A DESCRIPTION OF STOCHASTIC SYSTEMS USING CHAOTIC MAPS

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Received 28 August 2003 and in revised form 4 February 2004

Let $\rho(x,t)$ denote a family of probability density functions parameterized by time t. We show the existence of a family $\{\tau_t : t > 0\}$ of deterministic nonlinear (chaotic) point transformations whose invariant probability density functions are precisely $\rho(x,t)$. In particular, we are interested in the densities that arise from the diffusions. We derive a partial differential equation whose solution yields the family of chaotic maps whose density functions are precisely those of the diffusion.

1. Introduction

In this paper, we establish a method for describing flows of probability density functions by means of discrete-time chaotic maps. We start with a standard map whose invariant probability density function is known and then use it to derive other invariant probability density functions by a simple conjugation process which solves the inverse Perron-Frobenius problem [2, 3] in a time-varying setting.

2. Notation and preliminary results

In this paper, we consider space to consist of 1 dimension although the extension to 2 and 3 dimensions is straightforward. In the sequel, we will need some notions from ergodic theory and nonlinear dynamics, which can be found in [1].

Let $\mathbb{R} = (-\infty, \infty)$ and let $T : \mathbb{R} \to \mathbb{R}$ possess a unique absolutely continuous invariant measure μ which has the probability density function f, that is,

$$\int_A f\,dx = \int_{T^{-1}A} f\,dx \tag{2.1}$$

for any measurable set $A \subset \mathbb{R}$. The Perron-Frobenius operator P_T acting on the space of integrable functions is defined by

$$\int_A f\,dx = \int_{T^{-1}A} P_T f\,dx. \tag{2.2}$$

Journal of Applied Mathematics and Stochastic Analysis 2004:2 (2004) 137–141
2000 Mathematics Subject Classification: 28D05, 60G05
URL: http://dx.doi.org/10.1155/S1048953304308026

The operator P_T transforms probability density functions into probability density functions under the transformation T, where T is assumed to be nonsingular. One of the most important properties of P_T is that its fixed points are the densities of measures invariant under T [1].

Let $h : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism. Then $\tau = h^{-1} \circ T \circ h$ is a transformation from $\mathbb{R}$ into $\mathbb{R}$, which is differentiably conjugate to T and whose probability density function is given by

$$k = (f \circ h) \cdot |h'|. \tag{2.3}$$

We assume that T is a piecewise monotonic expanding C^1 map on $\mathbb{R}$ that admits a unique absolutely continuous invariant measure. Then the invariant density function $f(x)$ is a fixed point of the Perron-Frobenius operator P_T [1]. We now consider the inverse Perron-Frobenius problem: suppose we are given a probability density function $g(x)$ on $\mathbb{R}$, can we find a transformation τ such that $g(x)$ is the unique probability density function invariant under τ? This problem has been dealt with by Ershov and Malinetskiĭ [2] and in [3] from a computational perspective.

We solve the inverse Perron-Frobenius problem by applying (2.3), that is, we find h such that

$$(f \circ h) \cdot h' = g, \tag{2.4}$$

where we have assumed, without loss of generality, that h is an increasing function on $\mathbb{R}$. Now, let

$$F(x) = \int_{-\infty}^{x} f(y)dy \tag{2.5}$$

be the distribution function associated with f. Then, from (2.4) and the change-of-variable formula, we have

$$F(h(x)) = \int_{-\infty}^{x} g(y)dy. \tag{2.6}$$

Since F is a monotonically increasing function, it has a unique inverse and

$$h(x) = F^{-1}\left(\int_{-\infty}^{x} g(y)dy\right). \tag{2.7}$$

Thus, we have found $h(x)$ such that $\tau = h^{-1} \circ T \circ h$ has the probability density function $g(x)$. Summarizing, given any probability density function $g(x)$, we have proven the existence of a point transformation τ whose probability density function is $g(x)$.

Example 2.1. Let

$$T(x) = a \tan x, \quad x \neq \frac{k\pi}{2}, \ k = \pm 1, \pm 3, \ldots, \tag{2.8}$$

and $a>1$. Then the probability density function invariant under T is [1]

$$f(x) = \frac{p}{\pi(p^2+x^2)}, \tag{2.9}$$

where $p>0$ satisfies the equation $a\tanh(p) = p$. For $p>2$, $a \approx p$ and we can assume that $T(x) = p\tan x$. Hence, for $p=4$, say,

$$\begin{aligned} f(x) &= \frac{4}{\pi(16+x^2)}, \\ F(x) &= \int_{-\infty}^{x} \frac{4}{\pi(16+y^2)}dy = \frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{x}{4}\right), \\ F^{-1}(x) &= -4\cot(\pi x). \end{aligned} \tag{2.10}$$

Now, suppose we want to find a map τ whose unique invariant probability density function is given by $g(x) = (\cos 8x)\exp(-3x^2)$. We obtain

$$\int_{-\infty}^{x} (\cos 8x)\exp(-3x^2)dy = \frac{2\exp(-2x^2)(2/\pi)\cos^2(8x)}{1+\exp(-64/3)} \tag{2.11}$$

from which we can determine $h(x)$ using (2.7). Once h is known, so is $\tau = h^{-1}\circ T\circ h$ whose probability density function is $g(x)$.

The foregoing method can be extended to a family of probability density functions $\{g_t(y) : t \in I\}$. In this case, the homeomorphism h becomes a family of homeomorphisms $\{h_t : t \in I\}$ parameterized by t, where

$$h_t(x) = F^{-1}\left(\int_{-\infty}^{x} g_t(y)dy\right). \tag{2.12}$$

3. Chaotic map description of diffusions

Consider the diffusion equation

$$\frac{\partial\rho(x,t)}{\partial t} = -\nabla\left(v(x,t)\rho(x,t)\right) = -\frac{\partial}{\partial x}\left[b(x,t)\rho(x,t) - \frac{1}{2}\frac{\partial\rho(x,t)}{\partial x}\right], \tag{3.1}$$

where $b(x,t)$ is the forward drift coefficient. Our objective is to prove the existence of a family of point transformations $\{\tau_t \in \Gamma : t>0\}$ whose invariant probability density functions are $\{\rho_t : t>0\}$. To do this, we let T be the transformation defined by (2.8) and we derive a partial differential equation for $h_t(x)$ such that $\{\tau_t = h_t^{-1}\circ T\circ h_t\}$ possesses $\{\rho(x,t) : t>0\}$ as the associated family of invariant probability density functions.

Since

$$h_t(x) = F^{-1}\left(\int_{-\infty}^{x} \rho(y,t)dy\right), \tag{3.2}$$

we have

$$\frac{\partial}{\partial t}h_t(x) = (F^{-1})'\left(\int_{-\infty}^{x}\rho(y,t)dy\right)\left(\int_{-\infty}^{x}\frac{\partial}{\partial t}\rho(y,t)dy\right). \tag{3.3}$$

Noting that

$$(F^{-1})'\left(\int_{-\infty}^{x}\rho(y,t)dy\right) = \frac{1}{F'\left(F^{-1}\left(\int_{-\infty}^{x}\rho(y,t)dy\right)\right)} \tag{3.4}$$

and using (3.1), we obtain

$$\frac{\partial}{\partial t}h_t(x) = \frac{1}{f(h_t)}\int_{-\infty}^{x}\left[-\frac{\partial}{\partial y}b(y,t)\rho(y,t) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\rho(y,t))\right]dy. \tag{3.5}$$

Thus,

$$f(h_t(x))\frac{\partial}{\partial t}h_t(x) = -b(x,t)\rho(x,t) + \frac{1}{2}\frac{\partial}{\partial x}(\rho(x,t)) + q(t), \tag{3.6}$$

where $q(t)$ is an unknown function of t only. To find $q(t)$, we return to (3.3) and write

$$f(h_t(x))\frac{\partial}{\partial t}h_t(x) = \frac{\partial}{\partial t}\int_{-\infty}^{x}\rho(y,t)dy. \tag{3.7}$$

We assume that $\rho(y,t)$ and $(\partial/\partial x)(\rho(x,t))$ both go to 0 as $x \to \infty$; then the right-hand side of (3.7) also goes to 0 since

$$\int_{-\infty}^{\infty}\rho(y,t)dy = 1 \tag{3.8}$$

for all $t \geq 0$. Hence, $q(t) = 0$ for all $t \geq 0$. Thus, (3.6) reduces to

$$f(h_t(x))\frac{\partial}{\partial t}h_t(x) = -b(x,t)f(h_t(x))\frac{\partial}{\partial t}h_t(x) + \frac{1}{2}\frac{\partial}{\partial x}\left(f(h_t(x))\frac{\partial}{\partial t}h_t(x)\right) \tag{3.9}$$

or

$$(1+b(x,t))f(h_t(x))\frac{\partial}{\partial t}h_t(x) = \frac{1}{2}\frac{\partial}{\partial x}\left(f(h_t(x))\frac{\partial}{\partial t}h_t(x)\right), \tag{3.10}$$

whose solution is the family of homeomorphisms $\{h_t\}$ which determine the family of deterministic chaotic maps $\{\tau_t = h_t^{-1} \circ T \circ h_t\}$, whose probability density functions are equal to $\rho(x,t)$.

Acknowledgment

This research has been supported by Natural Sciences and Engineering Research Council of Canada (NSERC) grants.

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