ON FUZZY $\varphi\psi$-CONTINUOUS MULTIFUNCTION

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Here, we would study and characterize fuzzy $\varphi\psi$-continuity for fuzzy multifunctions which extend fuzzy $\varphi\psi$-continuity of fuzzy functions. Moreover, we obtain some results in fuzzy multifunctions.

1. Introduction and preliminaries

In the last three decades, the theory of multifunctions has advanced in a variety of ways and applications of this theory can be found, specially in functional analysis and fixed point theory. Recently many authors, for example, Albrycht and Matłoka [1] and Beg [3] have studied fuzzy multifunctions and have characterized some property of fuzzy multifunctions defined on a fuzzy topological space. Several authors have studied some type of fuzzy continuity for fuzzy functions and fuzzy multifunctions [1–5], [8–12]. In [3] fuzzy $\varphi\psi$-continuous functions have been studied. But this brand of fuzzy continuity has not considered for fuzzy multifunctions which we attempt to study and characterize.

The *fuzzy set* in (on) a universe X is a function with domain X and values in $I = [0,1]$. The class of all fuzzy sets on X will be denoted by I^X and symbols $A, B, \dots$ are used for fuzzy sets on X. 01_X is called *empty fuzzy set*, where 1_X is the characteristic function on X. For any fuzzy set A in X, the function value $A(x)$ is called the *grade of membership* of x in A. We write $x \in A$ if $A(x) > 0$. For any fuzzy set A, the fuzzy set $1 - A(x)$ is called the *complement* of A which is denoted by A^c. Let A and B be fuzzy sets in X, we write $A \leq B$ if $A(x) \leq B(x)$ for all x in X. For any family $\{A_\alpha\}_{\alpha \in \mathcal{A}}$ of fuzzy sets in X, $\bigvee_{\alpha \in \mathcal{A}} A_\alpha$ and $\bigwedge_{\alpha \in \mathcal{A}} A_\alpha$ are defined by $\sup_\alpha A_\alpha(x)$ and $\inf_\alpha A_\alpha(x)$, respectively. A family τ of fuzzy sets in X is called a *fuzzy topology* for X if (i) $\alpha 1_X \in \tau$ for each $\alpha \in I$; (ii) $A \wedge B \in \tau$ where $A, B \in \tau$ and (iii) $\bigvee_{\alpha \in \mathcal{A}} A_\alpha \in \tau$ whenever $A_\alpha \in \tau$ for all α in $\mathcal{A}$. The pair (X, τ) is called a *fuzzy topological space* [6]. Every member of τ is called *fuzzy open set* and its complements

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are called *fuzzy closed sets* [6]. In a fuzzy topological space X the *interior* and the *closure* of a fuzzy set A (simply int(A) and cl(B), resp.) are defined by

$$\begin{aligned} \operatorname{int}(A) &= \bigvee\{U : A \le U,\ U \text{ is a fuzzy open set}\}, \\ \operatorname{cl}(A) &= \bigwedge\{F : A \le F,\ F \text{ is a fuzzy closed set}\}. \end{aligned} \tag{1.1}$$

A *neighborhood of a fuzzy set* A in a fuzzy topological space X is any fuzzy set B for which there is a fuzzy open set V satisfying $A \le V \le B$. Any fuzzy open set V that satisfies $A \le V$ is called a *fuzzy open neighborhood* of A [10]. A fuzzy set A is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say $x \in X$. If its value at x is $\epsilon(0 \le \epsilon \le 1)$, we denote this fuzzy point by x_ϵ [11]. For any fuzzy point x_ϵ and any fuzzy set A we write $x_\epsilon \in A$ if and only if $\epsilon \le A(x)$. Let f be a function from X to Y. A fuzzy function $f : X \to Y$ is defined by

$$f(A)(y) = \begin{cases} \bigvee_{x \in f^{-1}(\{y\})} A(x) & f^{-1}(\{y\}) \neq \varnothing, \\ 0 & f^{-1}(\{y\}) = \varnothing, \end{cases} \tag{1.2}$$

for all y in Y, where A is an arbitrary fuzzy set in X [12]. A fuzzy function $f : X \to Y$ is called *fuzzy continuous* if for each $x_\epsilon \in X$ and each fuzzy neighborhood B of $f(x_\epsilon)$ there exists a fuzzy neighborhood A of x_ϵ such that $f(A) \le B$ [11]. A *fuzzy multifunction* $f : X \to\to Y$ assigns to each x in X a fuzzy set $f(x)$ of Y [2]. If A is a fuzzy set in X, then the fuzzy set $f(A)$ in Y is defined by

$$f(A)(y) = \bigvee_{x \in X} \big(f(x)(y) \wedge A(x)\big). \tag{1.3}$$

For more details about fuzzy multifunctions and their properties, the reader is referred to [1, 2, 10]. Throughout this paper, (X,τ) and (Y,ν) are fuzzy topological spaces. The symbol $f : X \to\to Y$ is used for a fuzzy multifunction from X to Y, while $f : X \to Y$ for a fuzzy function from X to Y.

Main results

Definition 1.1. (i) A fuzzy function φ on X is called a *fuzzy operation* on X, if $\varphi(01_X) = 01_X$ and int$(A) \le \varphi(A)$, where A is any nonempty fuzzy set in X. φ is called a *monotonous fuzzy operation*, if $\varphi(A) \le \varphi(B)$, whenever $A, B \in I^X$ and $A \le B$ [5].

(ii) $f : X \to Y$ is called a *fuzzy $\varphi\psi$-continuous function at $x_\epsilon \in X$* if for each fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \le \psi(B)$, where φ and ψ are fuzzy operation on X and Y, respectively. $f : X \to Y$ is said to be a *fuzzy $\varphi\psi$-continuous function* if it is a fuzzy $\varphi\psi$-continuous function at each $x_\epsilon \in X$.

Definition 1.2. (i)$f : X \to\to Y$ is called a *fuzzy $\varphi\psi$-continuous multifunction at $x_\epsilon \in X$* if for each fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \le \psi(B)$, where φ and ψ are fuzzy operation on X and Y, respectively. $f : X \to\to Y$ is said to be a *fuzzy $\varphi\psi$-continuous multifunction* if it is a fuzzy $\varphi\psi$-continuous multifunction at each $x_\epsilon \in X$.

(ii) $f : X \to\to Y$ is called a *single valued fuzzy multifunction* if f at each x is a fuzzy point $1_{\{y_x\}}$, where $y_x \in Y$. In this case it would induce a fuzzy function $\tilde{f} : X \to Y$ by $\tilde{f}(x) = y_x$. Therefore, $f(x) = 1_{\{\tilde{f}(x)\}}$.

Proposition 1.3. *Suppose $f : X \to\to Y$ be a single valued fuzzy multifunction. Then for any fuzzy set A in X; $f(A) = \tilde{f}(A)$. Therefore, f is a $\varphi\psi$-continuous multifunction if and only if $\tilde{f}$ is a fuzzy $\varphi\psi$-continuous function.*

Proof. The equivalence for any fuzzy set A of X can be derived from the following fact:

$$f(\varphi(A))(y) = \bigvee_{z\in X} (f(z)(y) \wedge \varphi(A)(z)) = \bigvee_{z\in X} (1_{\{\tilde{f}(z)\}}(y) \wedge \varphi(A)(z)). \tag{1.4}$$

On the other hand,

$$\tilde{f}(\varphi(A))(y) = \begin{cases} \bigvee_{h\in\tilde{f}^{-1}(\{y\})} \varphi(A)(h) & \tilde{f}^{-1}(\{y\}) \neq \emptyset, \\ 0 & \tilde{f}^{-1}(\{y\}) = \emptyset. \end{cases} \tag{1.5}$$

Therefore, $f(\varphi(A)) = \tilde{f}(\varphi(A))$. Now replacing identity function as a fuzzy operation on X instead of φ completes the proof. □

From the above result this brand of continuity for fuzzy multifunctions is in fact a generalization of $\varphi\psi$-continuity introduced in [3]. Next, we would like to present a result showing the relation between fuzzy $\varphi\psi$-continuity and fuzzy $\varphi\psi$-continuity in respect of nets. First we note to the following results.

Proposition 1.4. *Suppose $f : X \to\to Y$ be a fuzzy multifunction and $A, B \in I^X$ such that $A \leq B$. Then $f(A) \leq f(B)$.*

Proof.

$$f(A)(y) = \bigvee_{z\in X} (f(z)(y) \wedge A(z)) \leq \bigvee_{z\in X} (f(z)(y) \wedge B(z)) = f(B)(y). \tag{1.6}$$

□

Lemma 1.5. *Let $f : X \to\to Y$ be a fuzzy multifunction and let x_ϵ be a fuzzy point in X. Then $f(x_\epsilon) = f(x) \wedge \epsilon$.*

Proof. It is straightforward. □

We say that a net $(x^\alpha_{\epsilon_\alpha})_{\alpha\in\mathcal{A}}$ of fuzzy points in a fuzzy topological space X is *φ-convergent* to a fuzzy point x_ϵ (we will denote it by $x^\alpha_{\epsilon_\alpha} \xrightarrow{\varphi} x_\epsilon$) if for any neighborhood set A of x_ϵ, there is an $\alpha_0 \in \mathcal{A}$ in which $x^\alpha_{\epsilon_\alpha} \in \varphi(A)$ for all $\alpha \geq \alpha_0$.

Lemma 1.6. *Consider a fuzzy set A and a convergent net (A_α) of fuzzy sets which $A_\alpha \to A$ in X. Then $A_\alpha \xrightarrow{\varphi} A$.*

Proof. From the assumption given any fuzzy open neighborhood B of A, there is $\alpha_0 \in \mathcal{A}$ such that for all $\alpha \geq \alpha_0$,

$$A_\alpha \leq B = \operatorname{int}(B) \leq \varphi(B). \tag{1.7}$$

□

A fuzzy multifunction $f : X \to\to Y$ is called *net-fuzzy $\varphi\psi$-continuous* if for each net of fuzzy points $x^\alpha_{\epsilon_\alpha}$ and x_ϵ in X, $f(x^\alpha_{\epsilon_\alpha}) \xrightarrow{\psi} f(x_\epsilon)$, where $(x^\alpha_{\epsilon_\alpha}) \xrightarrow{\varphi} x_\epsilon$.

Theorem 1.7. *Let X be a fuzzy topological space. For any fuzzy multifunction $f : X \to\to Y$ the following are equivalent:*

(i) *f is a fuzzy $\varphi\psi$-continuous;*

(ii) *f is a net-fuzzy $\varphi\psi$-continuous.*

Proof. (i)⇒(ii).

For any fuzzy open neighborhood B of $f(x_\epsilon)$, there is a fuzzy open neighborhood A of x_ϵ such that

$$f(\varphi(A)) \leq \psi(B). \tag{1.8}$$

From the assumption, there is $\alpha_0 \in A$ for which

$$x^\alpha_{\epsilon_\alpha} \leq \varphi(A) \quad (\forall \alpha \geq \alpha_0). \tag{1.9}$$

According to Proposition 1.4, $f(x^\alpha_{\epsilon_\alpha}) \leq f(\varphi(A)) \leq \psi(B)$.

(ii)⇒(i).

On the contrary, there is a fuzzy point x_ϵ in X, a fuzzy open neighborhood B of $f(x_\epsilon)$ such that, there is not a fuzzy neighborhood A of x_ϵ satisfying in $f(\varphi(A)) \leq \psi(B)$. This means that there is $z_A \epsilon Y$ with the following property:

$$f(\varphi(A))(z_A) > \psi(B)(z_A). \tag{1.10}$$

Therefore,

$$\bigvee_{x\in X} (f(x)(z_A) \wedge \varphi(A)(x)) > \psi(B)(z_A). \tag{1.11}$$

Then $f(x_A)(z_A) \wedge \varphi(A)(z_A) > \psi(B)(z_A)$ for a suitable x_A of X. We conclude that

$$f(x_A)(z_A) > \psi(B)(z_A). \tag{1.12}$$

Consider $\{A_\alpha : \alpha \in \mathcal{A}\}$ as a system of fuzzy neighborhoods at x_ϵ. The following order makes $\mathcal{A}$ as a directed set and so it makes $\{A_\alpha : \alpha \in \mathcal{A}\}$ as a net:

$$\alpha \leq \beta \Longleftrightarrow A_\beta \leq A_\alpha. \tag{1.13}$$

Applying (1.12) for A_α instead of A, there is $x^\alpha_{\epsilon_\alpha}$ in A_α for which $f(x^\alpha_{\epsilon_\alpha}) > \psi(B)$. From the choice of $x^\alpha_{\epsilon_\alpha}$ in A_α, $x^\alpha_{\epsilon_\alpha} \to x_\epsilon$. Lemma 1.6 implies that $x^\alpha_{\epsilon_\alpha} \xrightarrow{\varphi} x_\epsilon$. Since $f(x^\alpha_{\epsilon_\alpha})(z_\alpha) = f(x_\alpha)(z_\alpha) \wedge \epsilon_\alpha \leq f(x_\alpha)(z_\alpha)$ so from (1.12), $f(x^\alpha_{\epsilon_\alpha})$ is not ψ-convergent to $f(x_\epsilon)$, which completes the proof. □

In the following result we show continuity of the composition of two fuzzy multifunction. Suppose $f : X \to\to Y$ and $g : Y \to\to Z$. We define *composition* $gof : X \to\to Z$ by $(gof)(x) = g(f(x)) = \bigcup_{t\in f(x)} g(t)$.

Corollary 1.8. *Suppose $f : X \to\to Y$ be a fuzzy $\varphi\psi$-continuous single valued multifunction and suppose $g : Y \to\to Z$ be $\psi\eta$-fuzzy continuous multifunction. Then, $gof : X \to\to Z$ is $\varphi\eta$-fuzzy continuous multifunction.*

Proof. Assume that $x^{\alpha}_{\epsilon_\alpha} \xrightarrow{\varphi} x_\epsilon$ in X. Since f is fuzzy $\varphi\psi$-continuous multifunction, so

$$f(x^{\alpha}_{\epsilon_\alpha}) \xrightarrow{\psi} f(x_\epsilon). \tag{1.14}$$

Assume that g is $\psi\eta$-fuzzy continuous multifunction and f is fuzzy $\varphi\psi$-continuous single valued multifunction, so

$$g(f(x^{\alpha}_{\epsilon_\alpha})) \xrightarrow{\eta} g(f(x_\epsilon)). \tag{1.15}$$

Therefore,

$$(gof)(x^{\alpha}_{\epsilon_\alpha}) \xrightarrow{\eta} (gof)(x_\epsilon). \tag{1.16}$$

Theorem 1.7 completes the proof. □

Definition 1.9. Let X_0 be a subset of X, let $i : X_0 \to X$ be the inclusion map, and let $f : X \to\to Y$ be a fuzzy multifunction. Say that foi is the *restriction* of f to X_0.

Lemma 1.10. *Assuming φ is a fuzzy operation on X and $X_0 \subseteq X$. Then $\widetilde{\varphi}(A) = \varphi(\widetilde{A})$ defines a fuzzy operation on X_0, where $\widetilde{A}$ is the extension of A by zero to X.*

Proof. It is easy to see that $\widetilde{\varphi}$ is a well-defined map and $\widetilde{\varphi}(01_X) = 01_X$. φ is a fuzzy operation, so $\operatorname{int}(\widetilde{A}) \le \varphi(\widetilde{A})$. But,

$$\begin{aligned} \operatorname{int}(A) &= \bigvee\{\acute{U} : \acute{U} \le A,\ \acute{U} \text{ is a fuzzy open set}\} \\ &= \bigvee\{Uoi : U \le A,\ U \text{ is a fuzzy open set}\}. \end{aligned} \tag{1.17}$$

For $x_0 \in X_0$, $\operatorname{int}(A)(x_0) = \operatorname{int}(\widetilde{A})(x_0)$. This shows that

$$\operatorname{int}(A) \le \varphi(\widetilde{A}) \le \widetilde{\varphi}(A). \tag{1.18}$$

□

The following result shows the fuzzy continuity of the restriction of fuzzy multifunction.

Theorem 1.11. *Suppose $f : X \to\to Y$ be a fuzzy $\varphi\psi$-continuous multifunction and $X_0 \subseteq X$. Then foi is a $\widetilde{\varphi}\psi$-fuzzy continuous multifunction, where φ is a monotonous fuzzy operation.*

Proof. For any fuzzy point x_ϵ in X_0, $j(x_\epsilon)$ is a fuzzy point in X. It shows that for any fuzzy open neighborhood B of $f(i(x_\epsilon))$, there is a fuzzy open neighborhood A of $i(x_\epsilon)$

for which $f(\varphi(A)) \le \psi(B)$. But Aoi is a fuzzy open neighborhood of x_ϵ in X_0, only we must show that $foi(\widetilde{\varphi}(Aoi)) \le \psi(B)$. To see this,

$$\begin{aligned} foi(\widetilde{\varphi}(Aoi))(y) &= \bigvee_{z\in X_0} ((foi)(z)(y) \wedge \widetilde{\varphi}(Aoi)(z)) \\ &= \bigvee_{z\in X_0} (f(z)(y) \wedge \varphi\widetilde{(Aoi)}oi(z)) \\ &\le \bigvee_{z\in X_0} (f(z)(y) \wedge \varphi(A)(z)) \\ &= f(\varphi(A))(y) \\ &\le \psi(B)(y). \end{aligned} \tag{1.19}$$

□

Proposition 1.12. *Suppose (X,τ) and (Y,η) be fuzzy topological spaces, φ and ψ are fuzzy operations on X and Y, respectively, where φ is a monotonous fuzzy operation. Let $f : X \to\to Y$ be any fuzzy multifunction and let $\mathcal{B}$ be a base for η. Then f is fuzzy $\varphi\psi$-continuous multifunction if and only if f is fuzzy $\varphi\psi$-continuous multifunction with respect to $\mathcal{B}$.*

Proof. $(\Rightarrow)$ is straightforward.

For $(\Leftarrow)$ consider any fuzzy point x_ϵ in X and any fuzzy open neighborhood B of $f(x_\epsilon)$. $C \in \mathcal{B}$ exists such that $f(x_\epsilon) \le C$ and $C \le B$. From the assumption there is a fuzzy open neighborhood A of x_ϵ such that $f(\varphi(A)) \le \psi(C)$. But ψ is monotonous so $f(\varphi(A)) \le \psi(B)$. □

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