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ON A GENERALIZED JACOBI TRANSFORM

JOSÉ SARABIA
UNEXPO
Sección de Matemática, Apartado Postal 352
Barquisimeto, Venezuela
E-mail: jsarabia196@hotmail.com.ve

S.L. KALLA
Kuwait University
Department of Mathematics and Computer Science
P.O. Box 5969, Safat 13060, Kuwait
E-mail: kalla@mcs.sci.kuniv.edu.kw

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In this paper, we study a generalized Jacobi transform and obtain images of certain functions under this transform. Furthermore, we define a Jacobi random variable and derive its moments, distribution function, and characteristic function.

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1. Introduction

Kalla et al. [4] have studied the following integral,

$$I^{a,b}_{\nu,\alpha,\beta} = \int_{-1}^{1} (1-x)^a(1+x)^b P_\nu^{(\alpha,\beta)}(x)dx \tag{1}$$

with $Re(a) > -1$, $Re(b) > -1$ and $P_\nu^{(\alpha,\beta)}$ is the Jacobi function, where

$$P_\nu^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_\nu}{\Gamma(\nu+1)} \cdot {}_2F_1\left(\begin{matrix} -\nu, \nu+\lambda \\ \alpha+1 \end{matrix} ; \frac{1-x}{2}\right) \tag{2}$$

and $\lambda = \alpha+\beta+1$. The authors considered its partial derivatives with respect to a and b. Some more general results were obtained in [6]. These results were extended by Sarabia [10] using the following integral:

$$I^{a,b,c,p}_{\nu,\alpha,\beta} = \int_{-1}^{1} (1-x)^a(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)dx, \tag{3}$$

where $P_\nu^{(\alpha,\beta)}$ is a generalized Jacobi function defined as

$$P_\nu^{(\alpha,\beta,c,p)}(x) = \frac{(\alpha+1)_\nu}{\Gamma(\nu+1)} \cdot {}_3F_2\left(\begin{matrix} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{matrix}; \frac{1-x}{2}\right), \tag{4}$$

where

$$P \in C - Z^- U\{0\}: \alpha, v \in C - Z^-; \beta \in C; Re(p-\beta-c) > 0. \tag{4.a}$$

Hence, $P_\nu^{(\alpha,\beta,c,p)}$ is continuous on $[-1,1]$. For $p = c$, (4) reduces to (2).

In this paper, we define a generalized Jacobi transform and obtain images of certain functions under this transform. Furthermore, we define a random variable and derive some statistical properties such as: moments, a distribution function, and characteristic function.

2. Generalized Jacobi Transform

Let f be a real-valued function defined on $[-1,1]$, with $Re(a) > -1$, $Re(b) > -1$, and conditions (4a) being held. Then the generalized Jacobi transform (GJT) of $f(x)$ is defined as

$$J_{\alpha,\beta}^{a,b,c,p}[f(x),\nu] = \int_{-1}^{1} (1-x)^a(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x) f(x)dx. \tag{5}$$

For continuous or sectionally continuous f on $[a,b]$, integral (5) exists. For $c = p$, (5) reduces to the well known Jacobi transform [3].

Now, we obtain images of some functions under the generalized Jacobi transform.

(i) For $f(x) = 1$, we have [9, 10]:

$$J_{\alpha,\beta}^{a,b,c,p}[1;\nu] = 2^{a+b+1}\frac{(\alpha+1)_\nu B(a+1,b+1)}{\Gamma(\nu+1)} \cdot {}_4F_3\left(\begin{matrix} -\nu, \nu+\lambda, c, a+1 \\ \alpha+1, p, a+b+2 \end{matrix}; 1\right), \tag{6}$$

(ii) $f(x) = ln(1-x)$. Since $f(x)$ is not piecewise continuous on $[-1,1]$, we have to demonstrate directly that (5) exists.

Indeed: $F(x,y) = (1-x)^y(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)$ and $D_2F(x,y) = ln(1-x)(1-x)^y (1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)$ are continuous on $D = [A,B] \times \bar{B}_R(a)$, where

$$[A,B] \subset (-1,1) \text{ and } \bar{B}_R(a) \subset S_\delta = \{a \in C: Re(a) \geq \delta > -1\}.$$

In addition, for $Re(y) > -1$ and $\bar{B}_R(a) \subset S_\delta$, $J_{\alpha,\beta}^{y,a,c,p}[1,\nu]$ exists.

Also, the constants $K_1 > 0$ and $K_2 > 0$ exist, such that:

$$|\, ln(1-x)(1-x)^y(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x) \,|$$

$$\leq M(x) = \begin{cases} -K_1 ln(1-x)(1-x)^{-\delta} & x \in (0,1] \\ K_2(1+x)^{Re(b)} & x \in [0,1]. \end{cases}$$

Then, by applying the M-Criterion of Weierstrass, we obtain:

$$\int_{-1}^{1} ln(1-x)(1-x)^y(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)dx,$$

which converges uniformly on $\bar{B}_R(a)$ and

$$D_2 \int_{-1}^{1} (1-x)^y (1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)dx$$

$$= \int_{-1}^{1} ln(1-x)(1-x)^y (1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)dx$$

on $\bar{B}_R(a)$. Then, for $a \in C$, so that $Re(a) > -1$, according to [13],

$$D_2 \int_{-1}^{1} (1-x)^a (1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)dx$$

$$= \int_{-1}^{1} ln(1-x)(1-x)^a (1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)dx. \tag{7}$$

By demonstrating that $J_{\alpha,\beta}^{a,b,c,p}[ln(1-x);\nu]$ exists and also

$$J_{\alpha,\beta}^{a,b,c,p}[ln(1-x);\nu] = \frac{\partial}{\partial \alpha} J_{\alpha,\beta}^{a,b,c,p}[1,\nu], \tag{8}$$

using (6) and (8), we get

$$J_{\alpha,\beta}^{a,b,c,p}[ln(1-x);\nu]$$

$$= \frac{\partial}{\partial \alpha}\left[\frac{(\alpha+1)_\nu \Gamma(b+1)2^{b+1}}{\Gamma(\nu+1)} \cdot 2^a \sum_{n=0}^{\infty} \frac{(-\nu)_n(\nu+\lambda)_n(c)_n\Gamma(a+1+n)}{(\alpha+1)_n(p)_n\Gamma(a+b+2+n)n!}\right],$$

and, hence

$$J_{\alpha,\beta}^{a,b,c,p}[ln(1-x);\nu] = ln2 \cdot J_{\alpha,\beta}^{a,b,c,p}[1;\nu]$$

$$+ \frac{(\alpha+1)_\nu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)} \cdot \sum_{n=0}^{\infty} \frac{(-\nu)_n(\nu+\lambda)_n(c_n(a+1)_n}{(\alpha+1)_n(p)_n n!(a+b+2)_n}$$

$$\cdot [\psi(a+1+n) - \psi(a+b+2+n)]. \tag{9}$$

We can see that (9) has as a particular case (2.2) from [4] when $c = p$. Likewise and relatively speaking, it is a more simple representation than [10, eq. (6)].

(iii) $f(x) = ln(1+x)$. Reasoning in a similar way as we did in the previous case, we have that if $Re(a)$, $Re(b) > -1$ and the conditions considered in (4.a) hold, then the existence of $J_{\alpha,\beta}^{a,b,c,p}[ln(1-x);v]$ can be easily proved. Also, by criterions of uniform convergence, we obtain:

$$J^{a,b,c,p}_{\alpha,\beta}[ln(1-x);\nu] = \frac{\partial}{\partial b}[J^{a,b,c,p}_{\alpha,\beta}[1;\nu]]$$

and

$$J^{a,b,c,p}_{\alpha,\beta}[ln(1-x);\nu] = [\phi(b+1) + ln2]J^{a,b,c,p}_{\alpha,\beta}[1;\nu]$$

$$-\frac{(\alpha+1)_\nu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n(a+1)_n}{(\alpha+1)_n(p)_n n!(a+b+2)_n}[\varphi(a+b+2)+S_n], \tag{10}$$

where $S_n = \sum_{k=0}^{n-1}\frac{1}{(a+b+2+k)}$ for $n \geq 1$, $S_0 = 0$.

Hence,

$$J^{a,b,c,p}_{\alpha,\beta}[ln(1-x);\nu] = [\phi(b+1) + ln2 - \phi(a+b+2)]J^{a,b,c,p}_{\alpha,\beta}[1;\nu]$$

$$-\frac{(\alpha+1)_\nu 2^{a+b+1}B(a+1,b+1)}{\Gamma(\nu+1)}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n(a+1)_n}{(\alpha+1)_n(p)_n n!(a+b+2)_n}S_n. \tag{11}$$

Clearly, the above results lead to

$$J^{a,b,c,p}_{\alpha,\beta}[ln(1-x);\nu] = [\Psi(b+1) + 2ln2 - \Psi(a+b+2)]J^{a,b,c,p}_{\alpha,\beta}[1;\nu]$$

$$+\frac{(\alpha+1)_\nu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n(a+1)_n}{(\alpha+1)_n(p)_n n!(a+b+2)_n}$$

$$[\Psi(a+b+2) - \Psi(a+b+2+n) - S_n] \tag{12}$$

and

$$J^{a,b,c,p}_{\alpha,\beta}[ln\left(\frac{1-x}{1+x}\right);\nu] = [\phi(a+b+2) - \phi(b+1)]]J^{a,b,c,p}_{\alpha,\beta}[1;\nu]$$

$$+\frac{(\alpha+1)_\nu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)}$$

$$\cdot\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n(a+1)_n}{(\alpha+1)_n(p)_n n!(a+b+2)_n}$$

$$[S_n + \Psi(a+1+n) - \Psi(a+b+2+n]. \tag{13}$$

(iv) $f(x) = (1-x)^A(1+x)^B$. Let $Re(a) > max\{-1, -Re(A)-1\}$, $Re(b) > max\{-1, -Re(B)-1\}$, that is to say $Re(a)$, $Re(b)$, $Re(a+A)$, $Re(b+B) > -1$; also let

$$Re(p-\beta-c) > 0;\ \alpha, \nu \in C - Z^-;\ \beta \in C. \tag{14}$$

Hence, equation (6) becomes:

$$J_{\alpha,\beta}^{a,b,c,p}[(1-x)^A(1+x)^B;\nu] = J_{\alpha,\beta}^{a+A,b+B,c,p}[1;\nu]. \tag{15}$$

Hence, we arrive at

$$J_{\alpha,\beta}^{a,b,c,p}[(1-x)^A(1+x)^B;\nu] = \frac{(\alpha+1)_\nu B(a+A+1,b+B+1)2^{a+A+b+B+1}}{\Gamma(\nu+1)}$$

$$\cdot\ {}_4F_3\left(\begin{matrix} -\nu, \nu+\lambda, c, a+A+1 \\ \alpha+1, p, a+A+b+B+2 \end{matrix}; 1\right). \tag{15.a}$$

(v) $f(x) = P_\mu^{(\gamma,\delta,d,q)}(x)$. Let

$$\eta = \gamma+\delta+1; q \in C - Z^- \cup \{O\}; \gamma, \mu \in C - Z; Re(q-\delta-d) > 0, \tag{16}$$

and let conditions of (4a) hold true. Then $f(x)$ is continuous on $[-1,1]$ and, therefore, its GJT exists. Consequently,

$$J_{\alpha,\beta}^{a,b,c,p}[P_\mu^{(\gamma,\delta,d,q)}(x);\nu] = \frac{(\gamma+1)_\mu}{\Gamma(\mu+1)}\int_{-1}^{1}(1-x)^a(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x)P_\mu^{(\gamma,\delta,d,q)}(x)dx.$$

Since the integral on interval $[r,R] \subset (-1,1)$ can be interchanged with the series (by uniform convergence) and since the transform exists, we can interchange the integral on $[-1,1]$ with the series by applying [1, (14.31)].

Hence,

$$J_{\alpha,\beta}^{a,b,c,p}[P_\mu^{(\gamma,\delta,d,q)}(x);\nu] = \frac{(\alpha+1)_\nu(\gamma+1)_\mu B(a+1,b+1)}{\Gamma(\nu+1)\Gamma(\mu+1)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{(-\mu)_n(\mu+\eta)_n(d)_n(a+1)_n 2^{a+b+n+1}}{(\gamma+1)_n(q)_n(a+b+2)_n n! 2^n}$$

$$\cdot\ {}_4F_3\left(\begin{matrix} -\nu, \nu+\lambda, c, a+n+1 \\ \alpha+1, p, a+b+n+2 \end{matrix}; 1\right),$$

and

$$J_{\alpha,\beta}^{a,b,c,p}[P_\mu^{(\gamma,\delta,d,q)}(x);\nu] = \frac{(\alpha+1)_\nu(\gamma+1)_\mu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)\Gamma(\mu+1)}$$

$$\cdot \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\mu)_n(\mu+\eta)_n(d)_n(a+1)_n(-\nu)_k(\nu+\eta)_k(c)_k(a+n+1)_k}{(\gamma+1)_n(q)_n(a+b+2)_n(\alpha+1)_k(p)_k(a+b+n+2)_k n!k!}. \tag{17}$$

Recollecting that $(h+n)_K = \frac{(h)_{n+K}}{(h)_n}$ and setting

$$H = \frac{(\alpha+1)_\nu(\gamma+1)_\mu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)\gamma(\mu+1)} \tag{18}$$

we get,

$$J_{\alpha,\beta}^{a,b,c,p}[P_\mu^{(\gamma,\delta,d,q)}(x);\nu]$$

$$= H\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-\mu)_n(\mu+\eta)_n(d)_n(a+1)_{n+k}(-\nu)_k(\nu+\lambda)_k(c)_k}{(\gamma+1)_n(q)_n(a+b+2)_{n+k}(\alpha+1)_k(p)_k n!k!}. \tag{19}$$

Using the Kampé de Fériet's hypergeometric function (see [8, p. 160]) and since the transform exists, we have:

$$J_{\alpha,\beta}^{a,b,c,p}[P_\mu^{(\gamma,\delta,d,q)}(x);\nu] = \frac{(\alpha+1)_\nu(\gamma+1)_\mu B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)\Gamma(\mu+1)}$$

$$\cdot F\left(\begin{array}{c|c|c} 1 & a+1 & \\ 3 & -\mu,-\nu;\mu+\eta,\nu+\lambda;d,c & \\ & & 1,1 \\ 1 & a+b+2 & \\ 2 & \gamma+1,\alpha+1;q,p & \end{array}\right). \tag{20}$$

3. Some Inequalities

In this section, we will obtain some inequalities to the GJT, based on the Luke's Inequality (see [7, p. 254], formula (6)):

$$(1-\theta z)^{-\sigma} < {}_{p+1}F_p\left(\begin{array}{c} \sigma;\alpha_1,\ldots,\alpha_p \\ \rho_1\cdots\rho_p \end{array};z\right) < 1-\theta+\theta(1-z)^{-\sigma}, \tag{21}$$

where $\theta = \prod_{j=1}^{p}\left(\frac{\alpha_j}{\rho_j}\right)$, with $0<z<1$; $\sigma>0$; $\rho_j \geq \alpha_j > 0 (j=1,\ldots,p)$.

We consider the following conditions:

$$0 > \nu > -\min\{1,1+\alpha\}; 0 < \lambda+\nu < \alpha+1;$$

$$0 < c \leq p; p > \beta + c; a+1, b > 0. \tag{22}$$

Then $0 < \theta = \frac{(\lambda+\nu)c}{(\alpha+1)p} < 1$, and since $0 \leq \frac{1-x}{2} \leq 1$, we obtain

$$1 \leq \left[1-\theta\left(\frac{1-x}{2}\right)\right]^{\nu} \leq {}_3F_2\left(\begin{matrix} -\nu, \lambda+\nu, c \\ \alpha+1, p \end{matrix}; \frac{1-x}{2}\right) \leq 1-\theta+\theta\left(\frac{1+x}{2}\right)^{\nu} \tag{23}$$

and

$$\frac{(\alpha+1)_{\nu}}{\Gamma(\nu+1)}(1-x)^a(1+x)^b \cdot {}_3F_2\left(\begin{matrix} -\nu, \lambda+\nu, c \\ \alpha+1, p \end{matrix}; \frac{1-x}{2}\right)$$

$$\leq \frac{(\alpha+1)_{\nu}}{\Gamma(\nu+1)}(1-x)^a(1+x)^b\left[1-\theta+\theta\left(\frac{1+x}{2}\right)^{\nu}\right]. \tag{24}$$

Integrating (24) between -1 and 1, and due to $0 \leq \theta\left(\frac{1-x}{2}\right) \leq \theta < 1$ we have the following, by applying criterions of uniform convergence and simplifying for $\frac{(\alpha+1)_{\nu}B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)}$, under the same conditions as that of (22),

$$1 \leq {}_2F_1\left(\begin{matrix} -\nu, a+1 \\ a+b+2 \end{matrix}; \theta\right) \leq {}_4F_3\left(\begin{matrix} -\nu, \nu+\lambda, c, a+1 \\ \alpha+1, p, a+b+2 \end{matrix}; 1\right)$$

$$\leq 1-\theta+\theta\frac{\Gamma(a+b+2)\Gamma(b+\nu+1)}{\Gamma(b+1)\Gamma(a+b+\nu+2)}. \tag{25}$$

Throughout this section, using (19) we will get a bound to the GJT of $f(x)$, which is continuous on $[-1,1]$.

Thus, let $M = \max\{\,|f(x)|: x \in [-1,1]\}$, then

$$|\,J_{\alpha,\beta}^{a,b,c,p}[f(x);\nu]\,| \leq MJ_{\alpha,\beta}^{a,b,c,p}[1;\nu]L \cdot {}_4F_3\left(\begin{matrix} -\nu, \nu+\lambda, c, a+1 \\ \alpha+1, p, a+b+2 \end{matrix}; 1\right)$$

$$\leq L\left[1-\theta-\theta\frac{\Gamma(a+b+2)\Gamma(b+\nu+1)}{\Gamma(b+1)\Gamma(a+b+\nu+2)}\right] \tag{26}$$

where,

$$L = \frac{M(\alpha+1)_{\nu}B(a+1,b+1)2^{a+b+1}}{\Gamma(\nu+1)}; \lambda = \alpha+\beta+1; \theta = \frac{(\lambda+\nu)c}{(\alpha+1)p},$$

along with conditions (22).

For example, let $f(x) = \frac{1}{(1+x^2)}$, $m = \frac{1}{2}$ (minimum) and $M = 1$ (maximum), with $a = 0$, $b = 1$; $-1 < \nu < -\frac{1}{2}$; $\alpha = \beta = 0$; $c = 1$, $p = 2$ and $0 < \theta < \frac{1}{4}$, then

$$1 \leq J_{0,0}^{0,1,1,2}\left[\frac{1}{1+x^2};\nu\right] \leq \frac{-\nu^2+\nu+4}{\nu+2}. \tag{27}$$

4. Generalized Jacobi Random Variable

Statistical distributions have been used in a variety of applications, including the field of reliability. Recently, Kalla et al. [5] have studied a unified form of gamma-type distributions. Here we define a Jacobi random variable and derive some statistical properties.

4.1 Density Function

We define the generalized Jacobi random variable with parameters $(\alpha, \beta, c, p, \nu; a, b)$ as a random variable, whose density function is given by

$$g_{a,b}^{(\alpha,\beta,c,p,\nu)}(x) = \begin{cases} K(1-x)^a(1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x) & \text{for } x \in [-1,1] \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

where

$$a, b > -1; \alpha, \beta, c > 0; p > \beta + c; -1 < \nu < 0. \tag{29}$$

Furthermore, $K = \frac{\Gamma(\nu+1)2^{-(a+b+1)}}{(\alpha+1)_\nu B(a+1,b+1)R}$ and

$$R = {}_4F_3\left(\begin{matrix} -\nu, \nu+\lambda, c, a+1 \\ \alpha+1, p, a+b+2 \end{matrix} ; 1 \right). \tag{30}$$

We can see that (27) becomes a density function of the family of β-random variables whose parameters are $(a+1, b+1)$, displaced on $x = 1 - 2u$, when $\nu = 0$.

Indeed

$$h(u) = g_{a,b}^{(\alpha,\beta,c,p,0)}(1-2u) = \frac{1}{B(a+1,b+1)} u^a (1-u)^b. \tag{31}$$

Hence, we can consider the generalized Jacobi random variable as a generalization of the β-random variable.

4.2 Moments of Order $m > 0$

Clearly, $E(X^0) = 1$. For this reason, we let $m > 0$ and condition (29) hold.

Thus, from [1, (14.31)], we have:

$$E(X^m) = K \cdot \sum_{s=0}^{\infty} \binom{m}{n} (-1)^s J_{\alpha,\beta}^{a,b,c,p}[(1-x)^s; \nu].$$

Then from (15a), for $A = s$, $B = 0$, we get:

$$E(X^m) = K\sum_{s=0}^{\infty}\frac{\binom{m}{s}(-1)^s(\alpha+1)_\nu B(a+s+1,b+1)2^{a+s+b+1}}{\Gamma(\nu+1)}$$
$$\cdot\, {}_4F_3\left(\begin{matrix}-\nu,\nu+\lambda,c,a+s+1\\ \alpha+1,p,a+s+b+2\end{matrix};1\right), \tag{32}$$

which can also be expressed as,

$$E(X^m) = \frac{1}{R}\cdot F\left(\begin{array}{c|c|c} 1 & a+1 & \\ 3 & -m,\,-\nu;1,\nu+\lambda;1,c & \\ & & 2,1 \\ 1 & a+b+2 & \\ 2 & 1,\alpha+1;1,p & \end{array}\right). \tag{33}$$

It is also possible to calculate the modified moment of the form, $E((I-X)^A(I+X)^B)$ with $A > -(1+a); B > -(1+b)$.

Indeed, from (15a), we have (with conditions (29)),

$$E((I-X)^A(I+B)^B) = \frac{2^{A+B}B(a+A+1,b+B+1)}{R\cdot B(a+1,b+1)}$$
$$\cdot\, {}_4F_3\left(\begin{matrix}-\nu,\nu+\lambda,c,a+A+1\\ \alpha+1,p,a+A+b+B+2\end{matrix};1\right). \tag{34}$$

4.3 Distribution Function

The distribution function of the generalized Jacobi random variable is defined as:

$$G_{a,b}^{(\alpha,\beta,c,p,\nu)}(x) = \int_{-\infty}^{x} g_{a,b}^{(\alpha,\beta,c,p,\nu)}(x)dx$$
$$= K\int_{-1}^{x}(1-t)^a(1+t)^b P_\nu^{(\alpha,\beta,c,p)}(t)dt,$$

for $|x| \leq 1$. Here and in the sequel, we abbreviate this distribution as $G(x)$.

Thus, by virtue of [1, (14.31)], we obtain:

$$G(x) = \frac{K(\alpha+1)_\nu}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n}{(\alpha+1)_n(p)_n n!2^n}\int_{-1}^{x}(1-t)^{a+n}(1+t)^b dt. \tag{35}$$

Moreover, since

$$\int_{-1}^{x}(1-t)^{a+n}(1+t)^b dt$$

$$= 2^{a+b+n+1}[B(a+n+1,b+1) - B_{\frac{1-x}{2}}(a+n+1,b+1)],$$

where $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function, $G(x) = G_1(x) - G_2(x)$, where:

$$G_1(x) = \frac{K(\alpha+1)_\nu 2^{a+b+1}B(a+1,b+1)}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n(a+1)_n}{(\alpha+1)_n(p)_n(a+b+2)_n n!} \tag{36}$$

$$= \frac{1}{R}\cdot {}_4F_3\left(\begin{matrix} -\nu,\nu+\lambda,c,a+1 \\ \alpha+1,p,a+b+2\end{matrix};1\right), \tag{37}$$

since $p - \beta - c + b + 1 > 0$.

On the other hand, due to [2, p. 87], we have that

$$B_x(p,q) = \frac{x^p}{p}\cdot {}_2F_1\left(\begin{matrix} p,1-q \\ p+1\end{matrix};x\right)(p,q>0;0<x<1). \tag{38}$$

Therefore,

$$G_2(x) = \frac{K(\alpha+1)_\nu}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{(-\nu)_n(\nu+\lambda)_n(c)_n 2^{a+b+n+1}}{(\alpha+1)_n(p)_n n! 2^n}B_{\frac{1-x}{2}}(a+n+1,b+1)$$

$$= \frac{K(\alpha+1)_\nu 2^{a+b+1}}{\Gamma(\nu+1)}\left(\frac{1-x}{2}\right)^{a+1}$$

$$\cdot\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a+1)_{n+k}(-\nu)_n(\nu+\lambda)_n(c)_n(-b)_k}{(a+2)_{n+k}(\alpha+1)_n(p)_n n! k!}\left(\frac{1-x}{2}\right)^{n+k}, \tag{39}$$

and by using Kampé de Feriét function, (39) can be expressed as,

$$G_2(x) = \frac{1}{B(a+1,b+1)R}\left(\frac{1-x}{2}\right)^{a+1}$$

$$\cdot F\left(\begin{matrix} 1 \\ 3 \\ \\ 1 \\ 2\end{matrix}\left|\begin{matrix} a+1 \\ -\nu,-b;1,\nu+\lambda;c,1 \\ \\ a+2 \\ \alpha+1,1;p,1\end{matrix}\right|\frac{1-x}{2};\frac{1-x}{2}\right). \tag{39.a}$$

Then from (36) and (39.a), the distribution function reduces to:

$$G_{a,b}^{(\alpha,\beta,c,p,\nu)}(x) = \frac{1}{R}\left[{}_4F_3\left(\begin{matrix} -\nu,\nu+\lambda,c,a+1 \\ \alpha+1,p,a+b+2\end{matrix};1\right)\right.$$

$$-\frac{1}{B(a+1,b+1)}\left(\frac{1-x}{2}\right)^{a+1}$$

$$\cdot F\left(\begin{array}{c|c|c} 1 & a+1 & \\ 3 & -\nu,\,-b;\nu+\lambda,1;c,1 & \\ & & \frac{1-x}{2};\frac{1-x}{2} \\ 1 & a+2 & \\ 2 & \alpha+1,1;p,1 & \end{array}\right)\Bigg]. \tag{40}$$

4.4 Characteristic Function

For a random variable X with continuous density function $f(x)$, the characteristic function of X is defined by:

$$\varphi_X(t)=\int_{-\infty}^{\infty} e^{itx}f(x)dx=\sqrt{2\pi}\mathfrak{F}[f(x);t], \tag{41}$$

where $\mathfrak{F}$ is the Fourier's transform of $f(x)$.

For the generalized Jacobi random variable, under the conditions given in (29), we have

$$\varphi_X(t)=\frac{K(\alpha+1)_\nu}{\Gamma(\nu+1)}\int_{-1}^{1} e^{itx}(1-x)^a(1+x)^b$$

$$\cdot\,{}_3F_2\left(\begin{array}{c} -\nu,\nu+\lambda,c \\ \alpha+1,p \end{array};\frac{1-x}{2}\right)dx. \tag{42}$$

If we set $u=\frac{1-x}{2}$ in (42), it becomes

$$\varphi_X(t)=\frac{K(\alpha+1)_\nu 2^{a+b+1}}{\Gamma(\nu+1)}\int_{-1}^{1} e^{it(1-2u)}u^a(1-u)^b$$

$$\cdot\,{}_3F_2\left(\begin{array}{c} -\nu,\nu+\lambda,c \\ \alpha+1,p \end{array};u\right)du. \tag{43}$$

As $F(u)={}_3F_2\left(\begin{array}{c} -\nu,\nu+\lambda,c \\ \alpha+1,p \end{array};u\right)$ and it is continuous on $[0,1]$, due to $p-\beta-c>0$, $M_1,M_2>0$ exist so that:

$$\mid e^{-2itu}u^a(1-u)^bF(u)\mid\ \le h(t)=\begin{cases} M_1(1-u)^b & \text{in } [\frac{1}{2},1] \\ M_2u^a & \text{in } [0,\frac{1}{2}] \end{cases}.$$

Furthermore, $h(t)$ is integrable on $[0,1]$, since $a>-1$ and $b>0>-1$, and this ensures the convergence of (43).

Also, from the uniform convergence of $\sum_{n=0}^{\infty}\frac{(-1)^n(2itu)^n}{n!}=e^{-2itu}$ on $[r,s]$, $\forall r,s$

with $0 < r \le s \le 1$, we have that [1, (14.31)] is applicable. Hence,

$$\varphi_X(t) = \frac{K(\alpha+1)_\nu 2^{a+b+1} e^{it}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(-1)^n (2it)^n}{n!} \int_0^1 u^{a+n}(1-u)^b$$

$$\cdot {}_3F_2\left(\begin{matrix} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{matrix} ; u \right) du. \tag{44}$$

Using [7, p. 161 (2)], with $p = 3 = q+1$, $a+1 > 0$, $b > 0$, and $p - \beta - c > 0$, we have the following:

$$\varphi_X(t) = \frac{K(\alpha+1)_\nu 2^{a+b+1} B(a+1, b+1) e^{it}}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(a+1)_n (-2it)^n}{(a+b+2)_n n!}$$

$$\cdot {}_4F_3\left(\begin{matrix} a+n+1, -\nu, \nu+\lambda, c \\ a+b+n+2, \alpha+1, p \end{matrix} ; 1 \right). \tag{45}$$

Finally, $\varphi_X(t)$ can be expressed in terms of the Kampé de Feriét's function,

$$\varphi_X(t) = \frac{e^{it}}{R} \cdot F^{1:3;0}_{1:2;0}\left[\begin{matrix} a+1; -\nu, \nu+\lambda, c; -; \\ a+b+2: \alpha+1, p; -; \end{matrix} \; 1, -2it \right]. \tag{46}$$

Observing that, with $t = -i\tau$ in (46), we have the moment generating function of X. Indeed,

$$M_X(\tau) = \frac{e^{\tau}}{R} \cdot F^{1:3;0}_{1:2;0}\left[\begin{matrix} a+1; -\nu, \nu+\lambda, c; -; \\ a+b+2: \alpha+1, p; -; \end{matrix} \; 1, -2\tau \right]. \tag{47}$$

The above expression can be reached by e^{xt} expansion in

$$M_X(t) = \int_{-1}^{1} e^{tx} K(1-x)^a (1+x)^b P_\nu^{(\alpha,\beta,c,p)}(x) dx.$$

Similarly, we have the Fourier transform of

$$(1-x)^a(1+x)^b {}_3F_2\left(\begin{matrix} -\nu, \nu+\lambda, c \\ \alpha+1, p \end{matrix} ; \frac{1-x}{2} \right)$$

from (46) and (41). Indeed:

$$\mathfrak{F}\left[(1-x)^a(1+x)^b{}_3F_2\left(\begin{matrix} -\nu,\nu+\lambda,c \\ \alpha+1,p \end{matrix};\frac{1-x}{2}\right);t\right]$$

$$=\frac{(\alpha+1)_\nu B(a+1,b+1)2^{a+b+1}e^{it}}{\sqrt{2\pi}\Gamma(\nu+1)}\cdot F^{1:3;0}_{1:2;0}\left[\begin{matrix} a+1:-\nu,\nu+\lambda,c-; \\ a+b+2:\alpha+1,p;-; \end{matrix} 1,-2\tau\right]. \tag{48}$$

4.5 The Distribution of $X+X'$, Where Both of Them are Independent Generalized Jacobi Random Variables

Let X be a generalized Jacobi random variable whose parameters are $(\alpha,\beta,c,p,\nu;a,b)$ and X' be a generalized Jacobi random variable whose parameters are $(\alpha',\beta',c',p',\nu';a',b')$, under the additional assumption that both group of parameters satisfy (29) and that these random variables are independent.

Since $\varphi_{X+X'}(t)=\varphi_X(t)\varphi_{X'}(t)$, from (43) we have:

$$\varphi_{X+X'}(t)=\frac{e^{2it}}{R\cdot R'}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}P(a,b,\nu,\lambda,c,\alpha,p,n,k)$$

$$P(a',b',\nu',\lambda',c',\alpha',n',k')(-2it)^{n+n'}. \tag{49}$$

Here the change of order is made possible due to the absolute convergence of the intermediate series of the quadruple series as well. Likewise:

$$P=P(a,b,\nu,\lambda,c,\alpha,p,n,k)=\frac{(a+1)_{n+k}(-\nu)_k(\nu+\lambda)_k(c)_k}{(a+b+2)_{n+k}(\alpha+1)_k(p)_k n!k!}. \tag{50}$$

Then, from (41), we have that the density function of $X+X'$ in the form:

$$f_{X+X'}(x)=\frac{1}{\sqrt{2\pi}}\mathfrak{F}^{-1}[\varphi_{X+X'}(t);x] \tag{51}$$

Because of uniform convergence, we have from (51) the following result:

$$f_{X+X'}(x)=\frac{1}{\sqrt{2\pi}RR'}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}(-2i)^{n+n'}PP'\mathfrak{F}^{-1}[e^{2it}t^{n+n'};x]. \tag{52}$$

Using [8, p. 517(19)], we have from (50) that

$$f_{X+X'}(x)=\frac{1}{RR'}\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\sum_{k=0}^{\infty}\sum_{k'=0}^{\infty}(-2)^{n+n'}PP'\delta^{(n+n')}(2-x), \tag{53}$$

where $\delta(x)$ is the Dirac generalized function (see [3, p. 11-13] and [11, p. 484-504]).

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