuous. □

5. Conclusions

Our results could be improved in two directions. One problem to consider is to establish the smoothness property of invariant densities for Lasota-Yorke maps in higher dimensions. Another problem would be to increase the degree of smoothness from C^{M-2} to C^{M-1}, (but as noted in the introduction this would in general require a finer partition). In one dimension, this was proved by [22]. Furthermore, ti seems possible to establish the existence of an **acim** for random maps (see [3] and [14]) composed of expanding transformations, and to derive smoothness properties of invariant densities based on the technique used in this paper.

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