A STROOCK FORMULA FOR A CERTAIN CLASS OF LÉVY PROCESSES AND APPLICATIONS TO FINANCE

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We find a Stroock formula in the setting of generalized chaos expansion introduced by Nualart and Schoutens for a certain class of Lévy processes, using a Malliavin-type derivative based on the chaotic approach. As applications, we get the chaotic decomposition of the local time of a simple Lévy process as well as the chaotic expansion of the price of a financial asset and of the price of a European call option. We also study the behavior of the tracking error in the discrete delta neutral hedging under both the equivalent martingale measure and the historical probability.

1. Introduction

In general, a Lévy process has not the chaotic representation property (say CRP for brevity) but Nualart and Schoutens have developed in [8] a kind of generalized CRP for a large class of Lévy processes. This work enabled to define a Malliavin-type derivative using the chaotic approach in a recent paper of Léon *et al.* [6]. The main goal of the present article is to get a Stroock formula in this setting. This formula gives the kernels of the chaotic decomposition of smooth random variables as functionals of the underlying Lévy process. For a complete survey on Lévy processes, we refer to [2, 14].

We apply this formula to obtain the chaos decomposition of some functionals of simple Lévy processes. These processes are the sum of a Wiener process and m independent Poisson processes. As it is pointed by the name, they are easy to handle and very useful for doing simulations since they approximate the square integrable compound Poisson process in the $L^2(\Omega \times [0,T])$ sense (see [6]). In particular, they approximate the process given by the sum of jumps with size greater than $\varepsilon > 0$.

In this paper, firstly, we obtain the decomposition of the local time $L(t,x)$ defined as the density of the occupation measure. Secondly, taking account that these simple Lévy processes have been studied in [6] for pricing and hedging options in financial markets driven by such processes, we apply the Stroock formula to obtain the chaos expansion of an asset price as well as the price of a European call option based on this asset. The chaotic approach enable us to study the asymptotic behavior of the variance, since the terms of the chaotic expansion are orthogonal and in particular uncorrelated, and this is useful in practical hedging.

The paper is organized as follows. The second section is devoted to recall some definitions and results related to Lévy chaotic calculus as well as to give some new remarks. In the third section, we get the Stroock formula. The fourth section is devoted to apply it in order to obtain the chaotic expansion of the local time of a simple Lévy process. Finally, the last three sections discuss chaotic expansions for the price of a financial asset and the price of a European call option, in a simple jump-diffusion model as well as the asymptotic behavior of the variance of the tracking error for discrete delta neutral hedging with respect to the mesh of the subdivision and under both the equivalent martingale measure and the historical probability.

2. Basic elements of Lévy chaotic calculus

2.1. The Teugels martingales family associated to a Lévy process. Let $X = \{X_t : t \geq 0\}$ be a real-valued Lévy process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Henceforth, we always assume that we are using the càdlàg version. Let $\{\mathcal{F}_t : t \geq 0\}$ be the natural filtration of X completed with the $\mathbb{P}$-null sets of $\mathcal{F}$. We also assume that the Lévy measure ν of X satisfies the following condition: there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\int_{(-\varepsilon,\varepsilon)^c} e^{\delta|x|} \nu(dx) < +\infty, \tag{2.1}$$

where $(-\varepsilon,\varepsilon)^c$ stands for the complement of the interval $(-\varepsilon,\varepsilon)$. This implies that X_t has moments of all orders and that the polynomials are dense in $L^2(\mathbb{R}, \mathbb{P} \circ X_1^{-1})$ (see [8]).
    Define

$$X_t^{(1)} = X_t, \qquad X_t^{(i)} = \sum_{0 < s \leq t} (\Delta X_s)^i, \quad i \geq 2. \tag{2.2}$$

We have the following.
   (i) The processes $X^{(i)} = \{X_t^{(i)} : t \geq 0\}$, $i = 1, 2, \ldots$, are Lévy processes that jump at the same points as X.
   (ii) $\mathbb{E}[X_t^{(i)}] = m_i t$, where $m_1 = \mathbb{E}[X_1]$ and $m_i = \int_{-\infty}^{\infty} x^i \nu(dx)$, $i \geq 2$.
   Now define

$$Y_t^{(i)} = X_t^{(i)} - m_i t, \quad i = 1, 2, \ldots. \tag{2.3}$$

 (iii) The processes $Y^{(i)} = \{Y_t^{(i)} : t \geq 0\}$ are martingales. The predictable quadratic covariation process of $Y^{(i)}$ and $Y^{(j)}$ is given by

$$\langle Y^{(i)}, Y^{(j)} \rangle_t = m_{i+j} t, \quad i, j \geq 2. \tag{2.4}$$

Now, we introduce the so-called Teugels's martingales,

$$H_t^{(i)} = \sum_{j=1}^{i} a_{i,j} Y_t^{(j)}, \quad i = 1, 2, \ldots, \tag{2.5}$$

where the constants $a_{i,j}$ are chosen in such a way that $a_{i,1} = 1$ and the martingales $H^{(i)}$, $i = 1, 2, \ldots$ are strongly orthogonal, that means, for $i \neq j$, the process $H^{(i)} H^{(j)}$ is a martingale.

In particular, since $\langle H^{(i)}, H^{(j)} \rangle_t$ is a predictable process such that $H_t^{(i)} H_t^{(j)} - \langle H^{(i)}, H^{(j)} \rangle_t$ is a martingale, we have that

$$\langle H^{(i)}, H^{(j)} \rangle_t = 0, \quad i \neq j. \tag{2.6}$$

Moreover, we have the following.

(iv) The processes $\{H_t^{(i)} : t \geq 0\}$ are martingales with predictable quadratic variation process given by

$$\langle H^{(i)}, H^{(i)} \rangle_t = q_i t, \tag{2.7}$$

where

$$q_i = \sum_{j,j'=1,\dots,i} a_{i,j} a_{i,j'} m_{j+j'} + a_{i,1}^2 \sigma^2. \tag{2.8}$$

2.2. Iterated integrals and a generalized CRP. Let $\Sigma_n = \{(t_1,\dots,t_n) \in \mathbb{R}_+^n : 0 < t_1 < t_2 < \cdots < t_n\}$ be the positive simplex of $\mathbb{R}^n$. Given $f \in L^2(\Sigma_n)$, we will denote by $J_n^{(i_1,\dots,i_n)}(f)$ the iterated integral of f with respect to $H^{(i_1)},\dots,H^{(i_n)}$:

$$J_n^{(i_1,\dots,i_n)}(f) = \int_0^\infty \left(\int_0^{t_n-} \cdots \left(\int_0^{t_2-} f(t_1,\dots,t_n) dH^{(i_1)}(t_1) \right) \cdots dH^{(i_{n-1})}(t_{n-1}) \right) dH^{(i_n)}(t_n). \tag{2.9}$$

We remark that all these integrals are well defined since all the processes $H^{(i)}$ for $i \geq 1$ are martingales with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$. Remark also that these iterated integrals are not the usual multiple stochastic integrals.

As a consequence of the strong orthogonality of the family of Teugels martingales, we have the following proposition.

PROPOSITION 2.1. *Let f and g belong to $L^2(\Sigma_n)$. Then*

$$\mathbb{E}\left[J_n^{(i_1,\dots,i_n)}(f) J_m^{(j_1,\dots,j_m)}(g) \right] = \begin{cases} q_{i_1} \cdots q_{i_n} \int_{\Sigma_n} f(t_1,\dots,t_n) g(t_1,\dots,t_n) dt_1 \cdots dt_n, \\ \quad \text{if } n = m \text{ and } (i_1,\dots,i_n) = (j_1,\dots,j_n), \\ 0, \quad \text{otherwise.} \end{cases} \tag{2.10}$$

Finally, we recall the chaotic representation property of the square integrable random variables, one of the main results in [8] (see also [6]).

PROPOSITION 2.2. *Let $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. Then F has a unique representation of the form*

$$F = \mathbb{E}[F] + \sum_{n=1}^\infty \sum_{i_1,\dots,i_n} J_n^{(i_1,\dots,i_n)} (f_{i_1,\dots,i_n}), \tag{2.11}$$

where $f_{i_1,\dots,i_n} \in L^2(\Sigma_n)$.

2.3. Derivative operators

Definition 2.3. Let $f \in L^2(\Sigma_n)$. Set

$$D_t^{(\ell)} J_n^{(i_1,\dots,i_n)}(f) = \sum_{k=1}^n \mathbf{1}_{\{i_k=\ell\}} J_{n-1}^{(i_1,\dots,\widehat{i_k},\dots,i_n)} \left(f(\underbrace{\cdots}_{k-1}, t, \cdots) \mathbf{1}_{\Sigma_n^{(k)}(t)}(\cdot) \right), \tag{2.12}$$

where

$$\Sigma_n^{(k)}(t) = \{(t_1,\dots,\widehat{t_k},\dots,t_n) \in \Sigma_{n-1} : 0 < t_1 < \cdots < t_{k-1} < t \le t_{k+1} < \cdots < t_n\} \tag{2.13}$$

and $\widehat{i}$ means that the ith index is omitted.

Observe that if $k \ne k'$, then $\Sigma_n^{(k)}(t) \cap \Sigma_n^{(k')}(t) = \varnothing$.

Now, we define the spaces of the random variables that are differentiable in the ℓth direction. For this, we define the following subset of $L^2(\Omega)$:

$$\mathbb{D}^{(\ell)_1,2} = \left\{ F \in L^2(\Omega) : \sum_{n=1}^\infty \sum_{i_1,\dots,i_n} \sum_{k=1}^n \mathbf{1}_{\{i_k=\ell\}} q_{i_1} \cdots \widehat{q}_{i_k} \cdots q_{i_n} \right.$$
$$\left. \times \int_0^\infty \|f_{i_1,\dots,i_n}(\cdot, t, \cdot) \mathbf{1}_{\Sigma_n^{(k)}(t)}\|_{L^2([0,\infty)^{n-1})}^2 \, dt < \infty \right\}. \tag{2.14}$$

Definition 2.4. Given $F \in \mathbb{D}^{(\ell)_1,2}$, we define the derivative of F in the ℓth direction as the element of $L^2(\Omega \times \mathbb{R})$ given by

$$D_t^{(\ell)} F = \sum_{n=1}^\infty \sum_{i_1,\dots,i_n} \sum_{k=1}^n \mathbf{1}_{\{i_k=\ell\}} J_{n-1}^{(i_1,\dots,\widehat{i_k},\dots,i_n)} \left(f_{i_1,\dots,i_n}(\cdots, t, \cdots) \mathbf{1}_{\Sigma_n^{(k)}(t)}(\cdot) \right). \tag{2.15}$$

Observe that, as in the classical situation for Gaussian processes, $\mathbb{D}^{(\ell)_1,2}$ is dense in $L^2(\Omega)$, since the elements of $L^2(\Omega)$ with a finite chaotic expansion are in $\mathbb{D}^{(\ell)_1,2}$.

Set $\mathbb{D}^{(\infty)_1,2} = \cap_{\ell \in \mathbb{N}} \mathbb{D}^{(\ell)_1,2}$. Define also for $r > 1$,

$$\mathbb{D}^{(\ell)_r,2} = \{F \in L^2(\Omega) : D_\cdot^{(\ell_1,\dots,\ell_r)} F \in L^2([0,\infty)^r \times \Omega)\} \tag{2.16}$$

and $\mathbb{D}^\infty = \cap_{r \in \mathbb{N}} \cap_{\ell \in \mathbb{N}^r} \mathbb{D}^{(\ell)_r,2}$.

2.4. Simple Lévy processes. A simple Lévy process is given by

$$X_t = \sigma W_t + \alpha_1 N_t^1 + \cdots + \alpha_m N_t^m, \quad t \ge 0, \tag{2.17}$$

where $\{W_t : t \ge 0\}$ is a standard Brownian motion, $\{N_t^j : t \ge 0\}$, $j = 1,\dots,m$, are independent Poisson processes of parameters $\lambda_1,\dots,\lambda_m$, respectively, $\sigma > 0$ and $\alpha_1,\dots,\alpha_m$ are different nonnull numbers. The Lévy measure of X is $\nu = \sum_{j=1}^m \lambda_j \delta_{\alpha_j}$ and satisfies the condition (2.1) of [8] for the validity of the chaotic representation property.

Observe that

$$X_t^{(1)} = X_t = \sigma W_t + \sum_{j=1}^{m} \alpha_j N_t^j,$$

$$X_t^{(i)} = \sum_{0 < s \le t} (\Delta X_s)^i = \sum_{j=1}^{m} \alpha_j^i \, N_t^j, \quad i \ge 2,$$

(2.18)

because the independence of two Poisson processes adapted to the same filtration implies that they do not jump at the same time (see [2]).

Since

$$Y_t^{(i)} = X_t^{(i)} - m_i t, \quad i \ge 1,$$

(2.19)

we have

$$Y_t^{(1)} = \sigma W_t + \sum_{j=1}^{m} \alpha_j (N_t^j - \lambda_j t),$$

$$Y_t^{(i)} = \sum_{j=1}^{m} \alpha_j^i (N_t^j - \lambda_j t), \quad i \ge 2.$$

(2.20)

Then, the martingales $Y^{(i)}$, $i \ge 1$, are linear combinations of $W_t, N_t^1 - \lambda_1 t, \dots, N_t^m - \lambda_m t$. Since the martingales $H^{(i)}$ are linear combinations of $Y^{(1)}, \dots, Y^{(i)}$, it follows that $H^{(i)}$ are also linear combinations of $W_t, \tilde{N}_t^1, \dots, \tilde{N}_t^m$, where $\tilde{N}_t^j = N_t^j - \lambda_j t$. Therefore, we have a CRP in terms of the iterated integrals with respect to $W_t, \tilde{N}_t^1, \dots, \tilde{N}_t^m$. Recall also that by [6, Proposition 1.10], $H^{(i)} = 0$, for $i \ge m + 2$. Furthermore, since we are assuming that $\alpha_1, \dots, \alpha_m$ are different, there is one and only one way to express $Y^{(1)}, \dots, Y^{(m+1)}$ as linear combinations of $W_t, \tilde{N}_t^1, \dots, \tilde{N}_t^m$ since the uniqueness of the CRP in terms of the $W_t, \tilde{N}_t^1, \dots, \tilde{N}_t^m$.

To unify the notations, we will write $G_0(t) = W_t$ and $G_j(t) = \tilde{N}_t^j$ for $j = 1, \dots, m$. Also, we will denote by $L_n^{(i_1, \dots, i_n)}(f)$ the iterated integral of f with respect to $G_{i_1}, \dots, G_{i_n}$:

$$L_n^{(i_1, \dots, i_n)}(f) = \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f(t_1, \dots, t_n) dG_{i_1}(t_1) \cdots dG_{i_{n-1}}(t_{n-1}) \, dG_{i_n}(t_n).$$

(2.21)

Thus, we have also the chaotic representation property in terms of the G_i's.

PROPOSITION 2.5. *Let $F \in L^2(\Omega, \mathscr{F}, \mathbb{P})$. Then F has a representation of the form*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{0 \le i_1, \dots, i_n \le m} L_n^{(i_1, \dots, i_n)}(f_{i_1, \dots, i_n}),$$

(2.22)

where $f_{i_1, \dots, i_n} \in L^2(\Sigma_n)$.

In this setting, we recall a result of Léon *et al.* [6] which says that is possible to compute the derivatives in the directions $W, N^1, \dots, N^m, \dots$, following the classical rules on each space. Recall that in Poisson setting the derivative is a difference operator (see [10]).

We use the space $\Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_m$, where Ω_0 is the canonical space of the Wiener process and Ω_i, $i = 1,\ldots,m$, are, respectively, the canonical spaces of Poisson processes N^i, $i = 1,\ldots,m$, that are the spaces of all possible paths of Poisson's.

Remark 2.6. If we iterate the derivative with respect to the Poisson's i and j ($i < j$), we obtain

$$
\begin{aligned}
D^i_{t_2} D^j_{t_1} F = {} & F(\omega_0,\ldots,\omega_i + \delta_{t_2},\ldots,\omega_j + \delta_{t_1},\ldots,\omega_m) \\
& - F(\omega_0,\ldots,\omega_i,\ldots,\omega_j + \delta_{t_1},\ldots,\omega_m) \\
& - F(\omega_0,\ldots,\omega_i + \delta_{t_2},\ldots,\omega_j,\ldots,\omega_m) + F(\omega_0,\ldots,\omega_m).
\end{aligned}
\tag{2.23}
$$

Observe that $D^i_{t_2} D^j_{t_1} F = D^j_{t_1} D^i_{t_2} F$, but this equality is not true in general if we only interchange the superindexes or the subindexes. If the iteration is done with respect to the same direction $i \geq 1$, we obtain that $D^{i,\ldots,i}_{t_1,\ldots,t_s} F$, denoted by $D^{i,s}_{J_s} F$ where $J_s = \{t_1,\ldots,t_s\}$, is equal to $\sum_{B \subset J_s} (-1)^{s-(\#B)} F(\omega_0,\ldots,\omega_i + \sum_{t_j \in B} \delta_{t_j},\ldots,\omega_m)$ where we have used the convention $\sum_{t_j \in \varnothing} \delta_{t_j} = 0$.

Remark 2.7. It is clear that $D^0_{t_1} D^i_{t_2} F = D^i_{t_2} D^0_{t_1} F$ for all $i \geq 1$. We want to compute $D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F$ for $i_1,\ldots,i_n \in \{0,1,\ldots,m\}$. To do this, we define $I_s = \{j, i_j = s\}$ and $J_s = \{t_j, j \in I_s\}$ for $s \in \{0,1,\ldots,m\}$. Then $k_s = \#I_s$ will be the order of derivation with respect to G_s, and hence $\sum_{i=0}^m k_i = n$ and $\cup_{i=0}^m J_i = \{t_1,\ldots,t_n\}$. Therefore,

$$
D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F = D^{0,k_0}_{J_0} D^{1,k_1}_{J_1} \cdots D^{m,k_m}_{J_m} F,
\tag{2.24}
$$

with the convention that $D^{i,0}_{J_i}$ is the identity, and we get

$$
D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F = D^{0,k_0}_{J_0} \left[\sum_{B_i \subset J_i, 1 \leq i \leq m} (-1)^{\sum_{i=1}^m (k_i - (\#B_i))} F\left(\omega_0, \omega_1 + \sum_{t_j \in B_1} \delta_{t_j}, \ldots, \omega_m + \sum_{t_j \in B_m} \delta_{t_j}\right) \right].
\tag{2.25}
$$

Remark 2.8. If $F = f(W_t, \tilde{N}^1_t, \ldots, \tilde{N}^m_t)$, where f is a smooth function, its derivative $D^i_s F$ for $i \geq 1$ and $s \leq t$ is the difference $f(W_t, \tilde{N}^1_t, \ldots, \tilde{N}^i_t + 1, \ldots, \tilde{N}^m_t) - f(W_t, \tilde{N}^1_t, \ldots, \tilde{N}^m_t)$. So in general

$$
\begin{aligned}
D^{i_1,\ldots,i_n}_{t_1,\ldots,t_n} F = {} & D^{0,k_0}_{J_0} D^{1,k_1}_{J_1} \cdots D^{m,k_m}_{J_m} F \\
= {} & D^{0,k_0}_{J_0} \left[\sum_{0 \leq j_i \leq k_i, 1 \leq i \leq m} (-1)^{n - k_0 - \sum_{i=1}^m j_i} \right. \\
& \left. \times \prod_{i=1}^m \binom{k_i}{j_i} f\left(W_t, \tilde{N}^1_t + j_1, \ldots, \tilde{N}^m_t + j_m\right) \right] \prod_{i=1}^m \mathbf{1}_{\{J_i \subset [0,t]^{k_i}\}} \\
= {} & \sum_{0 \leq j_i \leq k_i, 1 \leq i \leq m} (-1)^{n - k_0 - \sum_{i=1}^m j_i} \\
& \times \prod_{i=1}^m \binom{k_i}{j_i} \frac{\partial^{k_0} f}{\partial x_1^{k_0}}\left(W_t, \tilde{N}^1_t + j_1, \ldots, \tilde{N}^m_t + j_m\right) \prod_{i=0}^m \mathbf{1}_{\{J_i \subset [0,t]^{k_i}\}}.
\end{aligned}
\tag{2.26}
$$

Remark 2.9. If there's only one Poisson process and no Brownian part, different authors as Léon *et al.* [7] and Privault [12] have used the iterated derivatives to find the chaotic decomposition of functionals of the d first jump times of the process. In the present article, we consider different Poisson processes and a Brownian part. Moreover, we are not restricted to a specified finite number of jump times.

3. Stroock formula for Lévy processes

The aim of this section is to derive a Stroock formula for functionals of a Lévy process belonging to $\mathbb{D}^\infty$. First we deal with the case that F is an element of a specific chaos and finally we will extend the result to F in $\mathbb{D}^\infty$.

We start with the following lemma.

LEMMA 3.1. *If* $F = J_n^{i_1,\ldots,i_n}(f)$*, then*

$$D_{t_1,\ldots,t_n}^{j_1,\ldots,j_n} F = \sum_{\sigma \in \mathscr{P}\{1,\ldots,n\}} \delta_{j_{\sigma(1)},i_1} \cdots \delta_{j_{\sigma(n)},i_n} f\left(t_{\sigma(1)},\ldots,t_{\sigma(n)}\right) \mathbf{1}_{\{t_{\sigma(1)} < \cdots < t_{\sigma(n)}\}}, \quad (3.1)$$

where $\mathscr{P}\{1,\ldots,n\}$ *is the set of permutations of* $\{1,\ldots,n\}$.

Proof. First remark that, in the sum, only one term is different from zero. To prove the lemma, we use induction on n.

For $n = 2$, we have

$$D_{t_1}^{j_1}\left(\int_0^1 \int_0^{r_2-} f(r_1,r_2) dH^{(i_1)}(r_1) dH^{(i_2)}(r_2)\right)$$

$$= \delta_{j_1,i_1} \int_{t_1}^1 f(t_1,r_2) dH^{(i_2)}(r_2) + \delta_{j_1,i_2} \int_0^{t_1} f(r_1,t_1) dH^{(i_1)}(r_1)$$

$$= \delta_{j_1,i_1} \int_0^1 f(t_1,r_2) \mathbf{1}_{[t_1,1]}(r_2) dH^{(i_2)}(r_2) + \delta_{j_1,i_2} \int_0^1 f(r_1,t_1) \mathbf{1}_{[0,t_1]}(r_1) dH^{(i_1)}(r_1),$$
$$(3.2)$$

and applying now the operator $D_{t_2}^{j_2}$, we get

$$D_{t_2}^{j_2} D_{t_1}^{j_1}\left(J_2^{i_1,i_2}(f)\right) = \delta_{j_2,i_2} \delta_{j_1,i_1} f(t_1,t_2) \mathbf{1}_{[t_1,1]}(t_2) + \delta_{j_2,i_1} \delta_{j_1,i_2} f(t_2,t_1) \mathbf{1}_{[0,t_1]}(t_2). \quad (3.3)$$

Hence, if $t_1 < t_2$, we have

$$D_{t_2}^{j_2} D_{t_1}^{j_1}\left(\int_0^1 \int_0^{r_2-} f(r_1,r_2) dH^{(i_1)}(r_1) dH^{(i_2)}(r_2)\right) = \delta_{j_1,i_1} \delta_{j_2,i_2} f(t_1,t_2). \quad (3.4)$$

Therefore, the formula (3.1) is satisfied for $n = 2$.

Besides,

$$D_{t_1,\ldots,t_n}^{j_1,\ldots,j_n}\left(J_n^{i_1,\ldots,i_n}(f)\right) = D_{t_{j_n}}^{j_n}\left(\cdots\left(D_{t_{j_1}}^{j_1}\left(J_n^{i_1,\ldots,i_n}(f)\right)\right)\right)$$

$$= D_{t_{j_n}}^{j_n} \cdots D_{t_{j_2}}^{j_2}\left[\sum_{k=1}^n \delta_{i_k j_1} J_{n-1}^{i_1,\ldots,\widehat{i}_k,\ldots,i_n}\left(f(\ldots,t_1,\ldots,)\mathbf{1}_{\sum_n^{(k)}(t_1)}(\cdot)\right)\right]. \quad (3.5)$$

and the induction hypothesis yields

$$
\begin{aligned}
D^{j_1,\dots,j_n}_{t_1,\dots,t_n} & \left(J^{i_1,\dots,i_n}_n(f) \right) \\
&= \sum_{k=1}^{n} \delta_{i_k j_1} \sum_{\tau \in \mathscr{P}\{1,\dots,\hat{k},\dots,n\}} \delta_{j_{\tau(2)},i_1} \cdots \delta_{j_{\tau(k-1)},i_{k-1}} \delta_{j_{\tau(k+1)},i_{k+1}} \cdots \delta_{j_{\tau(n)},i_n} \\
&\quad \times f\left(t_{\tau(2)},\dots,t_{\tau(k-1)},t_1,t_{\tau(k+1)},\dots,t_{\tau(n)} \right) \mathbf{1}_{\{t_{\tau(2)}<\cdots<t_{\tau(k-1)}<t_1<t_{\tau(k+1)}\cdots<t_{\tau(n)}\}} \\
&= \sum_{\sigma \in \mathscr{P}\{1,\dots,n\}} \delta_{j_{\sigma(1)},i_1} \cdots \delta_{j_{\sigma(n)},i_n} f\left(t_{\sigma(1)},\dots,t_{\sigma(n)} \right) \mathbf{1}_{\{t_{\sigma(1)}<\cdots<t_{\sigma(n)}\}}.
\end{aligned}
\tag{3.6}
$$

$\square$

Remark 3.2. If $t_1 < \cdots < t_n$, we get the following equality:

$$
D^{j_1,\dots,j_n}_{t_1,\dots,t_n}\left(J^{i_1,\dots,i_n}_n(f) \right) = \delta_{j_1,i_1} \cdots \delta_{j_n,i_n} f(t_1,\dots,t_n).
\tag{3.7}
$$

Now, we apply Lemma 3.1 to obtain the Stroock formula.

THEOREM 3.3. *Let F be in $\mathbb{D}^\infty$ with chaotic expansion*

$$
F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{0 \le i_1,\dots,i_n \le m} J^{i_1,\dots,i_n}_n\left(f_{i_1,\dots,i_n} \right).
\tag{3.8}
$$

Then

$$
f_{i_1,\dots,i_n}(t_1,\dots,t_n) = \mathbb{E}[D^{i_1,\dots,i_n}_{t_1,\dots,t_n}F],
\tag{3.9}
$$

where $t_1 < \cdots < t_n$.

Proof. As the iterated derivatives of order n of elements belonging to chaos of order less than n are zero and applying the above lemma, we get

$$
D^{i_1,\dots,i_n}_{t_1,\dots,t_n}F = f_{i_1,\dots,i_n}(t_1,\dots,t_n) + M_n, \quad \text{with } t_1 < \cdots < t_n.
\tag{3.10}
$$

where M_n is a sum of variables in chaos of order greater than or equal to n. Taking the expectation, we obtain the desired formula. $\square$

4. Chaos expansion of the local time of a simple Lévy process

The aim of this section is to apply the Stroock formula to find the chaotic decomposition of the local time of a simple Lévy process.

We denote by H_n the nth Hermite polynomial defined by

$$
H_n(x) = \frac{(-1)^n}{\sqrt{n!}} \exp\left(\frac{x^2}{2} \right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2} \right) \right)
\tag{4.1}
$$

for $n \ge 1$ and $H_0(x) = 1$.

We define the local time $L(t,x)$ of a Lévy process X as the density of the occupation measure

$$m_t(A) = \int_0^t \mathbf{1}_A(X_s)\,ds, \quad A \in \mathcal{B}(\mathbb{R}). \tag{4.2}$$

It is well known that this density $L(t,x)$ exists and is a nondecreasing function of t and the measure $L(dt,x)$ is concentrated on the level set $\{s : X_s = x\}$. Moreover, Barlow [1] shows that $L(t,x)$ has an almost surely jointly continuous version.

It is well known also that we can write

$$L(t,x) = \int_0^t \delta_x(X_s)\,ds. \tag{4.3}$$

To apply the Stroock formula, we consider

$$L_\varepsilon(t,x) = \int_0^t p_\varepsilon(X_s - x)\,ds, \tag{4.4}$$

where p_ε is the centered Gaussian kernel with variance $\varepsilon > 0$. The classical idea of approximating the Dirac distribution δ_x by p_ε has been used to calculate the chaotic decomposition of the local time in the case of the Brownian motion by Nualart and Vives [11] and for the fractional Brownian motion by Coutin *et al.* [3] and Eddahbi *et al.* [4].

Before stating precise results of this section, we prove some technical lemmas.

LEMMA 4.1. *Let* $F = p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x)$. *Then*

$$D_{t_1,\dots,t_n}^{i_1,\dots,i_n} F$$

$$= \sum_{i=1}^m \sum_{\ell_i=0}^{k_i} \prod_{j=1}^m \binom{k_j}{\ell_j}(-1)^{n-k_0-\sum_{j=1}^m \ell_j}\sigma^{k_0} p_\varepsilon^{(k_0)}\left(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i + \alpha_i \ell_i - x\right)\prod_{r=1}^n \mathbf{1}_{[0,s]}(t_r), \tag{4.5}$$

where $k_i = \#\{j : i_j = i\}$, $i = 0,1,\dots,m$.

Proof. We only apply Remark 2.8 of Section 2. □

LEMMA 4.2. *Set* $g_n(y) = H_n(y)\exp(-y^2/2)$. *Then*

$$\mathbb{E}\left[p_\varepsilon^{(n)}(\sigma W_s - a)\right] = \frac{(-1)^n}{(s\sigma^2 + \varepsilon)^{(n+1)/2}}\frac{\sqrt{n!}}{\sqrt{2\pi}}g_n\left(\frac{a}{\sqrt{s\sigma^2 + \varepsilon}}\right). \tag{4.6}$$

Proof. As $p_\varepsilon^{(n)}(x) = (1/\sigma^{n+1})p_{\varepsilon/\sigma^2}^{(n)}(x/\sigma)$, we have

$$\mathbb{E}\left[p_\varepsilon^{(n)}(\sigma W_s - a)\right] = \frac{1}{\sigma^{n+1}}\mathbb{E}\left[p_{\varepsilon/\sigma^2}^{(n)}\left(W_s - \frac{a}{\sigma}\right)\right]. \tag{4.7}$$

Interchanging the derivative operator and the expectation, and using

$$p_\varepsilon^{(n)}(x) = (-1)^n\sqrt{n!}\frac{p_\varepsilon(x)}{\varepsilon^{n/2}}H_n\left(\frac{x}{\sqrt{\varepsilon}}\right), \qquad \mathbb{E}\left[p_\varepsilon(W_s - a)\right] = p_{s+\varepsilon}(a), \tag{4.8}$$

we get the formula of the lemma. □

Lemma 4.3. *For all $s > 0$ and for all $n \geq 0$,*

$$| \mathbb{E}[p_\varepsilon^{(n)}(\sigma W_s - a)] | \leq \frac{1}{2\pi} \Gamma\left(\frac{n+1}{2}\right)\left(\frac{s\sigma^2}{2}\right)^{-(n+1)/2}. \tag{4.9}$$

Proof. We apply Lemma 4.2 and the inequality $|g_n(y)| \leq ((2^{n/2})/\sqrt{\pi n!})\Gamma((n+1)/2)$ (see [4]).   □

Then, we can prove the following proposition.

Proposition 4.4. *The chaos expansion of $L_\varepsilon(t,x) = \int_0^t p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x)ds$ is*

$$L_\varepsilon(t,x) = \sum_{n=0}^\infty \sum_{0 \leq i_1,\dots,i_n \leq m} L_n^{i_1,\dots,i_n} \left[\sum_{i=1}^m \sum_{\ell_i=0}^{k_i} \prod_{j=1}^m \binom{k_j}{\ell_j} \sigma^{k_0}(-1)^{n-k_0-\sum_{j=1}^m \ell_j} \frac{\sqrt{k_0!}}{\sqrt{2\pi}} \right.$$
$$\times \sum_{r_1=0}^\infty \cdots \sum_{r_m=0}^\infty \int_{t_1 \vee \cdots \vee t_n}^t \frac{e^{-\lambda s}}{(s\sigma^2 + \varepsilon)^{(k_0+1)/2}}$$
$$\left. \times \prod_{j=1}^m \frac{(\lambda_j s)^{r_j}}{r_j!} g_n\left(\frac{x - \sum_{j=1}^m \alpha_j(r_j - \lambda_j s + \ell_j)}{\sqrt{s\sigma^2 + \varepsilon}}\right) ds \right], \tag{4.10}$$

where $\lambda = \sum_{i=1}^m \lambda_i$.

Proof. We apply the Stroock formula to $p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x)$ and use Lemma 4.1 for the expression of the iterated derivative and Lemma 4.2 for the expectation with respect to the Wiener part. Besides, we calculate the expectation with respect to the Poisson parts.   □

Remark 4.5. Note that the kernels depend only on $k_0, k_1, \dots, k_m$. Hence, it is the same for all sets of indexes $i_1, \dots, i_n$ that derive to equal $k_0, k_1, \dots, k_m$. Observe also that although the kernel is the same, the iterated integral $L_n^{i_1,\dots,i_n}$ can depend on the order of the indexes as we will see in the next example:

$$\int_0^1 \int_0^{u_2-} dW_{u_1} dN_{u_2} = \int_0^1 W_{u_2-} dN_{u_2} = \sum_{i=1}^{N_1} W_{T_i}, \tag{4.11}$$

where $T_i, i \in \mathbb{N}$, are the jump times of Poisson. But

$$\int_0^1 \int_0^{u_2-} dN_{u_1} dW_{u_2}$$
$$= \int_0^1 N_{u_2-} dW_{u_2}$$
$$= \left[N_{1-}(W_1 - W_{T_{N_1}}) + (N_{1-} - 1)(W_{T_{N_1}} - W_{T_{N_1-1}}) + \cdots + (W_{T_2} - W_{T_1}) \right]$$
$$= W_1 N_{1-} - \sum_{i=1}^{N_1} W_{T_i}. \tag{4.12}$$

In order to establish the chaotic expansion of the local time of a simple Lévy process we state the following lemma (see [11]).

LEMMA 4.6. *Let $\{F_\varepsilon\}_{\varepsilon>0}$ be a family of square integrable random variables with the expansions*

$$F_\varepsilon = \mathbb{E}[F_\varepsilon] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n}^\varepsilon), \tag{4.13}$$

where $f_{i_1,\ldots,i_n}^\varepsilon$ belongs to $L^2([0,\infty)^n)$.
    Assume that
    (i) *$f_{i_1,\ldots,i_n}^\varepsilon$ converges in $L^2([0,\infty)^n)$, when ε goes to zero, to some symmetric function $f_{i_1,\ldots,i_n} \in L^2([0,\infty)^n)$;*
    (ii) *$\sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} q_{i_1}\cdots q_{i_n} \sup_\varepsilon \|f_{i_1,\ldots,i_n}^\varepsilon(\cdot)\|_{L^2([0,\infty)^n)}^2$ is convergent.*
Then the family F_ε converges in $L^2(\Omega)$ to $F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} J_n^{(i_1,\ldots,i_n)}(f_{i_1,\ldots,i_n})$.

Then we can prove the following proposition.

PROPOSITION 4.7. *For each $x \in \mathbb{R}$ and $t \in [0,T]$, the random variables*

$$\int_0^t p_\varepsilon\left(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x\right) ds \tag{4.14}$$

converge to $L(t,x)$ in $L^2(\Omega)$ as ε tends to zero. Furthermore, with $\lambda = \sum_{i=1}^m \lambda_i$, the local time $L(t,x)$ has the following chaotic decomposition:

$$\begin{aligned}
L(t,x) = \sum_{n=0}^{\infty} \sum_{0 \le i_1,\ldots,i_n \le m} L_n^{i_1,\ldots,i_n} & \left[\sum_{i=1}^m \sum_{\ell_i=0}^{k_i}\left(\prod_{j=1}^m \binom{k_j}{\ell_j}\right)\sigma^{k_0}\right.\\
& \times (-1)^{n-k_0-\sum_{j=1}^m \ell_j}\frac{\sqrt{k_0!}}{\sqrt{2\pi}}\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\int_{t_1\vee\cdots\vee t_n}^t \frac{e^{-\lambda s}}{(s\sigma^2)^{(k_0+1)/2}}\\
& \left.\times \prod_{j=1}^m \frac{(\lambda_j s)^{r_j}}{r_j!}g_n\left(\frac{x-\sum_{j=1}^m \alpha_j(r_j-\lambda_j s+\ell_j)}{\sigma\sqrt{s}}\right)ds\right].
\end{aligned} \tag{4.15}$$

Remark 4.8. Note that Lemma 4.6, simultaneously, proves that the local time is in $L^2(\Omega)$ and gives its chaotic expansion.

Proof. We must check the two hypotheses of Lemma 4.6. We start with (ii). By Lemma 4.1, we have

$$\begin{aligned}
\left|\mathbb{E}[D_{t_1,\ldots,t_n}^{i_1,\ldots,i_n}L_\varepsilon(t,x)]\right| & \\
\le \sum_{i=1}^m \sum_{\ell_i=0}^{k_i}\prod_{j=1}^m & \binom{k_j}{\ell_j}\sum_{r_1=0}^{\infty}\cdots\sum_{r_m=0}^{\infty}\int_{t_1\vee\cdots\vee t_n}^t e^{-\lambda s}\prod_{j=1}^m \frac{(\lambda_j s)^{r_j}}{r_j!}\sigma^{k_0}\\
& \times \mathbb{E}_W\left[p_\varepsilon^{(k_0)}\left(\sigma W_s + \sum_{j=1}^m \alpha_j(r_j-\lambda_j s+\ell_j)-x\right)\right]\prod_{r=1}^n \mathbf{1}_{[0,s]}(t_r)\,ds,
\end{aligned} \tag{4.16}$$

and by Lemma 4.3, and since

$$
e^{-\lambda s} \prod_{j=1}^{m} \sum_{r_j=0}^{\infty} \frac{(\lambda_j s)^{r_j}}{r_j!} = 1,
\tag{4.17}
$$

the right-hand side of (4.16) is bounded by

$$
\sum_{i=1}^{m} \sum_{\ell_i=0}^{k_i} \frac{\sigma^{k_0}}{2\pi} \prod_{j=1}^{m} \binom{k_j}{\ell_j} \int_{t_1 \vee \cdots \vee t_n}^{t} \frac{\Gamma((k_0+1)/2)}{((s\sigma^2)/2)^{(k_0+1)/2}} ds
$$
$$
= \frac{2^{n-(k_0/2)+(1/2)}}{2\pi\sigma} \Gamma\left(\frac{k_0+1}{2}\right) \int_{t_1 \vee \cdots \vee t_n}^{t} \frac{ds}{s^{(k_0+1)/2}},
\tag{4.18}
$$

where we have used the identity

$$
\sum_{i=1}^{m} \sum_{\ell_i=0}^{k_i} \prod_{j=1}^{m} \binom{k_j}{\ell_j} = 2^{n-k_0}.
\tag{4.19}
$$

Therefore,

$$
\left| \mathbb{E}\left[D_{t_1,\ldots,t_n}^{i_1,\ldots,i_n} L_\varepsilon(t,x) \right] \right|^2 \le \frac{2^{2n-k_0+1}}{(2\pi\sigma)^2} \Gamma\left(\frac{k_0+1}{2}\right)^2 \left(\int_{t_1 \vee \cdots \vee t_n}^{t} \frac{ds}{s^{(k_0+1)/2}} \right)^2.
\tag{4.20}
$$

Then

$$
\left\| \mathbb{E}\left[D_{t_1,\ldots,t_n}^{i_1,\ldots,i_n} L_\varepsilon(t,x) \right] \right\|_{L^2([0,T]^n)}^2
$$
$$
\le \frac{2^{2n-k_0+1}}{(2\pi\sigma)^2} \Gamma\left(\frac{k_0+1}{2}\right)^2 \int_0^t \frac{y^{n-1} dy}{(n-1)!} \left(\int_y^t s^{(k_0+1)/2} ds \right)^2
$$
$$
= \frac{2^{2n-k_0+1}}{(2\pi\sigma)^2} \Gamma\left(\frac{k_0+1}{2}\right)^2 \int_0^t \int_0^t \int_0^t \frac{y^{n-1} \mathbf{1}_{[0,u \wedge v]}(y) dy}{(n-1)!(uv)^{(k_0+1)/2}} du\, dv
\tag{4.21}
$$
$$
= \frac{2^{2n-k_0}}{n!(\pi\sigma)^2} \Gamma\left(\frac{k_0+1}{2}\right)^2 \int_0^t v^{-(k_0+1)/2} dv \int_0^v u^{n-((k_0+1)/2)} du
$$
$$
= \frac{2^{2n-k_0}}{n!(\pi\sigma)^2} \Gamma\left(\frac{k_0+1}{2}\right)^2 \frac{2}{2n-k_0+1} \frac{t^{n-k_0+1}}{n-k_0+1}.
$$

Therefore,

$$
\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{0 \le i_1,\dots,i_n \le m} \left\| \mathbb{E}\big[D_{t_1,\dots,t_n}^{i_1,\dots,i_n} L_\varepsilon(t,x)\big]\right\|_{L^2([0,T]^n)}^2 \\
& \le \sum_{n=0}^{\infty} \sum_{k_0=0}^{n} \frac{t 2^{k_0+1}}{(\pi\sigma)^2 k_0!} \Gamma\left(\frac{k_0+1}{2}\right)^2 \sum_{k_1+\cdots+k_m=n-k_0} \frac{(4t)^{n-k_0}}{k_1!\cdots k_m!} \frac{1}{2n-k_0+1} \frac{1}{n-k_0+1} \\
& \le \sum_{n=0}^{\infty} \sum_{k_0=0}^{n} \frac{2^{k_0+1}}{m(\pi\sigma)^2 k_0!} \Gamma\left(\frac{k_0+1}{2}\right)^2 \frac{(4mt)^{n-k_0+1}}{(n-k_0+1)!} \frac{1}{2n-k_0+1} \\
& \le \sum_{k_0=0}^{\infty} \frac{2^{k_0+1}}{m(\pi\sigma)^2 k_0!} \Gamma\left(\frac{k_0+1}{2}\right)^2 \frac{e^{m4t}-1}{k_0+1},
\end{aligned}
$$
(4.22)

where we have used the facts that the bound of each term of the series depends only on $k_0, k_1, \dots, k_m$,

$$
\sum_{k_1+\cdots+k_m=r} \frac{1}{k_1!\cdots k_m!} = \frac{m^r}{r!}
$$
(4.23)

and $2n - k_0 + 1 \ge k_0 + 1$.

By Stirling formula,

$$
\frac{2^{k_0+1}}{\pi^2 (k_0+1)!} \Gamma\left(\frac{k_0+1}{2}\right)^2 \sim c k_0^{-3/2}, \quad \text{for } k_0 \text{ large.}
$$
(4.24)

Hence, the general term of the right-hand side of (4.22) behaves as $c k_0^{-3/2}$ and the corresponding series is convergent.

Note that in our setting $q_0 = \cdots = q_m = 1$, because we do not work with the $H^{(i)}$'s but use directly the Wiener and Poisson processes.

Now, it remains to check Lemma 4.6 (i). We have

$$
\begin{aligned}
& \left\| f_{i_1,\dots,i_n}^\varepsilon - f_{i_1,\dots,i_n} \right\|_{L^2([0,T]^n)}^2 \\
& = \left\| f_{i_1,\dots,i_n}^\varepsilon \right\|_{L^2([0,T]^n)}^2 + \left\| f_{i_1,\dots,i_n} \right\|_{L^2([0,T]^n)}^2 - 2\left\langle f_{i_1,\dots,i_n}^\varepsilon, f_{i_1,\dots,i_n} \right\rangle_{L^2([0,T]^n)}.
\end{aligned}
$$
(4.25)

It is clear that $f_{i_1,\dots,i_n}^\varepsilon$ converges to $f_{i_1,\dots,i_n}$ pointwise and using the dominated convergence theorem, we see easily that condition (i) holds.

Finally, we will show, following standard arguments, that the limit of $\int_0^t p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x)ds$, denoted by Λ_t^x, is the local time $L(t,x)$. The above estimates are uniform in $x \in \mathbb{R}$. Therefore, we can conclude that the convergence of $\int_0^t p_\varepsilon(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x)ds$ to Λ_t^x holds in $L^2(\Omega \times \mathbb{R}, \mathbb{P} \otimes \mu)$, for any finite measure μ. As a consequence, for any continuous function g in $\mathbb{R}$ with compact support, we have that

$$
\int_{\mathbb{R}} \left(\int_0^t p_\varepsilon\left(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i - x\right) ds \right) g(x) dx
$$
(4.26)

converges in $L^2(\Omega)$ to $\int_{\mathbb{R}} \Lambda_t^x g(x)dx$. But, this expression also converges to

$$\int_0^t g\left(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i\right)ds. \tag{4.27}$$

Hence,

$$\int_{\mathbb{R}} \Lambda_t^x g(x)dx = \int_0^t g\left(\sigma W_s + \sum_{i=1}^m \alpha_i \tilde{N}_s^i\right)ds, \tag{4.28}$$

which implies that $\Lambda_t^x = L(t,x)$. □

5. Chaos expansion of the price of a financial asset

In this section, we will use the Stroock formula to get the chaotic decomposition of the asset price S_t driven by the Lévy process $\tilde{X}_t = \mu t + \sigma W_t + \sum_{j=1}^m \alpha_j \tilde{N}_t^j$, where $\tilde{N}_t^j$, $j = 1,\dots,m$, are independent compensated Poisson processes with parameters λ_j, $j = 1,\dots,m$. This means that S_t satisfies the stochastic differential equation

$$dS_t = S_{t-} d\tilde{X}_t, \qquad S_0 = s_0, \tag{5.1}$$

with solution

$$S_t = s_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t - \sum_{i=1}^m \alpha_i\lambda_i t + \sigma W_t\right\}\prod_{i=1}^m (1+\alpha_i)^{N_t^i}$$
$$= s_0 \exp\left(ct + \sigma W_t + \sum_{i=1}^m \beta_i N_t^i\right), \tag{5.2}$$

where $c = \mu - \sigma^2/2 - \sum_{i=1}^m \alpha_i\lambda_i$ and $\beta_i = \log(1+\alpha_i)$ for $i = 1,\dots,m$ with the condition $\alpha_i > -1$ to guarantee the positivity of the price (see [13]).

Remark 5.1. If τ is a jump time of the Poisson process N^j, the relative jump $(S_\tau - S_{\tau-})/S_\tau$ is equal to α_j and the absolute jump $\log(S_\tau) - \log(S_{\tau-})$ of the log price process $\log(S_t)$ is β_j.

Now we compute the iterated derivatives of S_t and we obtain

$$D_{s_1,\dots,s_{k_0}}^{0,k_0} S_t = \sigma^{k_0} S_t, \tag{5.3}$$

for $s_1 < \cdots < s_{k_0} < t$,

$$D_s^\ell S_t = s_0 \exp\left(-\sigma^2 t/2 + \sigma W_t\right)\prod_{i=1,j\neq\ell}^m (1+\alpha_i)^{N_t^i}\left\{(1+\alpha_\ell)^{N_t^\ell+1} - (1+\alpha_\ell)^{N_t^\ell}\right\} = \alpha_\ell S_t, \tag{5.4}$$

for $s < t$, and

$$D_{s_1,\dots,s_\ell}^{\ell,k_\ell} S_t = \alpha_\ell^{k_\ell} S_t, \tag{5.5}$$

for $\ell = 1, \ldots, m$. Consequently,

$$D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} S_t = \sigma^{k_0} \alpha_1^{k_1} \cdots \alpha_m^{k_m} S_t \prod_{i=1}^{n} \mathbf{1}_{\{s_i < t\}}, \tag{5.6}$$

with $\sum_{i=0}^{m} k_i = n$.

Hence,

$$\mathbb{E}[D_{s_1,\ldots,s_n}^{k_0,\ldots,k_m} S_t] = \sigma^{k_0} \alpha_1^{k_1} \cdots \alpha_m^{k_m} s_0 e^{ct} \mathbb{E}_W [\exp(\sigma W_t)] \prod_{i=1}^{m} \mathbb{E}_{N_t^i} [(1+\alpha_i)^{N_t^i}] \prod_{k=1}^{n} \mathbf{1}_{\{0 < s_k < t\}}$$

$$= \sigma^{k_0} \alpha_1^{k_1} \cdots \alpha_m^{k_m} s_0 e^{ct} \exp(\sigma^2 t/2) \prod_{i=1}^{m} \exp(\alpha_i \lambda_i t) \prod_{k=1}^{n} \mathbf{1}_{\{0 < s_k < t\}},$$

$$f_{k_0,\ldots,k_m}(s_1,\ldots,s_n;t) = s_0 \prod_{i=0}^{m} \alpha_i^{k_i} e^{\gamma_i t} \mathbf{1}_{\{0 < s_1 < \cdots < s_n < t\}} = s_0 e^{\mu t} \prod_{i=0}^{m} \alpha_i^{k_i} \mathbf{1}_{\{0 < s_1 < \cdots < s_n < t\}},$$

$$\tag{5.7}$$

with $\alpha_0 = \sigma$, $\gamma_i = \alpha_i \lambda_i$, for $i = 1, \ldots, m$, $\gamma_0 = \mu - \sum_{i=1}^{m} \alpha_i \lambda_i$, and $\lambda_0 = 1$.

The chaotic decomposition of S_t is given by

$$S_t = \mathbb{E}[S_t] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} s_0 e^{\mu t} \prod_{j=0}^{m} \alpha_j^{k_j} L_n^{i_1,\ldots,i_n} (\mathbf{1}_{\{0 < s_1 < \cdots < s_n < t\}}). \tag{5.8}$$

As the terms of the chaotic decomposition are uncorrelated, the variance has the following expression:

$$\mathrm{Var}(S_t) = \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} s_0^2 e^{2\mu t} \prod_{j=0}^{m} \alpha_j^{2k_j} \mathbb{E}\left[L_n^{i_1,\ldots,i_n} (\mathbf{1}_{\{0 < s_1 < \cdots < s_n < t\}})^2\right]$$

$$= \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n} s_0^2 e^{2\mu t} \prod_{j=0}^{m} (\lambda_j \alpha_j^2)^{k_j} \frac{t^n}{n!}$$

$$= s_0^2 e^{2\mu t} \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{i_1,\ldots,i_n} \prod_{j=0}^{m} (\lambda_j \alpha_j^2)^{k_j} \tag{5.9}$$

$$= s_0^2 e^{2\mu t} \sum_{n=1}^{\infty} t^n \sum_{k_0 + \cdots + k_m = n} \prod_{j=0}^{m} \frac{(\lambda_j \alpha_j^2)^{k_j}}{k_j!}.$$

By the multinomial formula,

$$\mathrm{Var}(S_t) = s_0^2 e^{2\mu t} \sum_{n=1}^{\infty} \frac{(\sum_{j=0}^{m} t\lambda_j \alpha_j^2)^n}{n!} = s_0^2 e^{2t\mu} [e^{t \sum_{j=0}^{m} \lambda_j \alpha_j^2} - 1]. \tag{5.10}$$

If we only consider the first p chaos, the error of the variance is

$$s_0^2 \frac{t^{p+1}}{(p+1)!} e^{2\mu t} \left(\sum_{j=0}^{m} \lambda_j \alpha_j^2 \right)^{p+1} e^{\xi \sum_{j=0}^{m} \lambda_j \alpha_j^2}, \quad 0 < \xi < t, \tag{5.11}$$

which tends to 0 if p tends to infinity or t tends to zero.

In the case of the Brownian motion, $\gamma_j = \alpha_j = 0$ for $j = 1, \dots, m$ and

$$\mathrm{Var}\,(S_t) = s_0^2 e^{2\mu t} \sum_{n=1}^{\infty} t^n \frac{\sigma^{2n}}{n!} = s_0^2 e^{2t\mu} \left(e^{t\sigma^2} - 1 \right). \tag{5.12}$$

6. Chaos expansion of the price of a European call option

Let U_t be the price on t of a European call option written on the asset described in the last section. By the no-arbitrage theory, $U_t = \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)}g(S_T - K)/\mathscr{F}_t] = \mathbb{E}_{\mathbb{Q}}[g(S_T - K)/S_t]$, where $\mathbb{Q}$ is the unique martingale measure described in [6] and $g(x) = x \vee 0$. Note that in order to get the uniqueness of $\mathbb{Q}$, we have to add m additional assets defined by equations

$$dP_t^j = P_{t-}^j \, dX_t^j, \quad j = 1, \dots, m, \tag{6.1}$$

where

$$X_t^j = \mu_j t + \sigma_j W_t + \sum_{i=1}^{m} \alpha_{j,i} (N_t^i - \lambda_i t), \tag{6.2}$$

with the condition that there exist constants $M, L_1, \dots, L_m$, with $L_j \le \lambda_j$, such that

$$B \begin{pmatrix} M \\ L_1 \\ \vdots \\ L_m \end{pmatrix} = \begin{pmatrix} \mu - r \\ \mu_1 - r \\ \vdots \\ \mu_m - r \end{pmatrix}, \tag{6.3}$$

where $r > 0$ is the fixed interest rate and B is a matrix given by

$$B = \begin{pmatrix} \sigma & \alpha_1 & \cdots & \alpha_m \\ \sigma_1 & \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & \vdots & & \vdots \\ \sigma_m & \alpha_{m1} & \cdots & \alpha_{mm} \end{pmatrix}. \tag{6.4}$$

Using Girsanov's theorem, we find a $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ and m independent $\mathbb{Q}$-Poisson processes with parameters $\tilde{\lambda}_j = \lambda_j - L_j$, $N^{\mathbb{Q},1},\ldots,N^{\mathbb{Q},m}$, such that we can write

$$
\begin{aligned}
\frac{dS_t}{S_{t-}} &= \sigma dW_t^{\mathbb{Q}} + \sum_{i=1}^m \alpha_i (dN_t^{\mathbb{Q},i} - \tilde{\lambda}_i dt) + rdt, \\
\frac{dP_t^1}{P_{t-}^1} &= \sigma_1 dW_t^{\mathbb{Q}} + \sum_{i=1}^m \alpha_{1,i} (dN_t^{\mathbb{Q},i} - \tilde{\lambda}_i dt) + rdt, \\
&\vdots \\
\frac{dP_t^m}{P_{t-}^m} &= \sigma_m dW_t^{\mathbb{Q}} + \sum_{i=1}^m \alpha_{m,i} (dN_t^{\mathbb{Q},i} - \tilde{\lambda}_i dt) + rdt.
\end{aligned}
\tag{6.5}
$$

The explicit solution of (6.5) is

$$
S_t = s_0 \exp\left(ct + \sigma W_t^{\mathbb{Q}}\right) \prod_{i=1}^m (1+\alpha_i)^{N_t^{\mathbb{Q},i}} = s_0 \exp\left(ct + \sigma W_t^{\mathbb{Q}} + \sum_{i=1}^m \beta_i N_t^{\mathbb{Q},i}\right),
\tag{6.6}
$$

where $c = r - (\sigma^2/2) - \sum_{i=1}^m \alpha_i \tilde{\lambda}_i$.

The chaotic expansion of U_t is given by

$$
U_t = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n} L_n^{i_1,\ldots,i_n}\left(f_{i_1,\ldots,i_n}(\cdot,t)\right),
\tag{6.7}
$$

and using the Stroock formula, we get

$$
\begin{aligned}
f_{i_1,\ldots,i_n}(s_1,\ldots,s_n,t) &= \mathbb{E}_{\mathbb{Q}}[D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} U_t] = \mathbb{E}_{\mathbb{Q}}[D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} \mathbb{E}_{\mathbb{Q}}[e^{-r(T-t)} g(S_T - K)/\mathscr{F}_t]] \\
&= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}} D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} [g(S_T - K) \mathbf{1}_{\{s_1 < \cdots < s_n < t\}}/\mathscr{F}_t]] \\
&= \mathbb{E}_{\mathbb{Q}}[D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} g(S_T - K)] e^{-r(T-t)} \mathbf{1}_{\{0 < s_1 < \cdots < s_n < t\}},
\end{aligned}
\tag{6.8}
$$

where we have used the known commutative property between the derivative operator and the conditional expectation (see [10]).

Now, we have to compute $\mathbb{E}_{\mathbb{Q}}[D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} g(S_T - K)]$. In order to do this, we approximate the function g, uniformly, by the sequence of $\mathscr{C}^{\infty}$ functions:

$$
g_{\varepsilon}(x) = \int_{-\infty}^x \Phi_{\varepsilon}(y) dy,
\tag{6.9}
$$

where Φ_{ε} is the cumulative probability function of the centered normal law with variance ε.

Observe that the first derivative of g_{ε} is Φ_{ε} and the high-order derivatives are the functions $p_{\varepsilon}^{(j-2)}(x)$ for $j \geq 2$, that are for every j, the $(j-2)$-order derivatives of the centered Gaussian kernel of variance ε, denoted by p_{ε}.

Let $F_\varepsilon = g_\varepsilon(S_T - K)$. Clearly, F_ε converges in L^2 to $F = (S_T - K)^+ = U_T$. Then, applying Remark 2.8 to g_ε and assuming $s_1 < s_2 < \cdots < s_n < T$,

$$D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} F_\varepsilon = D_{J_0}^{0,k_0} D_{J_1}^{1,k_1} \cdots D_{J_m}^{m,k_m} F_\varepsilon$$
$$= \sum_{i=1}^{m} \sum_{\ell_i=0}^{k_i} \prod_{j=1}^{m} \binom{k_j}{\ell_j} (-1)^{n-k_0-\sum_{j=1}^{m}\ell_j} \partial_W^{(k_0)} g_\varepsilon(\widetilde{S}_T - K), \tag{6.10}$$

where $\widetilde{S}_T = S_T e^{\sum_{j=1}^{m} \beta_j \ell_j T}$.

Using the formula $f(ae^{bx} - c)^{(n)} = \sum_{j=1}^{n} f^{(j)}(ae^{bx} - c) b^n a^j e^{jbx} c_{n,j}$ with $c_{n,n} = c_{n,1} = 1$, we get the following.

(i) If $k_0 = 0$,

$$D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} F_\varepsilon = \sum_{i=1}^{m} \sum_{0 \leq \ell_i \leq k_i} (-1)^{n-\sum_{i=1}^{m}\ell_i} \prod_{i=1}^{m} \binom{k_i}{\ell_i} g_\varepsilon(\widetilde{S}_T - K). \tag{6.11}$$

(ii) If $k_0 > 0$,

$$D_{s_1,\ldots,s_n}^{i_1,\ldots,i_n} F_\varepsilon = \sum_{i=1}^{m} \sum_{0 \leq \ell_i \leq k_i} (-1)^{n-k_0-\sum_{i=1}^{m}\ell_i} \prod_{i=1}^{m} \binom{k_i}{\ell_i} \sigma^{k_0} \sum_{j=1}^{k_0} g_\varepsilon^{(j)}(\widetilde{S}_T - K) \widetilde{S}_T^j c_{k_0,j}. \tag{6.12}$$

Therefore, we have to compute and pass to limit, when ε tends to zero, three types of expectations:

$$\mathbb{E}[g_\varepsilon(\widetilde{S}_T - K)],$$
$$\mathbb{E}[\Phi_\epsilon(\widetilde{S}_T - K)\widetilde{S}_T], \tag{6.13}$$
$$\mathbb{E}\left[p_\varepsilon^{(\ell)}(\widetilde{S}_T - K)\widetilde{S}_T^{\ell+2}\right]$$

for $\ell \geq 0$.

In the first case, using the uniform convergence of g_ε to g, we obtain as a limit

$$\mathbb{E}[g(\widetilde{S}_T - K)] = \mathbb{E}[(\widetilde{S}_T - K)^+] = \mathbb{E}[\widetilde{S}_T \mathbf{1}_{\{\widetilde{S}_T > K\}}] - K\mathbb{P}\{\widetilde{S}_T > K\}. \tag{6.14}$$

For the second case, using the dominated convergence theorem, we have that the limit is

$$\mathbb{E}[\widetilde{S}_T \mathbf{1}_{\{\widetilde{S}_T > K\}}]. \tag{6.15}$$

So for the two first expectations, the problem reduces to compute explicitly $\mathbb{E}[\widetilde{S}_T \mathbf{1}_{\{\widetilde{S}_T > K\}}]$ and $\mathbb{P}\{\widetilde{S}_T > K\}$.

Fixing Poisson processes, we define

$$C = s_0 \exp\left\{ \sum_{j=1}^{m} \beta_j N_T^{(j)} + \left(r - \frac{\sigma^2}{2} + \sum_{j=1}^{m} (\beta_j \ell_j - \alpha_j \widetilde{\lambda}_j) \right) T \right\}. \tag{6.16}$$

In order to get the expectations with respect to the Gaussian part, we need only the fact that

$$\mathbb{P}_{W_T}\{e^{\sigma W_T} \geq \gamma\} = 1 - \Phi\left(\frac{\log(\gamma)}{\sigma\sqrt{T}}\right) \tag{6.17}$$

and the following lemma.

LEMMA 6.1. *Let $\gamma \geq 0$. Then*

$$\mathbb{E}_{W_T}\left[e^{\sigma W_T}\mathbf{1}_{\{e^{\sigma W_T}>\gamma\}}\right] = e^{\sigma^2 T/2}\Phi\left(\sigma\sqrt{T} - \frac{\log(\gamma)}{\sigma\sqrt{T}}\right). \tag{6.18}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}_{W_T}\left[e^{\sigma W_T}\mathbf{1}_{\{e^{\sigma W_T}>\gamma\}}\right] &= \int_{\mathbb{R}} e^{x\sigma}\mathbf{1}_{\{e^{x\sigma}>\gamma\}}\exp\left(-\frac{x^2}{2T}\right)\frac{dx}{\sqrt{2\pi T}} \\
&= \int_{(\log\gamma)/\sigma}^{\infty}\exp\left(-\frac{x^2}{2T}+\sigma x\right)\frac{dx}{\sqrt{2\pi T}} \\
&= e^{\sigma^2 T/2}\int_{(\log\gamma)/\sigma\sqrt{T}}^{\infty}\exp\left(-\frac{(y-\sigma\sqrt{T})^2}{2}\right)\frac{dy}{\sqrt{2\pi}} \\
&= e^{\sigma^2 T/2}\int_{((\log\gamma)/\sigma\sqrt{T})-\sigma\sqrt{T}}^{\infty}\exp\left(-\frac{x^2}{2}\right)\frac{dx}{\sqrt{2\pi}}.
\end{aligned} \tag{6.19}$$

$\square$

Finally, for the expectations of third type, we need the following result.

LEMMA 6.2.

$$\lim_{\varepsilon\to 0}\mathbb{E}_{W_T}\left[p_\varepsilon^{(\ell)}(Ce^{\sigma W_T}-K)(Ce^{\sigma W_T})^{\ell+2}\right] = (-1)^\ell h_\ell^{(\ell)}(K), \tag{6.20}$$

where

$$h_\ell(x) = x^{\ell+2}f(x) = x^{\ell+1}p_{\sigma\sqrt{T}}\left(\log\left(\frac{x}{C}\right)\right), \tag{6.21}$$

with f being the density of the log-normal law with parameters $\log(C)$ and $\sigma\sqrt{T}$.

Proof. We have

$$\mathbb{E}_{W_T}\left[p_\varepsilon^{(\ell)}(Ce^{\sigma W_T}-K)(Ce^{\sigma W_T})^{\ell+2}\right] = \int_{\mathbb{R}} p_\varepsilon^{(\ell)}(x-K)x^{\ell+2}f(x)dx. \tag{6.22}$$

Taking $h_\ell(x) = x^{\ell+2}f(x)$ and integrating by parts, using that h_ℓ and all its derivatives are fast decreasing functions, the last integral is equal to

$$(-1)^\ell\int_{\mathbb{R}} p_\varepsilon(x-K)h_\ell^{(\ell)}(x)dx = (-1)^\ell\left(p_\varepsilon * h_\ell^{(\ell)}\right)(K), \tag{6.23}$$

where $*$ denotes the convolution. Using that p_ε is an approximation of the identity, we obtain the limit

$$(-1)^\ell h_\ell^{(\ell)}(K). \tag{6.24}$$

$\square$

As a conclusion, we have

$$f_{i_1,\ldots,i_n}(s_1,\ldots,s_n;t)$$

$$= \sum_{\ell_1=0}^{k_1} \cdots \sum_{\ell_m=0}^{k_m} (-1)^{n-k_0-\ell_1-\cdots-\ell_m} \binom{k_1}{\ell_1} \cdots \binom{k_m}{\ell_m} \sigma^{k_0}$$

$$\times \sum_{j_1,\ldots,j_m=0}^{\infty} e^{-\tilde{\lambda}_1 T} \cdots e^{-\tilde{\lambda}_m T} \frac{\tilde{\lambda}_1^{j_1} \cdots \tilde{\lambda}_m^{j_m} T^{j_1+\cdots+j_m}}{j_1! \cdots j_m!} \tag{6.25}$$

$$\times \left[\left(C(j_1,\ldots,j_m) e^{(\sigma^2 T)/2} - K\mathbf{1}_{\{k_0=0\}} \right) \Phi\left(\frac{\log C(j_1,\ldots,j_m) - \log K}{\sigma\sqrt{T}} \right) \right.$$

$$\left. + \sum_{l=0}^{k_0-2} c_{k_0,l+2} (-1)^l h_l^{(l)}(K) \mathbf{1}_{\{k_0\geq 2\}} \right] e^{-r(T-t)} \mathbf{1}_{\{0<s_1<\cdots<s_n<t\}},$$

where

$$h_l(x) = x^{l+1} p_{\sigma\sqrt{T}}(\log x - \log C(j_1,\ldots,j_m)),$$

$$C(j_1,\ldots,j_m) = s_0 \exp\left\{ \sum_{i=1}^m \beta_i j_i + \left(r - \frac{\sigma^2}{2} + \sum_{i=1}^m (\beta_i \ell_i - \alpha_i \tilde{\lambda}_i) \right) T \right\}. \tag{6.26}$$

In the case of no jumps in the model, we obtain

$$f_n(s_1,\ldots,s_n;t) = \sigma^n \left[\left(S_0 e^{rT} - K\mathbf{1}_{\{n=0\}} \right) \Phi\left(\frac{\log C - \log K}{\sigma\sqrt{T}} \right) \right.$$

$$\left. + \sum_{l=0}^{n-2} c_{n,l+2} (-1)^l h_l^{(l)}(K) \mathbf{1}_{\{n\geq 2\}} \right] e^{-r(T-t)} \mathbf{1}_{\{0<s_1<\cdots<s_n<t\}}, \tag{6.27}$$

where now C reduces to $s_0 \exp\{(r - (\sigma^2/2))T\}$.

7. Chaos expansion of the tracking error in discrete delta-neutral hedging

Assume that we are in a neutral risk environment. That is, from now on, we are concerned with the unique probability $\mathbb{Q}$ that makes $e^{-rt}S_t$ a martingale. Here we follow some ideas from [5].

Let U_t be the prize of a derivative security. Its actualized price $\tilde{U}_t$ is given by

$$e^{-rt}U_t = E[e^{-rT}h(S_T)/\mathscr{F}_t]. \tag{7.1}$$

By the Markov property of S_t, the process $\tilde{U}_t$ can be written as a function of t and S_t. Moreover thanks to (6.6), we can write $\tilde{U}_t = u(t, Y_t)$, where $Y_t = ct + \sigma W_t^{\mathbb{Q}} + \sum_{i=1}^m \beta_i N_t^{\mathbb{Q},i}$ and $\beta_i = \ln(1 + \alpha_i)$. By its definition, $\tilde{U}_t$ is a martingale.

We set

$$u^1(s, Y_{s-}, y) := u(s, Y_{s-} + y) - u(s, Y_{s-}) - \frac{\partial u}{\partial x}(s, Y_{s-})y. \tag{7.2}$$

If the function u is regular, the Itô formula (see Protter [13, pages 81–82]) applied to $u(t, Y_t)$ yields

$$u(t, Y_t) - u(0, Y_0) = \int_0^t \left(\frac{\partial u}{\partial s}(s, Y_s) + c\frac{\partial u}{\partial x}(s, Y_s) \right) ds + \frac{1}{2} \int_0^t \sigma^2 \frac{\partial^2 u}{\partial x^2}(s, Y_s) ds$$

$$+ \int_0^t \sigma \frac{\partial u}{\partial x}(s, Y_{s-}) dW_s^{\mathbb{Q}} + \sum_{j=1}^m \beta_j \int_0^t \frac{\partial u}{\partial x}(s, Y_{s-}) dN_s^{\mathbb{Q},j} \tag{7.3}$$

$$+ \sum_{0 < s \leq t} u^1(s, Y_{s-}, \Delta Y_s).$$

Now, [9, Lemma 5, page 773] due to Nualart and Schoutens (2001) applied to $u^1(s, Y_{s-}, y)$ gives

$$\sum_{0 < s \leq t} u^1(s, Y_{s-}, \Delta Y_s) = \sum_{i=1}^{m+1} \int_0^t \left(\int_{\mathbb{R}} u^1(s, Y_{s-}, y) p_i(y) \nu(dy) \right) dH_s^{(i)}$$

$$+ \int_0^t \int_{\mathbb{R}} u^1(s, Y_{s-}, y) \nu(dy) ds, \tag{7.4}$$

where $p_i(y) = a_{i,i} x^i + a_{i,i-1} x^{i-1} + \cdots + a_{i,1} x$. Note that there are only $m+1$ processes H^i associated to Y which are different from zero (see [6]).

In our case the explicit expression of the right-hand side of the last equation is a martingale plus the following term:

$$\int_0^t \sum_{j=1}^m \left(\tau_{\beta_j} u(s, Y_{s-}) - \frac{\partial u}{\partial x}(s, Y_{s-}) \beta_j \right) \tilde{\lambda}_j ds, \tag{7.5}$$

where $\tau_\beta u(t, x) = u(t, x + \beta) - u(t, x)$.

Hence, the martingale property of $u(t, Y_t)$ is equivalent to the condition that u satisfies the following partial differential integral equation:

$$\frac{\partial u}{\partial t}(t, x) + c\frac{\partial u}{\partial x}(t, x) + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}(t, x) + \sum_{j=1}^m \tilde{\lambda}_j \tau_{\beta_j} u(t, x) = 0,$$

$$u(0, x) = u_0(x), \tag{7.6}$$

where u_0 is a given function.

Moreover $\partial_x^{n-j} \circ \tau_{\beta_{i_1}} \circ \cdots \circ \tau_{\beta_{i_j}} u(t, Y_t)$, $i_1, \ldots, i_j \in \{1, 2, \ldots, m\}$ and $j \in \{0, 1, \ldots, n\}$, is also a martingale since $\partial_x^{n-j} \circ \tau_{\beta_{i_1}} \circ \cdots \circ \tau_{\beta_{i_j}} u(t, x)$ satisfy (7.6).

Being $u(t, Y_t)$ a functional assumed to be square integrable, by the Stroock formula, its chaos expansion is

$$
\begin{aligned}
& u(t, Y_t) \\
& = \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n} J_n^{i_1, \ldots, i_n} \left(E[D_{s_1, \ldots, s_n}^{i_1, \ldots, i_n} u(t, Y_t)] \right) \\
& = \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n, k_0 + \cdots + k_m = n} E\left[\sigma^{k_0} \partial_x^{k_0} \circ \tau_{\beta_1}^{k_1} \circ \cdots \circ \tau_{\beta_m}^{k_m} u(t, Y_t) \right] J_n^{i_1, \ldots, i_n} \left(\mathbf{1}_{\{0 < s_1 < \cdots < s_n \leq t\}} \right) \\
& = \sum_{n=0}^{\infty} \sum_{i_1, \ldots, i_n} \sigma^{k_0} \partial_x^{k_0} \circ \tau_{\beta_1}^{k_1} \circ \cdots \circ \tau_{\beta_m}^{k_m} u(0, 0) J_n^{i_1, \ldots, i_n} \left(\mathbf{1}_{\{0 < s_1 < \cdots < s_n \leq t\}} \right).
\end{aligned}
\tag{7.7}
$$

From the Clark-Ocone formula, we have

$$
e^{rT} U_0 - U_T + \int_0^T \psi_0(s) \, d\widetilde{S}(s) + \sum_{j=1}^{m} \int_0^T \psi_j(s) \, d\widetilde{P}_j(s) = 0,
\tag{7.8}
$$

with

$$
(\psi_0(s), \ldots, \psi_m(s)) = (\phi_0(s), \ldots, \phi_m(s)) B^{-1}
\begin{pmatrix}
\dfrac{1}{S^*(s-)} & 0 & \cdots & 0 \\
0 & \dfrac{1}{P_1^*(s-)} & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & \dfrac{1}{P_m^*(s-)}
\end{pmatrix},
$$

$$
\phi_j(s) = E\left[D_s^{(j)} e^{-rT} U_T / \mathscr{F}_{s-} \right], \quad j = 0, \ldots, m.
\tag{7.9}
$$

It represents the value of a dynamic portfolio with a short position in one derivative and a suitable large position on the risky assets.

Let $\{t_0 = 0 < t_1 < \cdots < t_p = T\}$ be a partition of $[0, T]$ such that $t_{j+1} - t_j = \delta$ for all $j = 0, \ldots, p-1$, and hence $p\delta = T$.

Discrete hedging corresponds to the value process

$$
e^{rT} U_0 - U_T + \sum_{i=0}^{p-1} \int_{t_i}^{t_{i+1}} \psi_0(t_i) \, d\widetilde{S}(s) + \sum_{j=1}^{m} \sum_{i=0}^{p-1} \int_{t_i}^{t_{i+1}} \psi_j(t_i) \, d\widetilde{P}_j(s),
\tag{7.10}
$$

and the difference between values of the two portfolios is the so-called tracking error that can be written in this case as

$$E_T = \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} e^{-rT}\left(E[D_s^0 U_T/\mathscr{F}_s] - E[D_{t_j}^0 U_T]/\mathscr{F}_{t_j}\right)dW_s^{\mathbb{Q}}$$

$$+ \sum_{k=1}^{m} \sum_{j=0}^{p-1} \int_{t_j}^{t_{j+1}} e^{-rT}\left(E[D_s^k U_T/\mathscr{F}_s] - E[D_{t_j}^k U_T]/\mathscr{F}_{t_j}\right)d\tilde{N}_s^{\mathbb{Q},k}.$$

(7.11)

Of course, this is a centered random variable that converges to zero when the partition becomes finer and finer but, moreover, it has no first chaos. In fact

$$E_T = \sum_{n=2}^{\infty} \sum_{i_1,\dots,i_n} \sigma^{k_0}\partial_x^{k_0} \circ \tau_{\beta_1}^{k_1} \circ \cdots \circ \tau_{\beta_m}^{k_m} u(0,0) J_n^{i_1,\dots,i_n}\left(1 - \sum_{j=0}^{p-1} \mathbf{1}_{\{s_{n-1}<t_j<s_n\leq t_{j+1}\}}\right). \qquad (7.12)$$

Then

$$\mathrm{Var}(E_T) = \sum_{n=2}^{\infty} \sum_{i_1,\dots,i_n} \left[\sigma^{k_0}\partial_x^{k_0} \circ \tau_{\beta_1}^{k_1} \circ \cdots \circ \tau_{\beta_m}^{k_m} u(0,0)\right]^2 \int_0^T \int_0^{s_n} \cdots \int_0^{s_2} \mathbf{1}_{B_n(\pi)} ds_1 \cdots ds_n,$$

(7.13)

where $B_n(\pi)$ is the set of n-tuples $(s_1,\dots,s_n)$ such that s_{n-1} and s_n become in the same interval of the partition. On one hand,

$$\mathrm{Var}(E_T) \leq \sum_{n=2}^{\infty} \sum_{i_1,\dots,i_n} \left[\sigma^{k_0}\partial_x^{k_0} \circ \tau_{\beta_1}^{k_1} \circ \cdots \circ \tau_{\beta_m}^{k_m}(u)(0,0)\right]^2 \frac{T^n}{n!}. \qquad (7.14)$$

On the other hand, it is easy to see that

$$\int_0^T \int_0^{s_n} \cdots \int_0^{s_2} \mathbf{1}_{B_n(\pi)} ds_1 \cdots ds_n = \sum_{j=0}^{p-1}\left((j+1)^n - j^n - nj^{n-1}\right)\frac{\delta^n}{n!}$$

$$= P_{n-1}(p)\frac{\delta^n}{n!},$$

(7.15)

where P_{n-1} denotes a polynomial with degree $n-1$. Hence, $\mathrm{Var}(E_T) = O(\delta)$.

Under the historical probability, $W_t = W_t^{\mathbb{Q}} - \nu_0 t$ is a Brownian motion and $\tilde{N}_t^i = \tilde{N}_t^{\mathbb{Q},i} - \nu_i t$, for $i = 1,\dots,m$, are Poisson processes, where $\nu_0 = (\mu - r)/\sigma$ and $\nu_i = ((\lambda_i - \tilde{\lambda}_i)(\beta_i - \alpha_i)/\beta_i$.

If we define $Z_t = Y_t - \gamma t = ct + \sigma W_t + \sum_{i=1}^{m}\beta_i N_t^i$, with $\gamma = \sigma\nu_0 + \sum_{i=1}^{m}\beta_i\nu_i$, then we can write $u(t,Y_t) = u(t,Z_t + \gamma t)$. Note that $u(t,Z_t)$ is a $\mathbb{P}$-martingale since $u(t,x)$ satisfies (7.6) but $u(t,Z_t + \gamma t)$ is not.

Using the classical Taylor formula on $u(t,Z_t + \gamma t)$, we obtain

$$u(t,Z_t + \gamma t) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\partial^\ell u(t,Z_t)\gamma^\ell t^\ell. \qquad (7.16)$$

Therefore, the chaos expansion of the tracking error becomes

$$
E_T = \sum_{n=2}^{\infty} \sum_{i_1,\ldots,i_n} \sum_{\ell=0}^{\infty} \frac{\gamma^\ell T^\ell}{\ell!} \sigma^{k_0} \partial_x^{k_0+\ell} \circ \tau_{\beta_1}^{k_1} \circ \cdots \circ \tau_{\beta_m}^{k_m} u(0,0) J_n^{i_1,\ldots,i_n}(\mathbf{1}_{B_n(\pi)})
\tag{7.17}
$$

and has the same behavior as before.

Acknowledgments

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