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ON THE STABILITY OF STATIONARY SOLUTIONS OF A LINEAR INTEGRO-DIFFERENTIAL EQUATION

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In this paper the following two connected problems are discussed. The problem of the existence of a stationary solution for the abstract equation

$$\epsilon x''(t) + x'(t) = Ax(t) + \int_{-\infty}^{t} E(t - s(x(s)ds + \xi(t), t \in \boldsymbol{R} \qquad (1)$$

containing a small parameter ϵ in Banach space B is considered. Here $A \in \mathcal{L}(B)$ is a fixed operator, $E \in C([0, +\infty), \mathcal{L}(B))$ and ξ is a stationary process. The asymptotic expansion of the stationary solution for equation (1) in the series on degrees of ϵ is given.

We have proved also the existence of a stationary with respect to time solution of the boundary value problem in B for a telegraph equation (6) containing the small parameter ϵ. The asymptotic expansion of this solution is also obtained.

Key words: Stationary Solutions, Singular Perturbations, Telegraph Equation, Time-Stationary Solutions, Asymptotic Expansions.

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1. Introduction

Let $(B \| \cdot \|)$ be a complex Banach space, $\overline{0}$ the zero element in B, and $\mathcal{L}(B)$ the Banach space of bounded linear operators on B with the operator norm, denoted also by the symbol $\| \cdot \|$. For a B-valued function, continuity and differentiability refer to continuity and differentiability in the B-norm. For an $\mathcal{L}(B)$-valued function, continuity is the continuity in the operator norm. For operator A, the sets $\sigma(A)$ and $\rho(A)$ are its spectrum and resolvent set, respectively.

In the following, we will consider random element son the same complete probability space $(\Omega, \mathfrak{F}, P)$. The uniqueness of a random process that satisfies an equation, is its uniqueness up to stochastic equivalence. We consider only B-valued random functions which are continuous with a probability of one. All equalities with random elements in this article are always equalities with a probability one. For a given equation, we consider only solutions which are measurable with respect to the right-hand side random process.

It is well known that the stationary solutions of difference and differential equations are steady with respect to various perturbations of the right-hand side and perturbation of coefficients. For example, see [5]. In the present work, it is shown that stability has a place with respect to perturbations such as degeneracy of the equation.

In the first part of this paper, we consider the following equation

$$\epsilon x''(t) + x'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s)x(s)ds + \xi(t), t \in \boldsymbol{R} \tag{1}$$

containing a parameter ϵ in B. Here $A \in \mathfrak{L}(B)$ is fixed operator, ξ is a stationary process in B and $E \in C([0, +\infty), \mathfrak{L}(B))$ is a function satisfying the condition

$$a: = \int_{0}^{+\infty} \| E(s) \| \, ds < +\infty.$$

We suppose that the following condition

$$\sigma(A) \cap i\boldsymbol{R} = \emptyset \tag{2}$$

holds. Under condition (2) the function

$$G(t): = \begin{cases} -e^{At}P_+, & t < 0; \\ e^{At}P_-, & t > 0 \end{cases}$$

satisfies the inequality

$$b: = \int_{\boldsymbol{R}} \| G(s) \| \, ds < +\infty.$$

Here P_- and P_+ are Riesz spectral projectors corresponding to the spectral sets $\sigma(A) \cap \{z \mid Re z < 0\}$ and $\sigma(A) \cap \{z \mid Re z > 0\}$, respectively.

Let S be the class of all stationary B-valued processes $\{\xi(t): t \in \boldsymbol{R}\}$ which possess continuous derivatives of all orders on $\boldsymbol{R}$ with a probability one and such that, for some numbers $L = L_\xi > 0$, $C = C_\xi > 0$, $\delta > 0$, the following inequalities

$$\forall n \geq 0: \boldsymbol{E}\{ \sup_{0 \leq s \leq \delta} \| \xi^{(n)}(s) \| \} \leq LC^n$$

hold. The notations $\xi \in S(L, C, \delta)$ and $\xi \in S$ will be used. Then we have the following result.

Theorem 1: *Let $A \in \mathcal{L}(B)$ be an operator satisfying* (2). *Suppose that $\xi \in S$ and $ab < 1$. Then there exists $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$, the equation* (1) *has a stationary solution $x_\epsilon \in S$, which for every bounded subset J of $\boldsymbol{R}$, satisfies*

$$\boldsymbol{E}\Big(\sup_{s \in J} \| x_\epsilon(s) - y_0(s) \| \Big) \to 0, \quad \epsilon \to 0,$$

where y_0 is a unique stationary solution of the equation

$$x'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s)x(s)ds + \xi(t), t \in \boldsymbol{R}. \tag{3}$$

The process x_ϵ is a unique solution of (1) *in the class of all stationary connected processes in S.*

This theorem is proved in Section 2. The method of proof uses a modification of the proof of Theorem 1 in [7] about the stability of stationary solutions for equation (1) with $E \equiv 0$.

Remark 1: The asymptotic expansion for a stationary solution of (1) is obtained.

Remark 2: The assumption (2) is equivalent to the existence of a unique stationary solution $\{x(t) \mid t \in \boldsymbol{R}\}$ with $\boldsymbol{E} \| x(0) \| < +\infty$ of the equation

$$x'(t) = Ax(t) + \xi(t), \ t \in \boldsymbol{R}$$

for every stationary process $\{\xi(t) \mid t \in \boldsymbol{R}\}$ with $\boldsymbol{E} \| \xi(0) \| < +\infty$, see [3, pp. 201-202].

Remark 3: The general approach to the analysis of the Cauchy problem for deterministic differential equations containing a small parameter leads to the appearance of boundary layer summands in the asymptotic expansion of solution [10]. These summands are absent in the asymptotic expansion of the stationary solution in the considered problem.

Remark 4: The problem of the existence of stationary solutions for difference and differential stochastic equations has been investigated by many authors. See, for example, monograph [1], surveys [2, 4] and article [6].

Corollary 1: *Let $A \in \mathcal{L}(B)$ be an operator satisfying* (1). *Suppose that $\xi \in S$. Then there exists $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$, the equation*

$$\epsilon x''(t) + x'(t) = Ax(t) + \xi(t), \quad t \in \boldsymbol{R} \tag{4}$$

has a unique stationary solution $x_\epsilon \in S$, which, for every bounded subset J of $\boldsymbol{R}$, satisfies

$$\boldsymbol{E}\Big(\sup_{s \in J} \| x_\epsilon(s) - x_0(s) \| \Big) \to 0, \quad \epsilon \to 0,$$

where x_0 is a unique stationary solution of the equation

$$x'(t) = Ax(t) + \xi(t), \quad t \in \boldsymbol{R}.$$

The second part of this paper deals with the asymptotic expansion of the stationary with respect to time solution of a boundary value problem containing a small parameter. The following definition is necessary.

Definition 1: A B-valued random function u defined on $Q: = \boldsymbol{R} \times [0,\pi]$ is time-stationary if

$$\forall t \in \boldsymbol{R} \forall n \in \boldsymbol{N} \forall \{(t_1,x_1),\dots,(t_n,x_n)\} \subset Q \forall \{D_1,\dots,D_n\} \subset \mathfrak{B}(B):$$

$$P\left\{\bigcap_{k=1}^{n}\{\omega: u(\omega; t_k + t, x_k) \in D_k\}\right\} = P\left\{\bigcap_{k=1}^{n}\{\omega: u(\omega; t_k, x_k) \in D_k\}\right\},$$

where $\mathfrak{B}(B)$ is the Borel σ-algebra of B.

Let

$$C_0^3: = \{g:[0,\pi] \rightarrow \boldsymbol{C} \mid g^{(k)}(0) = g^{(k)}(\pi) = 0, k = 0,1,2\} \cap C^3([0,\pi]).$$

Theorem 2: *Let $A \in \mathfrak{L}(B)$ be an operator satisfying the following condition*

$$\{k^2 + i\alpha \mid k \in \boldsymbol{N}, \alpha \in \boldsymbol{R}\} \subset \rho(A). \tag{5}$$

Suppose that $g \in C_0^3$ and $\xi \in S$ with a number $\delta > 0$ and $ab < 1$. Then there exists $\epsilon_0 > 0$ such that for every ϵ with $|\epsilon| < \epsilon_0$, the boundary value problem

$$\begin{cases} \epsilon u''_{tt}(t,x;\epsilon) + u'_t(t,x;\epsilon) - u''_{xx}(t,x;\epsilon) \\ \qquad = Au(t,x;\epsilon) + g(x)\xi(t), \quad t \in \boldsymbol{R},\ x \in [0,\pi] \\ u(t,0;\epsilon) = u(t,\pi;\epsilon) = \bar{0}\,, t \in \boldsymbol{R} \end{cases} \tag{6}$$

has a unique time-stationary solution $u(\cdot,\cdot;\epsilon)$ with

$$\boldsymbol{E}\left(\sup_{0 \le s \le \delta, 0 \le x \le \pi} \| u(s,x;\epsilon) \|\right) + \boldsymbol{E}\left(\sup_{0 \le s \le \delta, 0 \le x \le \pi} \| u'_t(s,x;\epsilon) \|\right) < +\infty,$$

which, for every $t \in \boldsymbol{R}$, satisfies

$$\boldsymbol{E}\left(\sup_{t \le s \le t+\delta, 0 \le x \le \pi} \| u(t,x;\epsilon) - v(t,x) \|\right) \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where v is the unique time-stationary solution of the following boundary value problem for a heat equation

$$\begin{cases} v'_t(t,x) - v''_{xx}(t,x) = Av(t,x) + g(x)\xi(t), \quad t \in Q \\ \qquad v(t,0) = v(t,\pi) = 0, \quad t \in \boldsymbol{R} \end{cases} \tag{7}$$

with

$$\sup_{0 \le x \le \pi} \boldsymbol{E} \| v(0,x) \| < +\infty.$$

This theorem is proved in Section 3.

Remark 5: Condition (5) is a necessary and sufficient condition of the existence of a time-stationary solution for boundary value problem (7) [8].

Remark 6: Note that, if $\epsilon > 0$, problem (6) is a boundary value problem for a hyperbolic equation and that, if $\epsilon = 0$, we have a boundary value problem for a parabolic equation.

Remark 7: The study of the asymptotic behavior of a solution $u(\cdot,\cdot;\epsilon)$ of the telegraph equation from (6) as $\epsilon \rightarrow 0+$ has also physical sense [9].

2. Asymptotic Expansion of the Stationary Solution of Equation (1)

In order to prove Theorem 1, a few lemmas will be needed.

Lemma 1: *Let $A \in \mathcal{L}(B)$ be an operator satisfying* (2). *Suppose that $\xi \in S$. Then the equation*

$$x'(t) = Ax(t) + \xi(t), t \in \boldsymbol{R}$$

has a unique stationary solution $x \in S$, which can be presented in the form

$$x(t) = \int_{\boldsymbol{R}} G(t-s)\xi(s)ds = \int_{\boldsymbol{R}} G(s)\xi(t-s)ds, t \in \boldsymbol{R}.$$

Proof: This is the corollary of Theorem 1 in [3, pp. 201-202].

Lemma 2: *Let $A \in \mathcal{L}(B)$ be an operator satisfying* (2). *Suppose that $\xi \in S$. The following two statements are equivalent:*

(*i*) *A stationary process $x \in S$ is a unique stationary solution of the equation* (3).

(*ii*) *A stationary process $x \in S$ is a unique stationary solution of the equation*

$$x(t) = \int_{\boldsymbol{R}} G(t-s) \int_{-\infty}^{s} E(s-u)x(u)duds + \int_{\boldsymbol{R}} G(t-s)\xi(s)ds, t \in \boldsymbol{R}. \tag{8}$$

Proof: The result is a consequence of Lemma 1.

Lemma 3: *Let $A \in \mathcal{L}(B)$ be an operator satisfying* (2) *and $ab < 1$. Suppose that ξ is a stationary process in B, which, for some $\delta > 0$, satisfies*

$$\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| \xi(t) \| \Big) < +\infty.$$

Then the equation (8) *has a unique stationary solution x, which satisfies*

$$\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| x(t) \| \Big) < +\infty. \tag{9}$$

Proof: Let S_0 be the class of all stationary connected B-valued processes x which are stationary connected with ξ and, for given $\delta > 0$, satisfy (9). Let us introduce the operator

$$(Tx)(t) := \int_{\boldsymbol{R}} G(t-s) \int_{-\infty}^{s} E(s-u)x(u)duds + \int_{\boldsymbol{R}} G(t-s)\xi(s)ds, t \in \boldsymbol{R}.$$

Then $Tx \in S_0$ and

$$\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| (Tx)(t) - (Ty)(t) \| \Big) \le ab\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| x(t) - y(t) \| \Big),$$

therefore T is a continuous operator on S_0. Set

$$x_0(t) := \int_{\boldsymbol{R}} G(t-s)\xi(s)ds, \quad t \in \boldsymbol{R},$$

then $x_0 \in S_0$ and

$$\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| x_0(t) \| \Big) \le b\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| \xi(t) \| \Big).$$

Introduce the sequences of random processes

$$x_0, x_1 := Tx_0, x_2 := Tx_1, \ldots, x_n := Tx_{n-1}, \ldots.$$

It is clear that

$$x_n \in S_0, n \in \boldsymbol{N}; \quad x_{n+1} = Tx_n, n \ge 0$$

and for every $t \in \boldsymbol{R}$

$$\boldsymbol{E} \| x_{n+1}(t) - x_n(t) \| \le \boldsymbol{E}\Big(\sup_{t \le s \le t+\delta} \| x_{n+1}(s) - x_n(s) \| \Big)$$

$$\le a(ab)^{n+1}\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| \xi(s) \| \Big), \quad n \ge 0.$$

Hence, the series

$$x(t) := x_0(t) + [x_1(t) - x_0(t)] + \ldots + [x_n(t) - x_{n-1}(t)] + \ldots$$

converges with a probability one for every $t \in \boldsymbol{R}$ and this convergence is uniform over the bounded subset of $\boldsymbol{R}$ with a probability one. By continuity of T we have $x = Tx$. The solution x of (8) is unique.

Lemma 4: *Let $A \in \mathfrak{L}(B)$ be an operator satisfying* (2) *and $ab < 1$. Suppose that ξ is a stationary process in B, which, for some $\delta > 0$, satisfies*

$$\boldsymbol{E}\Big(\sup_{0 \le t \le \delta} \| \xi(t) \| \Big) < +\infty.$$

Then equation (3) *has a unique stationary solution x, which satisfies* (9).

Proof: The result is an immediate consequence of Lemma 2 and Lemma 3.

Set $c := (1 - ab)^{-1}$.

Lemma 5: *Let $A \in \mathfrak{L}(B)$ be an operator satisfying* (2) *and $ab < 1$. Suppose that $\xi \in S(L, C, \delta)$. The equation* (3) *has a unique stationary solution $x \in S(bcL, C, \delta)$.*

Proof: We return to the proof of Lemma 3 where the stationary solution x for equation (3) was given. From the inclusion $\xi \in S(L, C, \delta)$ and representation

$$x_0(t) = \int_{\boldsymbol{R}} G(s)\xi(t-s)ds, t \in \boldsymbol{R}$$

it follows that

$$x_0^{(k)}(t) := \int_{\boldsymbol{R}} G(s)\xi^{(k)}(t-s)ds, \quad t \in \boldsymbol{R}$$

for every $k \ge 0$ and $x_0 \in S(bL, C, \delta)$. For the process $x_1 - x_0$, we have

$$x_1(t) - x_0(t) = \int_{\boldsymbol{R}} G(u) \int_0^{+\infty} E(v)x_0(t-u-v)dudv, t \in \boldsymbol{R}.$$

Hence, for every $k \ge 0$, we have

$$x_1^{(k)}(t) - x_0^{(k)}(t) = \int_{\boldsymbol{R}} G(u) \int_0^{+\infty} E(v)x_0^{(k)}(t-u-v)dudv, t \in \boldsymbol{R},$$

and $(x_1 - x_0) \in S(ab^2L, C, \delta)$. By induction, we find

$$(x_n - x_{n-1}) \in S(b(ab)^n L, C, \delta), n \geq 1.$$

Therefore,

$$x \in S(bcL, C, \delta).$$

Lemma 5 is proved.

Proof of Theorem 1: Let $\xi \in S(L, C, \delta)$. We shall construct the asymptotic expansion for a solution of (1) in the following way. From Lemma 5, equation (3) has a unique stationary solution $y_0 \in S(bcL, C, \delta)$. Note that $y_0'' \in S(bcLC^2, C, \delta)$. Let y_1 be a unique stationary solution for equation

$$y_1'(t) = Ay_1(t) + \int_{-\infty}^{t} E(t-s)y_1(s)ds - y_0''(t), t \in \boldsymbol{R}.$$

This solution exists from Lemma 5 and

$$y_1 \in S(b^2c^2LC^2, C, \delta).$$

By analogy with y_1, let y_2 be a unique stationary solution for equation

$$y_2'(t) = Ay_2(t) + \int_{-\infty}^{t} E(t-s)y_2(s)ds - y_1''(t), t \in \boldsymbol{R}.$$

For this solution, we have $y_2 \in S(b^3c^3LC^4, C, \delta)$.

If the processes $y_0, y_1, \ldots, y_{n-1}$ for $n \geq 1$ are already constructed we will define process y_n as a unique stationary solution of the equation

$$y_n'(t) = Ay_n(t) + \int_{-\infty}^{t} E(t-s)y_n(s)ds - y_{n-1}''(t), \quad t \in \boldsymbol{R},$$

which satisfies

$$y_n \in S(b^{n+1}c^{n+1}LC^{2n}, C, \delta).$$

It is clear that the processes y_n, $n \geq 0$ are stationary connected [3].

Set

$$y_\epsilon(t) := \sum_{n=0}^{\infty} \epsilon^n y_n(t), \; t \in \boldsymbol{R}. \tag{10}$$

Since

$$\sum_{n=0}^{\infty} |\epsilon^n| \boldsymbol{E}\left(\sup_{t \leq s \leq t+\delta} \| y_n(s) \|\right) \leq \sum_{n=0}^{\infty} |\epsilon|^n \frac{b^{n+1}LC^{2n}}{(1-ab)^{n+1}} \leq \frac{2bL}{1-ab}$$

for every $t \in \boldsymbol{R}$ and $|\epsilon| \leq \epsilon_0 := (1-ab)/(2bC^2)$, the series for y_ϵ converges uniformly on bounded subsets of $\boldsymbol{R}$ with a probability one. This shows that y_ϵ is continuous on $\boldsymbol{R}$ with a probability one stationary process.

By exactly the same arguments as those used above, we claim that the series for y'_ϵ, y''_ϵ are also absolutely and uniform convergent on bounded subsets of $\boldsymbol{R}$ with a probability one and we have

$$\epsilon y''_\epsilon(t) + y'_\epsilon(t) = \sum_{n=0}^{\infty} (\epsilon^{n+1} y''_n(t) + \epsilon^n y'_n(t))$$

$$= \sum_{n=0}^{\infty} \left[\epsilon^{n+1} \left(Ay_{n+1}(t) + \int_{-\infty}^{t} E(t-s) y_{n+1}(s) ds - y'_{n+1}(t) \right) + \epsilon^n y'_n(t) \right]$$

$$= \sum_{n=0}^{\infty} \epsilon^{n+1} A y_{n+1} + \sum_{n=0}^{\infty} \epsilon^{n+1} \int_{-\infty}^{t} E(t-s) y_{n+1} ds - \sum_{m=1}^{\infty} \epsilon^m y'_m(t) + \sum_{n=0}^{\infty} \epsilon^n y'_n(t)$$

$$= A \left(\sum_{m=1}^{\infty} \epsilon^m y_m(t) \right) + \int_{-\infty}^{t} E(t-s) \left(\sum_{m=1}^{\infty} \epsilon^m y_m(s) \right) ds + y_0(t)$$

$$= Ay_\epsilon(t) + \int_{-\infty}^{t} E(t-s) y_\epsilon(s) ds - Ay_0(t) - \int_{-\infty}^{t} E(t-s) y_0(s) ds + y'_0(t)$$

$$= Ay_\epsilon(t) + \int_{-\infty}^{t} E(t-s) y_\epsilon(s) ds + \xi(t), t \in \boldsymbol{R}.$$

Moreover, for every $t \in \boldsymbol{R}$, we have

$$\boldsymbol{E} \left(\sup_{t \le s \le t+\delta} \| y_\epsilon(s) - y_0(s) \| \right) \le \sum_{m=1}^{\infty} |\epsilon|^{m-1} \frac{b^{m+1} L C^{2m}}{(1-ab)^{m+1}} \le \frac{2b^2 L C^2}{(1-ab)^2} \epsilon,$$

if $|\epsilon| \le \epsilon_0$.

To complete the proof of Theorem 1 we need show only the uniqueness. It is sufficient to prove the following fact. If z is stationary connected with the process x_ϵ solution of (1), which satisfies

$$\boldsymbol{E} \left(\sup_{0 \le t \le \delta} \| z(t) \| \right) < +\infty, \quad \boldsymbol{E} \left(\sup_{0 \le t \le \delta} \| z'(t) \| \right) < +\infty,$$

then $z = x_\epsilon$. We apply Lemma 4 in the following way. The difference $u := x_\epsilon - z$ is a stationary process which satisfies the equation

$$\epsilon u''(t) + u'(t) = Ax(t) + \int_{-\infty}^{t} E(t-s) u(s) ds, \quad t \in \boldsymbol{R} \tag{11}$$

and

$$\boldsymbol{E}\left(\sup_{0 \le t \le \delta} \| u(t) \|\right) < +\infty, \quad \boldsymbol{E}\left(\sup_{0 \le t \le \delta} \| u'(t) \|\right) < +\infty.$$

Let us consider a Banach space B^2 of two vectors equipped with term-by-term linear operations and with the norm which is equal to the sum of the norms of the coordinates. Let

$$\boldsymbol{u}(t) := \begin{pmatrix} u' \\ u \end{pmatrix}, \quad \mathbb{A} := \begin{pmatrix} -\epsilon^{-1} & \epsilon^{-1}A \\ I & \Theta \end{pmatrix}, \quad \mathbb{E} := \begin{pmatrix} \Theta & \epsilon^{-1}E \\ \Theta & \Theta \end{pmatrix},$$

where Θ and I are the zero operator and identity operator on B, respectively. Then the following equation in B^2

$$\boldsymbol{u}'(t) = \mathbb{A}\boldsymbol{u}(t) + \int_{-\infty}^{t} \mathbb{E}(t-s)\boldsymbol{u}ds, \quad t \in \boldsymbol{R} \tag{12}$$

is equivalent to the equation (11) in B. By direct computation we obtain that condition

$$\sigma(\mathbb{A}) \cap i\boldsymbol{R} = \emptyset$$

is fulfilled if, for every $\alpha \in \boldsymbol{R}$, an operator $A - (i\alpha + \alpha^2\epsilon)I$ has a bounded inverse. For the justification of this assertion for all small $|\epsilon|$ it suffices to make use of condition (2) and the boundedness of operator A. Then, by Lemma 4, the equation (12) has a unique stationary solution and hence $\boldsymbol{u}(t) = \overline{0}$, $t \in \boldsymbol{R}$ with the probability of of one.

The proof is complete.

Remark 8: Let $B = \boldsymbol{R}$. It can be proven that the existence of expansion (10) for the solution of equation (4) leads to condition $\xi = C^{\infty}(\boldsymbol{R})$.

3. Time-Stationary Solutions of the Boundary Value Problem for PDE Containing a Parameter

Proof of Theorem 2: Let a process $\xi \in S(L, C, \delta)$ and a function $g \in C_0^3$ be given. Then, one can expand g as

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx, \quad x \in [0, \pi]; \quad \{g_k : k \ge 1\} \subset \boldsymbol{C},$$

where the series on the right-hand since is uniformly convergent. Note that

$$g_k = \frac{2}{k} \int_0^{\pi} g(x) \sin kx \, dx, k \ge 1.$$

Let $k \ge 1$ be fixed. From assumption (5) and Corollary 1, it follows that there is $\xi_k > 0$ such that for every ϵ with $|\epsilon| < \epsilon_k$, the equation

$$\epsilon v_k''(t;\epsilon) + v_k'(t;\epsilon) + k^2 v_k(t;\epsilon) = A v_k(t;\epsilon) + g_k \xi(t), t \in \boldsymbol{R} \tag{13}$$

has a unique stationary solution $v_k(\cdot\,;\epsilon)$ such that

$$\boldsymbol{E}\left(\sup_{t \in J} \| v_k(t;\epsilon) - v_k(t) \|\right) \rightarrow 0, \quad \epsilon \rightarrow 0,$$

where v_k is a unique stationary solution of the equation

$$v_k'(t) + k^2 v_k(t) = A v_k(t) + g_k \xi(t), t \in \boldsymbol{R},$$

and J is a bounded subset of $\boldsymbol{R}$. Moreover, for every $t \in \boldsymbol{R}$, we have

$$\boldsymbol{E}\left(\sup_{t \leq s \leq t+\delta} \| v_k(s;\epsilon) \|\right) \leq 2 \mid g_k \mid L L_{1,k} \tag{14}$$

and

$$\boldsymbol{E}\left(\sup_{t \leq s \leq t+\delta} \| v_k(s;\epsilon) - v_k(s) \|\right) \leq 2 \mid g_k \mid L L_{1,k}^2 C^2 \mid \epsilon \mid, \tag{15}$$

if $\mid \epsilon \mid \leq \epsilon_k$, where

$$L_{1,k} := \int_{\boldsymbol{R}} \| G_k(s) \| \, ds < +\infty$$

and G_k is Green's function for operator $A - k^2 I$; $k \geq 1$. It follows from the properties of G_k that

$$L_{1,k} \leq \frac{\widetilde{L}}{k^2 - k_0^2}, \quad k > k_0, \tag{16}$$

where a number $\widetilde{L}$ can be chosen to be independent of k.

Now we shall remark, that by virtue of boundedness of an operator A, the numbers ϵ_k, $k \geq 1$ are identifiable and not depending on k. Really, let k_0 be the least natural number such that a spectrum of an operator $A - (\alpha^2\epsilon - k_0^2)I$ resides in the left half-plane. Then the spectrum of an operator $A - (\alpha^2\epsilon - k^2)I$, $k \geq k_0$ also resides in the left half-plane and it is possible to put $\epsilon_0 := \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_{k_0}\} > 0$. Thus, for every ϵ, $\mid \epsilon \mid < \epsilon_0$, all equations (13) have a unique stationary solution.

Let us consider the series

$$u(t,x;\epsilon) := \sum_{k=1}^{\infty} v_k(t;\epsilon) \sin kx, \quad (t,x) \in Q \tag{17}$$

for $\mid \epsilon \mid \leq \epsilon_0$. It follows from (14) and (16) that

$$\sum_{k=1}^{\infty} \boldsymbol{E}\left(\sup_{t \leq s \leq t+\delta, 0 \leq x \leq \pi} \| v_k(t;\epsilon) \sin kx \|\right) \leq \sum_{k=1}^{\infty} 2 \mid g_k \mid L L_{1,k} < +\infty,$$

for every $t \in \boldsymbol{R}$ and $\mid \epsilon \mid \leq \epsilon_0$. This implies that the series (17) converges absolutely and uniformly on $[t, t+\delta] \times [0,\pi]$ with the probability one and the random function $u(\cdot\,, \cdot\,;\epsilon)$ is a continuous, time-stationary with respect of time variable, random functions. In addition,

$$\boldsymbol{E}\left(\sup_{0 \leq s \leq \delta, 0 \leq x \leq \pi} \| u(s,x;\epsilon) \|\right) < +\infty.$$

Using the above-mentioned reasoning, the following equalities are installed

$$u'_t(t,x;\epsilon):=\sum_{k=1}^{\infty}v'_k(t,\epsilon)\sin kx,$$

$$u''_{tt}(t,x;\epsilon):=\sum_{k=1}^{\infty}v''_k(t;\epsilon)\sin kx, \tag{18}$$

$$u''_{xx}(t,x;\epsilon):=\sum_{k=1}^{\infty}(-k^2)v_k(t;\epsilon)\sin kx,$$

for $(t,x)\in Q$ and uniform on $[t,t+\delta]\times[0,\pi]$ convergence with the probability one of an appropriate series for any $t\in \boldsymbol{R}$ and $|\epsilon|\le\epsilon_0$. We have also

$$\boldsymbol{E}\left(\sup_{0\le s\le\delta,0\le x\le\pi}\|u'_t(s,x;\epsilon)\|\right)<+\infty.$$

From (17), (18), and (13), it follows that

$$\epsilon u''_{tt}(t,x;\epsilon)+u'_t(t,x;\epsilon)-u''_{xx}(t,x;\epsilon)$$
$$=\sum_{k=1}^{\infty}(\epsilon v''_k(t;\epsilon)+v'_k(t;\epsilon)+k^2v_k(t;\epsilon))\sin kx$$
$$=\sum_{k=1}^{\infty}(Av_k(t;\epsilon)+g_k\xi(t))\sin kx$$
$$=Au(t,x;\epsilon)+g(x)\xi(t),(t,x)\in Q.$$

Hence, the random function $u(\cdot,\cdot;\epsilon)$ for ϵ with $|\epsilon|<\epsilon_0$ is a time-stationary solution of (6).

This solution is unique. To see this, we observe that for any $t\in\boldsymbol{R}$, the elements $\{v_k(t;\epsilon)\}$ are Fourier coefficients of $u(t,\cdot;\epsilon)\in C^2([0,\pi],B)$ which determine $u(t,\cdot;\epsilon)$ uniquely with the probability one. See, for example [3] for details. By Corollary 1, the solutions of (13) are also determined uniquely with a probability one.

Similarly, by repeating the above arguments, we conclude that random function

$$v(t,x):=\sum_{k=1}^{\infty}v_k(t)\sin kx,\quad (t,x)\in Q$$

is a unique, stationary with respect to time variable, solution of (7) and

$$\boldsymbol{E}\left(\sup_{t\le s\le t+\delta,0\le x\le\pi}\|v(s,x;\epsilon)\|\right)<+\infty$$

for every $t\in\boldsymbol{R}$. Note that the random functions $u(\cdot,\cdot;\epsilon)$, $|\epsilon|\le\epsilon_0$ and v are time-stationary connected.

Finally, let us consider the difference $u(\cdot,\cdot;\epsilon)-v(\cdot,\cdot)$ for $|\epsilon|<\epsilon_0$. By Corollary 1, the following inequalities

$$\boldsymbol{E}\left(\sup_{t\le s\le t+\delta,0\le x\le\pi}\|u(t,x;\epsilon)=v(t,x)\|\right)$$

$$\leq \sum_{k=1}^{\infty} \boldsymbol{E}\left(\sup_{t \leq s \leq t+\delta} \| v(t;\epsilon) - v_k(t) \| \right) \leq \sum_{k=1}^{\infty} 2L \mid g_k \mid L_{1,k}^2 C^2 \mid \epsilon \mid$$

hold.

Theorem 2 is proved.

References

[1] Arato, M., *Linear Stochastic Systems with Constant Coefficients. A Statistical Approach*, Springer-Verlag, Berlin-Heidelberg 1982.

[2] Bainov, D.D. and Kolmanovskii, V.B., Periodic solutions of stochastic functional equations, *Math. J. Toyama Univ.* **14**:1 (1991), 1-39.

[3] Dorogovtsev, A. Ya., *Periodic and Stationary Regimes of Infinite-Dimensional Deterministic and Stochastic Dynamically Systems*, Vissha Shkola, Kiev 1992 (in Russian).

[4] Dorogovtsev, A. Ya., Periodic processes: a survey of results, *Theory of Stoch. Proc.* **2**(18):3-4 (1996), 36-53.

[5] Dorogovtsev, A. Ya., Stability of stationary and periodic solution equations in Banach space, *J. of Appl. Math and Stoch. Anal.* **10**:3 (1997), 249-255.

[6] Dorogovtsev, A. Ya., Periodic distribution solution for a telegraph equation, *J. of Appl. Math and Stoch. Anal.* **12**:2 (1999), 121-131.

[7] Dorogovtsev, A. Ya., Stability of stationary solutions, *Dokl. Acad. Nauk* (Moscow) **369** (1999), 309-310.

[8] Dorogovtsev, A. Ya., Stationary solutions to boundary problem for the heat equations, *Hiroshima Math. J.* **30**:2 (2000), 191-203.

[9] Tolubinskii, E.V., *The Theory of Transposition Processes*, Naukova Dumka, Kiev 1969 (in Russian).

[10] Vishik, M.I. and Lusternik, L.A., Regular degeneration and boundary layer for linear differential equations with a small parameter, *Math. Surveys* **12**:5 (1957), 3-122 (in Russian).