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INTEGRODIFFERENTIAL EQUATIONS WITH ANALYTIC SEMIGROUPS [1]

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In this paper we study a class of integrodifferential equations considered in an arbitrary Banach space. Using the theory of analytic semigroups we establish the existence, uniqueness, regularity and continuation of solutions to these integrodifferential equations.

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1 Introduction

In this paper we are concerned with the following integrodifferential equation in a Banach space X:

$$\begin{aligned} \frac{du(t)}{dt} + Au(t) &= f(t, u(t)) + K(u)(t), \quad t > t_0, \\ u(t_0) &= u_0, \end{aligned} \tag{1.1}$$

where

$$K(u)(t) = \int_{t_0}^{t} a(t-s)g(s, u(s))ds. \tag{1.2}$$

In (1.1), we assume that $-A$ generates an analytic semigroup, $S(t)$, $t \geq 0$ on X, the function a is real-valued and locally integrable on $[0, \infty)$, and the nonlinear maps f and g are defined on $[0, \infty) \times X$ into X. We first establish that under **Assumption F0**, stated below, there exists a unique local mild solution to (1.1). Then under **Assumption F**, stated below, we study the regularity of the mild solution to (1.1) and show under additional condition of Hölder continuity on a that the mild solution to (1.1) is in fact the classical solution. Further, we analyze the continuation of the solutions to (1.1)

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under different conditions. Finally, at the end we give an example of a class of parabolic integrodifferential equations as an application of the results obtained for the abstract integrodifferential equation (1.1).

Equation (1.1) represents an abstract formulation of certain classes of parabolic integrodifferential equations. These type of equations model the physical phenomena involving certain type of memory effects. For instance, Nohel [12] has considered a nonlinear Volterra equation of the type (1.1), in which $g(t, u(t)) = Bu(t)$, where $-B$ is a nonlinear accretive operator. For more details on such formulations and corresponding techniques used to study such problems, we refer to Bahuguna and Pani [2], Barbu [4, 5, 6], Crandall, Londen and Nohel [7].

Heard and Rankin [9] have considered the following integrodifferential equation in a Banach space X:

$$\begin{aligned} \frac{du(t)}{dt} + A(t)u(t) &= f(t, u(t)) + \int_{t_0}^{t} a(t, s)g(s, u(s))ds, \quad t_0 < t < T, \\ u(t_0) &= u_0 \end{aligned} \tag{1.3}$$

where the linear operator $-A(t)$ for each $t \geq 0$ is the infinitesimal generator of an analytic semigroup in X, the nonlinear map g, defined on $[0, \infty) \times D(A(0))$ into X, is such that $g(t, .)$ is Lipschitz continuous on the domain $D(A(0))$ of $A(0)$ into X with respect to the graph norm of $A(0)$, the nonlinear map f, defined on $[0, \infty) \times X_\alpha$ into X, satisfies the condition that there exist constants $L > 0$, $0 < \eta, \gamma \leq 1$ and $0 < \alpha < 1$ such that

$$\|f(t, x) - f(s, y)\| \leq L[|t - s|^\eta + \|x - y\|_\alpha^\gamma] \tag{1.4}$$

for all $(t, x), (t, y) \in [0, \infty) \times X_\alpha$. Here X_α for $0 \leq \alpha \leq 1$ is the Banach space $D(A^\alpha)$ endowed with the norm $\|u\|_\alpha = \|A^\alpha u\|$.

Webb [15] has also considered (1.3) and has assumed that f maps $\mathbf{R} \times X_1$ into X_α and for each $t \in \mathbf{R}$ there exists a positive constant $C(t)$ such that

$$\|f(t, x) - f(t, y)\|_\alpha \leq C(t)\|x - y\|_1. \tag{1.5}$$

for all $x, y \in X_1$.

The existence result is proved by first solving the following integrodifferential equation uniquely:

$$\begin{aligned} \frac{du_v(t)}{dt} + A(t)u_v(t) &= f(t, v(t)) + \int_{t_0}^{t} a(t, s)g(s, u_v(s))ds, \quad t_0 < t < T, \\ u_v(t_0) &= u_0 \end{aligned} \tag{1.6}$$

where $v(t)$ is chosen from a closed, bounded, convex subset S of an appropriate Banach space. Existence of a unique $u(t)$ is established by proving that the mapping $K(v) = u_v$ is a strict contraction from S into S. This is possible because of the extra smoothness assumption (1.5) on f. Since Heard and Rankin [9] assumed a weaker condition (1.4), they require an estimate of the following type on the map K:

$$\|K(v_1) - K(v_2)\|_{C(J;X_1)} \leq C\|v_1 - v_2\|_{C(J;X_\alpha)}^\gamma + \epsilon \tag{1.7}$$

and they use Schauder's fixed point theorem to establish the existence.

When (1.4) is replaced by the stronger assumption (2.1), stated below, the methods used by Heard and Rankin [9] do not automatically imply that the solution is unique. They are able to prove uniqueness in the case in which X is a Hilbert space. Furthermore, the nonlinear map g is assumed to be defined from $[t_0, T) \times W$ into X where W is an open subset of X_1 and satisfies the local Lipschitz condition:

$$\|g(t,x) - g(s,y)\| \leq b_0 \|x - y\| \tag{1.8}$$

for all $t, s \in [t_0, T)$ and $x, y \in B_1(y_0; r) = \{z \in X_1 \ : \ \|z - y_0\|_1 \leq r\}$.

2 Preliminaries

Let X denote a Banach space and let J denote the closure of the interval $[t_0, T)$ $t_0 < T \leq \infty$. Let $-A$ be the infinitesimal generator of an analytic semigroup $S(t),\ t \geq 0$ in X. We note that if $-A$ is the infinitesimal generator of an analytic semigroup then $-(A + \alpha I)$ is invertible and generates a bounded analytic semigroup for $\alpha > 0$ large enough. This allows us to reduce the general case in which $-A$ is the infinitesimal generator of an analytic semigroup to the case in which the semigroup is bounded and the generator is invertible. Hence for convenience, we suppose that

$$\|S(t)\| \leq M \quad \text{for} \quad t \geq 0$$

and

$$0 \in \rho(-A),$$

where $\rho(-A)$ is the resolvent set of $-A$. It follows that for $0 \leq \alpha \leq 1$, A^α can be defined as a closed linear invertible operator with its domain $D(A^\alpha)$ being dense in X. We denote by X_α the Banach space $D(A^\alpha)$ equipped with norm

$$\|x\|_\alpha = \|A^\alpha x\|$$

which is equivalent to the graph norm of A^α. We have

$$X_\beta \hookrightarrow X_\alpha \quad \text{for} \quad 0 < \alpha < \beta$$

and the embedding is continuous.

By a *classical solution* to (1.1) on J, we mean a function $u \in C(J; X) \cap C^1(J \backslash \{t_0\}; X)$ satisfying (1.1) on J. By a *local classical solution* to (1.1) on J we mean that there exist a T_0, $t_0 < T_0 < T$, and a function u defined from $J_0 = [t_0, T_0]$ into X such that u is a classical solution to (1.1) on J_0.

To establish the existence of a unique classical solution to (1.1) in later sections, we shall require the following assumption on the maps f and g.

Assumption F: Let U be an open subset of $[0, \infty) \times X_\alpha$. A function f is said to satisfy **Assumption F** if for every $(t, x) \in U$ there exist a neighborhood $V \subset U$ of (t, x) and constants $L > 0,\ 0 < \theta < 1$ such that

$$\|f(s_1, u) - f(s_2, v)\| \leq L[|s_1 - s_2|^\theta + \|u - v\|_\alpha] \tag{2.1}$$

for all (s_1, u) and (s_2, v) in V.

By a *mild solution* to (1.1) on J we mean a continuous function u defined from J into X satisfying the following integral equation

$$u(t) = S(t-t_0)u_0 + \int_{t_0}^{t} S(t-s)[f(s,u(s) + K(u)(s)]ds, \quad t \in J. \tag{2.2}$$

We say that (1.1) has a *local mild solution* if there exist a T_0, $0 < T_0 < T$ and a continuous function u defined from $J_0 = [t_0, T_0]$ into X such that u is a mild solution to (1.1) on J_0.

To establish the existence of a unique local mild solution, we only need the following assumptions on f and g.

Assumption F0: Let U be an open subset of $[0,\infty) \times X_\alpha$. The function f is said to satisfy **Assumption F0** if for every $(t,x) \in U$ there exist a neighborhood $V \subset U$ of (t,x) and a constant $L_0 > 0$ such that

$$\|f(s,u) - f(s,v)\| \leq L_0 \|u - v\|_\alpha \tag{2.3}$$

for all (s,u) and (s,v) in V.

3 Local Existence of Mild Solutions

As pointed out earlier, we may suppose without loss of generality that the analytic semigroup generated by $-A$ is bounded and that $-A$ is invertible. Furthermore, we assume that $0 < T < \infty$ to establish local existence. With these simplifications we have the following theorem.

Theorem 3.1: *Suppose that the operator $-A$ generates the analytic semigroup $S(t)$ with $\|S(t)\| \leq M$, $t \geq 0$ and that $0 \in \rho(-A)$. If the maps f and g satisfy* **Assumption F0** *and the real valued map a is integrable on J, then* (1.1) *has a unique local mild solution for every $u_0 \in X_\alpha$.*

Proof: We shall use the notions and notations introduced in the preceding section. We fix a point (t_0, u_0) in the open subset U of $[0,\infty) \times X_\alpha$ and choose $t_1' > t_0$ and $\delta > 0$ such that (2.3), with some constant $L_0 > 0$ holds for the functions f and g on the set

$$V = \{(t,x) \in U \;:\; t_0 \leq t \leq t_1',\; \|x - u_0\|_\alpha \leq \delta\}. \tag{3.1}$$

Let

$$B_1 = \sup_{t_0 \leq t \leq t_1'} \|f(t,u_0)\|$$

and

$$B_2 = \sup_{t_0 \leq t \leq t_1'} \|g(t,u_0)\|.$$

Choose $t_1 > t_0$ such that

$$\|S(t-t_0) - I\| \|A^\alpha u_0\| \leq \frac{1}{2}\delta, \quad \text{for} \quad t_0 \leq t \leq t_1 \tag{3.2}$$

and

$$t_1 - t_0 < \min\left\{t_1' - t_0, [\frac{\delta}{2}{C_\alpha}^{-1}(1-\alpha)\{(L_0\delta + B_1) + a_T(L_0\delta + B_2)\}^{-1}]^{\frac{1}{1-\alpha}}\right\} \tag{3.3}$$

where C_α is a positive constant depending on α satisfying

$$\|A^\alpha S(t)\| \le C_\alpha t^{-\alpha}, \quad \text{for} \quad t > t_0, \tag{3.4}$$

and

$$a_T = \int_0^T |a(s)| ds. \tag{3.5}$$

Let $Y = C([t_0, t_1]; X)$ be endowed with the supremum norm

$$\|y\|_Y = \sup_{t_0 \le t \le t_1} \|y(t)\|.$$

Then Y is a Banach space. We define a map on Y by $Fy = \tilde{y}$ where $\tilde{y}$ is given by

$$\tilde{y}(t) = S(t-t_0)A^\alpha u_0 + \int_{t_0}^t A^\alpha S(t-s)[f(s, A^{-\alpha}y(s)) + \int_{t_0}^s a(s-\tau) g(\tau, A^{-\alpha}y(\tau)) d\tau] ds.$$

Now, for every $y \in Y$, $Fy(t_0) = A^\alpha u_0$ and for $t_0 \le s \le t \le t_1$ we have

$$\begin{aligned} Fy(t) - Fy(s) &= [S(t-t_0) - S(s-t_0)]A^\alpha u_0 \\ &\quad + \int_s^t A^\alpha S(t-\tau)[f(\tau, A^{-\alpha}y(\tau)) \\ &\quad + \int_{t_0}^\tau a(\tau-\eta) g(\eta, A^{-\alpha}y(\eta)) d\eta] d\tau \\ &\quad + \int_{t_0}^s A^\alpha [S(t-\tau) - S(s-\tau)][f(\tau, A^{-\alpha}y(\tau)) \\ &\quad + \int_{t_0}^\tau a(\tau-\eta) g(\eta, A^{-\alpha}y(\eta)) d\eta] d\tau. \end{aligned} \tag{3.6}$$

It follows from **Assumption F0** on the functions f and g, (3.4) and (3.5) that $F : Y \longrightarrow Y$.

Let $\mathcal{S}$ be the nonempty closed and bounded set given by

$$\mathcal{S} = \{ y \in Y \;:\; y(t_0) = A^\alpha u_0, \; \|y(t) - A^\alpha u_0\| \le \delta \}. \tag{3.7}$$

Then for $y \in \mathcal{S}$ we have

$$\begin{aligned} \|Fy(t) - A^\alpha u_0\| &\le \|S(t-t_0) - I\| \, \|A^\alpha u_0\| \\ &\quad + \int_{t_0}^t \|A^\alpha S(t-s)\| \, \|f(s, A^{-\alpha}y(s)) - f(s, u_0)\| ds \\ &\quad + \int_{t_0}^t \|A^\alpha S(t-s)\| \\ &\qquad [\int_{t_0}^s |a(s-\tau)| \, \|g(\tau, A^{-\alpha}y(\tau)) - g(\tau, u_0)\| d\tau] ds \\ &\quad + \int_{t_0}^t \|A^\alpha S(t-s)\| \, \|f(s, u_0)\| ds \\ &\quad + \int_{t_0}^t \|A^\alpha S(t-s)\| \, [\int_{t_0}^s |a(s-\tau)| \, \|g(\tau, u_0)\| d\tau] ds \\ &\le \frac{1}{2}\delta + C_\alpha (1-\alpha)^{-1} [(L_0\delta + B_1) + a_T(L_0\delta + B_2)](t_1 - t_0)^{1-\alpha} \\ &\le \delta \end{aligned} \tag{3.8}$$

where the last two inequalities follow from (3.2) and (3.3). Thus, we have that $F : \mathcal{S} \longrightarrow \mathcal{S}$. Now we show that F is a strict contraction on $\mathcal{S}$ which will ensure the existence of a unique continuous function satisfying equation (2.2). Let y and z be in $\mathcal{S}$; then

$$\begin{aligned} \|Fy(t) &- Fz(t)\| \\ &= \|\tilde{y}(t) - \tilde{z}(t)\| \\ &\leq \int_{t_0}^{t} \|A^{\alpha} S(t-s)\| \, \|f(s, A^{-\alpha}y(s)) - f(s, A^{-\alpha}z(s))\| ds \\ &\quad + \int_{t_0}^{t} \|A^{\alpha} S(t-s)\| \\ &\quad\quad [\int_{t_0}^{s} |a(s-\tau)| \, \|g(\tau, A^{-\alpha}y(\tau)) - g(\tau, A^{-\alpha}z(\tau))\| d\tau] ds. \end{aligned} \tag{3.9}$$

Using **Assumption F0** on f and g and (3.4), (3.5), we get

$$\begin{aligned} \|Fy(t) - Fz(t)\| &\leq L_0[(1 + a_T) \int_{t_0}^{t} \|A^{\alpha} S(t-s)\| ds] \|y - z\|_Y \\ &\leq L_0 (1 + a_T) C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \|y - z\|_Y \\ &\leq \frac{1}{\delta} L_0 \delta (1 + a_T) C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \|y - z\|_Y \\ &\leq \frac{1}{\delta} [L_0 \delta + B_1 + a_T (L_0 \delta + B_2)] C_{\alpha} (1-\alpha)^{-1} (t_1 - t_0)^{1-\alpha} \|y - z\|_Y \\ &\leq \frac{1}{2} \|y - z\|_Y, \end{aligned} \tag{3.10}$$

using (3.3) in the last inequality. Thus F is a strict contraction map from $\mathcal{S}$ into $\mathcal{S}$ and therefore by the Banach contraction principle there exists a unique fixed point y of F in $\mathcal{S}$, i.e., there is a unique $y \in \mathcal{S}$ such that

$$Fy = y = \tilde{y}. \tag{3.11}$$

Let $u = A^{-\alpha} y$. Then for $t \in [t_0, t_1]$, we have

$$\begin{aligned} u(t) &= A^{-\alpha} y(t) \\ &= S(t - t_0) u_0 + \int_{t_0}^{t} S(t-s)[f(s, u(s)) + K(u)(s)] ds. \end{aligned} \tag{3.12}$$

Hence u is a unique local mild solution to (1.1).

4 Regularity of Mild Solutions

In this section we establish the regularity of the mild solutions to (1.1). Again, let J denote the closure of the interval $[t_0, T)$, $t_0 < T \leq \infty$. In addition to the hypotheses mentioned in the earlier sections, we assume the following on the kernel a:

(H) There exist constants $C_0 \geq 0$ and $0 < \beta \leq 1$ such that

$$|a(t) - a(s)| \leq C_0 |t - s|^{\beta}$$

for all $t, s \in J$.

Theorem 4.1: *Suppose that $-A$ generates the analytic semigroup $S(t)$ such that $\|S(t)\| \leq M$ for $t \geq 0$, and $0 \in \rho(-A)$. Further, suppose that the maps f and g satisfy* **Assumption F** *and the kernel a satisfies* **(H)**. *Then* (1.1) *has a unique local classical solution for each $u_0 \in X_\alpha$.*

Proof: From Theorem 3.1, it follows that there exist T_0, $t_0 < T_0 < T$ and a function u such that u is a unique mild solution to (1.1) on $J_0 = [t_0, T_0)$ given by

$$u(t) = S(t-t_0)u_0 + \int_{t_0}^{t} S(t-s)[f(s,u(s)) + K(u)(s)]ds, \quad t \in J_0, \tag{4.1}$$

where

$$K(u)(t) = \int_{t_0}^{t} a(t-s)g(s,u(s))ds.$$

Let

$$v(t) = A^{\alpha}u(t). \tag{4.2}$$

Then

$$\begin{aligned} v(t) &= S(t-t_0)A^{\alpha}u_0 \\ &\quad + \int_{t_0}^{t} A^{\alpha}S(t-s)[f(s,A^{-\alpha}v(s)) + \int_{t_0}^{s} a(s-\tau)g(\tau,A^{-\alpha}v(\tau))d\tau]ds. \end{aligned} \tag{4.3}$$

For simplification, we set

$$\tilde{f}(t) = f(t, A^{-\alpha}v(t)), \tilde{g}(t) = g(t, A^{-\alpha}v(t)). \tag{4.4}$$

Then (4.3) can be rewritten as

$$\begin{aligned} v(t) &= S(t-t_0)A^{\alpha}u_0 \\ &\quad + \int_{t_0}^{t} A^{\alpha}S(t-s)[\tilde{f}(s) + \int_{t_0}^{s} a(s-\tau)\tilde{g}(\tau)d\tau]ds. \end{aligned} \tag{4.5}$$

Since $u(t)$ is continuous on J_0 and the maps f and g satisfy **Assumption F**, it follows that $\tilde{f}$ and $\tilde{g}$ are continuous, and therefore bounded on J_0. Let

$$N_1 = \sup_{t \in J_0} \|\tilde{f}(t)\| \text{ and } N_2 = \sup_{t \in J_0} \|\tilde{g}(t)\|. \tag{4.6}$$

We show that $\tilde{f}$ and $\tilde{g}$ are locally Hölder continuous on J_0. For this, we first show that $v(t)$ is locally Hölder continuous on J_0. From Theorem 2.6.13 in Pazy [13], it follows that for every $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have

$$\begin{aligned} \|(S(h)-I)A^{\alpha}S(t-s)\| &\leq C_\beta h^\beta \|A^{\alpha+\beta}S(t-s)\| \\ &\leq Ch^\beta (t-s)^{-(\alpha+\beta)}. \end{aligned} \tag{4.7}$$

Next, we have

$$\begin{aligned} \|v(t+h) - v(t)\| &\leq \|(S(h)-I)S(t-t_0)A^{\alpha}u_0\| \\ &\quad + \int_{t_0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\| \, \|\tilde{f}(s) + \int_{t_0}^{s} a(s-\tau)\tilde{g}(\tau)d\tau\|ds \\ &\quad + \int_{t}^{t+h} \|A^{\alpha}S(t+h-s)\| \, \|\tilde{f}(s) + \int_{t_0}^{s} a(s-\tau)\tilde{g}(\tau)d\tau\|ds. \end{aligned} \tag{4.8}$$

Now,

$$\|(S(h)-I)S(t-t_0)A^{\alpha}u_0\| \le C(t-t_0)^{-(\alpha+\beta)}h^{\beta} \le M_1 h^{\beta}, \tag{4.9}$$

where M_1 depends on t and blows up as t decreases to t_0. Furthermore

$$\begin{aligned} & \int_{t_0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\| \, \|\tilde{f}(s)+\int_{t_0}^{s} a(s-\tau)\tilde{g}(\tau)d\tau\|ds \\ \le & \quad [N_1 + a_{T_0}N_2T_0]h^{\beta}C_{\beta}\int_{t_0}^{t}(t-s)^{-(\alpha+\beta)}ds \\ \le & \quad M_2 h^{\beta}, \end{aligned} \tag{4.10}$$

where M_2 is independent of t. Also, we have

$$\begin{aligned} & \int_{t}^{t+h} \|A^{\alpha}S(t+h-s)\| \, \|\tilde{f}(s)+\int_{t_0}^{s} a(s-\tau)\tilde{g}(\tau)d\tau\|ds \\ \le & \quad [N_1 + a_{T_0}N_2T_0]C_{\alpha}\int_{t}^{t+h}(t+h-s)^{-\alpha}ds \\ \le & \quad M_3 h^{\beta}, \end{aligned} \tag{4.11}$$

where M_3 is also independent of t. From the estimates (4.9)–(4.11), it follows that there exists a constant C_1 such that for every $t_0' > t_0$, we have

$$\|v(t)-v(s)\| \le C_1|t-s|^{\beta}, \tag{4.12}$$

for all $t_0 < t_0' < t, s < T_0$. Now, **Assumption F** together with (4.12) implies that there exist constants $C_2, C_3 \ge 0$ and $0 < \gamma, \eta < 1$ such that for all $t_0 < t_0' < t, s < T_0$, we have

$$\begin{aligned} \|\tilde{f}(t)-\tilde{f}(s)\| & \le C_2|t-s|^{\gamma}, \\ \|\tilde{g}(t)-\tilde{g}(s)\| & \le C_3|t-s|^{\eta}. \end{aligned} \tag{4.13}$$

Let

$$h(t) = \tilde{f}(t) + \int_{t_0}^{t} a(t-\tau)\tilde{g}(\tau)d\tau. \tag{4.14}$$

Now we show that $h(t)$ is locally Hölder continuous on J_0. For $s \le t$, we have

$$\begin{aligned} \|h(t)-h(s)\| &= \|\tilde{f}(t)-\tilde{f}(s)\| + \int_{t_0}^{s} |a(t-\tau)-a(s-\tau)| \, \|\tilde{g}(\tau)\| \, d\tau \\ & \quad + \int_{s}^{t} |a(t-\tau)| \, \|\tilde{g}(\tau)\| \, d\tau \\ &\le C_2|t-s|^{\gamma} + N_2C_0T_0|t-s|^{\beta} + N_2a_{T_0}(2T_0)^{1-\beta}|t-s|^{\beta} \\ &\le C_4|t-s|^{\delta} \end{aligned} \tag{4.15}$$

for some constants $C_4 \ge 0$ and $0 < \delta < 1$. Consider the following initial value problem

$$\begin{aligned} \frac{dv(t)}{dt} + Av(t) &= h(t), \quad t > t_0, \\ v(t_0) &= u_0. \end{aligned} \tag{4.16}$$

By Corollary 4.3.3 in Pazy [13], (4.16) has a unique solution $v \in C^1((t_0, T_0]; X)$ given by

$$v(t) = S(t - t_0)u_0 + \int_{t_0}^{t} S(t - s)h(s)ds. \tag{4.17}$$

For $t > t_0$, each term on the right hand side belongs to $D(A)$ and hence belongs to $D(A^\alpha)$. Applying A^α to both sides of (4.17) and using the uniqueness of $v(t)$, we have that $A^\alpha v(t) = u(t)$. Thus, it follows that u is the classical solution to (1.1) on J_0.

5 Global Existence

In order to establish the global existence of classical solutions to (1.1), we need the following lemma.

Lemma 5.1: *Let $\phi(t, s) \geq 0$ be continuous on $0 \leq s \leq t \leq T < \infty$. If there are positive constants A, B and α such that*

$$\phi(t, s) \leq A + B \int_{s}^{t} (t - \sigma)^{\alpha - 1} \phi(\sigma, s) d\sigma, \tag{5.1}$$

for $0 \leq s < t \leq T$, then there is a constant C such that

$$\phi(t, s) \leq C.$$

Proof: For $0 \leq s < t \leq T$, we have

$$\int_{s}^{t} (t - \tau)^{\alpha - 1} (\tau - s) d\tau = (t - s)^{\alpha + \beta - 1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \tag{5.2}$$

which holds for every $\alpha, \beta > 0$. Integrating (5.1) $n - 1$ times using (5.2) and replacing $t - s$ by T, we get

$$\phi(t, s) \leq A \sum_{j=0}^{n-1} \left(\frac{BT^\alpha}{\alpha} \right)^j + \frac{(B\Gamma(\alpha))^n}{\Gamma(n\alpha)} \int_{s}^{t} (t - \sigma)^{n\alpha - 1} \phi(\sigma, s) d\sigma. \tag{5.3}$$

Let n be large enough so that $n\alpha > 1$. We majorize $(t - \sigma)^{n\alpha - 1}$ by $T^{n\alpha - 1}$ to obtain

$$\phi(t, s) \leq c_1 + c_2 \int_{s}^{t} \phi(\sigma, s) d\sigma. \tag{5.4}$$

Application of Gronwall's inequality leads to

$$\phi(t, s) \leq c_1 e^{c_2(t-s)} \leq c_1 e^{c_2 T} \leq C. \tag{5.5}$$

This completes the proof of the lemma.

The following theorem establishes the global existence of classical solutions to (1.1).

Theorem 5.2: *Let $0 \in D(A)$ and let $-A$ be the infinitesimal generator of an analytic semigroup $S(t)$ satisfying*

$$\|S(t)\| \leq M$$

for $t \geq t_0$. *Let* $f, g : [t_0, \infty) \times X_\alpha \longrightarrow X$ *satisfy* **Assumption F** *and let the kernel* a *satisfy* **(H)**. *If there exist continuous nondecreasing functions* k_1 *and* k_2 *from* $[t_0, \infty)$ *into* $[0, \infty)$ *such that*

$$\begin{aligned} \|f(t,x)\| &\leq k_1(t)(1 + \|x\|_\alpha) \quad \text{for} \quad t \geq t_0, x \in X_\alpha, \\ \|g(t,x)\| &\leq k_2(t)(1 + \|x\|_\alpha) \quad \text{for} \quad t \geq t_0, x \in X_\alpha, \end{aligned} \tag{5.6}$$

then the initial value problem (1.1) *has a unique classical solution* u *on* $[t_0, \infty)$ *for every* $u_0 \in X_\alpha$.

Proof: From Theorem 4.1 it follows that there exist a T_0, $t_0 < T_0$ and a unique classical solution u on $J_0 = [t_0, T_0]$. If

$$\|u(t)\|_\alpha \leq C \tag{5.7}$$

for $t \in J_0$ for some positive constant C, then the solution $u(t)$ may be continued further on the right of T_0. Therefore it suffices to prove that if a classical solution u to (1.1) exists on $[t_0, T]$, $t_0 < T < \infty$ then $\|u(t)\|_\alpha$ is bounded as $t \uparrow T$. Since $u(t)$ is a classical solution as well it is a mild solution. Therefore we have

$$\begin{aligned} u(t) &= S(t - t_0)u_0 \\ &+ \int_{t_0}^{t} S(t-s)[f(s, u(s)) + \int_{t_0}^{s} a(s-\tau) g(\tau, u(\tau)) d\tau] ds. \end{aligned} \tag{5.8}$$

Making use of the fact that $S(t)$ commutes with A and that

$$\|S(t)\| \leq M,$$
$$\|A^\alpha S(t)\| \leq C_\alpha t^{-\alpha}$$

for $t \geq t_0$ in (5.8), after applying A^α and taking norms on both sides, we get

$$\begin{aligned} \|u(t)\|_\alpha &\leq M\|A^\alpha u_0\| \\ &+C_\alpha \int_{t_0}^{t} (t-s)^{-\alpha} \|f(s, u(s)) + \int_{t_0}^{s} a(s-\tau) g(\tau, u(\tau)) d\tau\| \, ds. \end{aligned} \tag{5.9}$$

For the last term in (5.9), we have the estimate

$$\int_{t_0}^{s} |a(s-\tau)| \; \|g(\tau, u(\tau)) d\tau\| \leq a_T k_2(T) \int_{t_0}^{s} [1 + \|u(\tau)\|_\alpha] d\tau. \tag{5.10}$$

Incorporating the estimate of (5.10) in (5.9), we get

$$\begin{aligned} \|u(t)\|_\alpha &\leq M\|A^\alpha u_0\| \\ &+C_\alpha [k_1(T) + a_T k_2(T)] \int_{t_0}^{t} (t-s)^{-\alpha} \\ &\qquad [1 + \|u(s)\|_\alpha + \int_{t_0}^{s} (1 + \|u(\tau)\|_\alpha) d\tau] ds. \end{aligned} \tag{5.11}$$

After a slight modification in (5.11), we get

$$\begin{aligned} \|u(t)\|_\alpha &\leq M\|A^\alpha u_0\| \\ &+C_\alpha (1+T)[k_1(T) + a_T k_2(T)] \frac{T^{1-\alpha}}{1-\alpha} \\ &+ \int_{t_0}^{t} (t-s)^{-\alpha} [\|u(s)\|_\alpha + \int_{t_0}^{s} \|u(\tau)\|_\alpha d\tau] ds. \end{aligned} \tag{5.12}$$

The estimate in (5.12) is of the type

$$\begin{aligned}\|u(t)\|_\alpha &\le C_1 \\ &\quad + C_2\int_{t_0}^{t}(t-s)^{-\alpha}[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]ds, \qquad (5.13)\end{aligned}$$

for some positive constants C_1 and C_2 depending on α and T only. Integrating (5.13) over (t_0, t), we get

$$\begin{aligned}\int_{t_0}^{t}\|u(\xi)\|_\alpha d\xi &\le C_1 T \\ &\quad + C_2\int_{t_0}^{t}\int_{t_0}^{\xi}(\xi-s)^{-\alpha}[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]ds d\xi. \quad (5.14)\end{aligned}$$

Changing the order of integration in (5.14), we obtain

$$\begin{aligned}\int_{t_0}^{t}\|u(\xi)\|_\alpha d\xi &\le C_1 T \\ &\quad + C_2\int_{t_0}^{t}\int_{s}^{t}(\xi-s)^{-\alpha}[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]d\xi ds. \quad (5.15)\end{aligned}$$

We rewrite (5.15) as

$$\begin{aligned}\int_{t_0}^{t}\|u(\xi)\|_\alpha d\xi &\le C_1 T \\ &\quad +C_2\int_{t_0}^{t}(\int_{t_0}^{\xi}(\xi-s)^{-\alpha}d\xi)[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]ds \\ &\le C_1 T + \frac{C_2 T}{1-\alpha}\int_{t_0}^{t}(t-s)^{-\alpha}[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]ds. \quad (5.16)\end{aligned}$$

The estimate (5.16) is of the form

$$\begin{aligned}\int_{t_0}^{t}\|u(\xi)\|_\alpha d\xi &\le C_3 \\ &\quad +C_4\int_{t_0}^{t}(t-s)^{-\alpha}[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]ds, \qquad (5.17)\end{aligned}$$

for some positive constants C_3 and C_4, depending on α and T only. Adding (5.13) and (5.17), we have

$$\begin{aligned}\|u(t)\|_\alpha + \int_{t_0}^{t}\|u(\xi)\|_\alpha d\xi &\le C_5 \\ &\quad +C_6\int_{t_0}^{t}(t-s)^{-\alpha}[\|u(s)\|_\alpha + \int_{t_0}^{s}\|u(\tau)\|_\alpha d\tau]ds, \quad (5.18)\end{aligned}$$

for some positive constants C_5 and C_6, depending on α and T only. Applying Lemma 5.1 to (5.18), we conclude that

$$\|u(t)\| \le C$$

on $[t_0, T]$. This completes the proof of the theorem.

6 Applications

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the linear partial differential operator

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha. \tag{6.1}$$

where $a_\alpha(x)$ is a real or complex valued function defined on $\bar{\Omega}$ for each multi-index α. We assume that $A(x, D)$ is strongly elliptic, i.e., there exists a constant $c > 0$ such that

$$\sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha \geq c|\xi|^{2m} \tag{6.2}$$

for all $x \in \bar{\Omega}$ and $\xi \in \mathbf{R}^n$. Consider the parabolic integrodifferential equation

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &+ A(x, D)u(x, t) = f(x, t, u(x, t), Du(x, t), \cdots, D^{2m-1}u(x, t)) \\
&+ \int_{t_0}^{t} a(t - s) g(x, s, u(x, s), Du(x, s), \cdots, D^{2m-1}u(x, s))\, ds, \qquad (6.3) \\
& \quad x \in \Omega,\ t > t_0, \\
u(x, t_0) &= u_0(x) \quad x \in \Omega \\
u(x, t) &= 0, \quad x \in \Omega,\ t \in [t_0, T),\ t_0 < T \leq \infty,
\end{aligned}$$

where D^j stands for any j-th order derivative. We assume that f and g are continuously differentiable functions of all their variables, except possibly in x.

The parabolic integrodifferential equation (6.3) can be reformulated as the following abstract integrodifferential equation in $X = L^p(\Omega)$:

$$\begin{aligned}
\frac{du(t)}{dt} + A_p u(t) &= F(t, u(t)) + \int_{t_0}^{t} a(t - s) G(s, u(s))\, ds, \quad t > t_0, \qquad (6.4) \\
u(t_0) &= u_0,
\end{aligned}$$

where $A_p : D(A_p) \subset X \longrightarrow X$ given by

$$D(A_p) = W^{2m,p}(\Omega) \cap W_0^{2m,p}(\Omega), \quad A_p u = A(x, D)u + \lambda u \quad \text{for} \quad u \in D(A_p),\ \lambda > 0$$

and $F, G : [t_0, T) \times D(A_p) \longrightarrow X$ are the Nemyckii operators given by

$$F(t, u)(x) = f(x, t, u(x, t), Du(x, t), ..., D^{2m-1}u(x, t)) \tag{6.5}$$

$$G(t, u)(x) = g(x, t, u(x, t), Du(x, t), ..., D^{2m-1}u(x, t)) \tag{6.6}$$

where we assume the usual sufficient Caratheodory and growth conditions on the functions f and g for the Nemyckii operators in (6.5) and (6.6) to be well defined. Here we assume that λ is large enough so that A_p is invertible. It follows that $-A_p$ is the infinitesimal generator of an analytic semigroup on X. Also, from imbedding theorems it follows that X_α is continuously imbedded in $C^{2m-1}(\bar{\Omega})$ for $1 - 1/2m < \alpha < 1$ and p large enough. It can be verified that the **Assumption F** is satisfied by F and G. Under suitable assumptions on the kernel a, Theorem 4.1 assures the existence of a unique global classical solution to (6.4) for p large enough which in turn guarantees the existence of a unique global classical solution to (6.3).

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