Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2007, Article ID 56924, 6 pages
doi:10.1155/2007/56924

Research Article

Laws of Large Numbers for Asymmetrical Cauchy Random Variables

André Adler

Received 16 August 2006; Accepted 20 November 2006

We generalize the Cauchy distribution so that we can have asymmetrical tails. This allows us to obtain unusual laws of large numbers involving weighted sums of these random variables. Unusual in the sense that even though in every case $E|X| = \infty$, we can still obtain a nonzero limit for these weighted sums.

1. Introduction

In this paper, we observe weighted sums of what we call asymmetrical Cauchy random variables. They are the usual Cauchy random variables with a slight twist. If our random variables were symmetrical, then the limit to all of our theorems would be zero, which certainly holds true in the case where our two parameters p and q are equal, but is of minor interest. The density we use is

$$f(x) = \begin{cases} \dfrac{p}{\pi(1+x^2)}, & \text{if } x \geq 0, \\ \dfrac{q}{\pi(1+x^2)}, & \text{if } x < 0, \end{cases} \tag{1.1}$$

where $p+q=2$. If we let $p=q=1$, we get the usual Cauchy distribution.

Our goal is to establish laws of large numbers for weighted sums of these random variables. It should be noted that $E|X| = \infty$ in every case. We will show which sequences of constants $\{a_n,\ n \geq 1\}$ and $\{b_n,\ n \geq 1\}$ will allow our partial sums $\sum_{n=1}^{N} a_n X_n / b_N$ to converge to a nonzero constant.

As usual, we define $\lg x = \log(\max\{e,x\})$ and $\lg_2 x = \lg(\lg x)$. We use the constant C to denote a generic real number that is not necessarily the same in each appearance. Before we present our theorems there is a crucial lemma that all our results hinge upon. The proof follows directly from L'Hopital's Rule, so we will omit it.

Lemma 1.1.

$$\lim_{x\to\infty}\frac{\pi-2\arctan x}{x^{-1}}=2. \tag{1.2}$$

2. Strong law

From past results we know that only certain types of coefficients will allow us to establish a strong law. It turns out that a_n must be of the order n^{-1}. Naturally we can multiply this weight by a slowly varying function such as our logarithms, but we must be careful how far we go with that.

Theorem 2.1. *If $\{X_n,\ n\ge 1\}$ are i.i.d. asymmetrical Cauchy random variables, then for all $\beta>0$ one has*

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{N}((\lg n)^{\beta-2}/n)X_n}{(\lg N)^{\beta}}=\frac{p-q}{\pi\beta}\quad \textit{almost surely.} \tag{2.1}$$

Proof. Let $a_n=(\lg n)^{\beta-2}/n$, $b_n=(\lg n)^{\beta}$, and $c_n=b_n/a_n=n(\lg n)^2$. We use the partition

$$\begin{aligned}\frac{1}{b_N}\sum_{n=1}^{N}a_nX_n&=\frac{1}{b_N}\sum_{n=1}^{N}a_n\big[X_nI(|X_n|\le c_n)-EXI(|X|\le c_n)\big]\\&\quad+\frac{1}{b_N}\sum_{n=1}^{N}a_nX_nI(|X_n|>c_n)+\frac{1}{b_N}\sum_{n=1}^{N}a_nEXI(|X|\le c_n).\end{aligned} \tag{2.2}$$

The first term vanishes almost surely by the Khintchine-Kolmogorov convergence theorem, see [1, page 113], and Kronecker's lemma since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{c_n^2}EX^2I(|X|\le c_n)&=\sum_{n=1}^{\infty}\frac{1}{c_n^2}\left[\int_{-c_n}^{0}\frac{qx^2dx}{\pi(1+x^2)}+\int_{0}^{c_n}\frac{px^2dx}{\pi(1+x^2)}\right]\\&\le C\sum_{n=1}^{\infty}\frac{1}{c_n^2}\left[\int_{-c_n}^{0}dx+\int_{0}^{c_n}dx\right]\\&\le C\sum_{n=1}^{\infty}\frac{1}{c_n}=C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.\end{aligned} \tag{2.3}$$

The second term vanishes, with probability one, by the Borel-Cantelli lemma and our lemma since

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\{|X|>c_n\} &= \sum_{n=1}^{\infty}\left[\int_{-\infty}^{-c_n}\frac{q\,dx}{\pi(1+x^2)}+\int_{c_n}^{\infty}\frac{p\,dx}{\pi(1+x^2)}\right] \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[-q\arctan c_n+\frac{q\pi}{2}+\frac{p\pi}{2}-p\arctan c_n\right] \\
&= \frac{p+q}{2\pi}\sum_{n=1}^{\infty}\left[\pi-2\arctan c_n\right]\le C\sum_{n=1}^{\infty}\frac{1}{c_n}=C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.
\end{aligned}
\tag{2.4}
$$

The truncated first moment is

$$
\begin{aligned}
EXI(|X|\le c_n) &= \left[\int_{-c_n}^{0}\frac{qx\,dx}{\pi(1+x^2)}+\int_{0}^{c_n}\frac{px\,dx}{\pi(1+x^2)}\right] \\
&= \frac{1}{2\pi}\left[-q\lg\left(1+c_n^2\right)+p\lg\left(1+c_n^2\right)\right] \\
&= \frac{p-q}{2\pi}\lg\left(1+c_n^2\right)\sim\frac{p-q}{\pi}\lg c_n\sim\frac{p-q}{\pi}\lg n.
\end{aligned}
\tag{2.5}
$$

Therefore

$$
\frac{\sum_{n=1}^{N}a_n EXI(|X|\le c_n)}{b_N}\sim\frac{(p-q)\sum_{n=1}^{N}(\lg n)^{\beta-1}/n}{\pi(\lg N)^{\beta}}\longrightarrow\frac{p-q}{\pi\beta},
\tag{2.6}
$$

which completes the proof. □

3. Weak law

In order to establish a strong law, with nonzero limit, for these types of random variables one is forced to set a_n to be some slowly varying function divided by n, while b_n must also be slowly varying. If one wants to try more conventional constants such as $a_n = 1$ and $b_n = n$, we will have to set our sights a bit lower and settle for a weak law.

Theorem 3.1. *If $\{X_n,\ n\ge 1\}$ are i.i.d. asymmetrical Cauchy random variables, then for all $\alpha > -1$ and any slowly varying function $L(\cdot)$ one has as $N\to\infty$*

$$
\frac{\sum_{n=1}^{N}n^{\alpha}L(n)X_n}{N^{\alpha+1}L(N)\lg N}\xrightarrow{P}\frac{p-q}{\pi(\alpha+1)}.
\tag{3.1}
$$

Proof. This proof is a consequence of the degenerate convergence theorem which can be found on [1, page 356]. As usual, set $a_n = n^{\alpha}L(n)$ and $b_n = n^{\alpha+1}L(n)\lg n$. By choosing N

sufficiently large, we have b_N/a_n as large as we wish, thus for all $\epsilon > 0$, we have via our lemma

$$\begin{aligned}
\sum_{n=1}^{N} P\left\{|X| \geq \frac{\epsilon b_N}{a_n}\right\} &= \sum_{n=1}^{N}\left[\int_{-\infty}^{-\epsilon b_N/a_n} \frac{q\,dx}{\pi(1+x^2)} + \int_{\epsilon b_N/a_n}^{\infty} \frac{p\,dx}{\pi(1+x^2)}\right] \\
&= \frac{1}{\pi}\sum_{n=1}^{N}\left[-q\arctan\frac{\epsilon b_N}{a_n} + \frac{q\pi}{2} + \frac{p\pi}{2} - p\arctan\frac{\epsilon b_N}{a_n}\right] \\
&= \frac{1}{\pi}\sum_{n=1}^{N}\left[(p+q)\frac{\pi}{2} - (p+q)\arctan\frac{\epsilon b_N}{a_n}\right] \\
&= \frac{1}{\pi}\sum_{n=1}^{N}\left[\pi - 2\arctan\frac{\epsilon b_N}{a_n}\right] \\
&< \frac{C\sum_{n=1}^{N} a_n}{b_N} = \frac{C\sum_{n=1}^{N} n^\alpha L(n)}{N^{\alpha+1}L(N)\lg N} < \frac{C}{\lg N} \longrightarrow 0,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\sum_{n=1}^{N} \frac{a_n^2}{b_N^2} EX^2 I\left(|X| \leq \frac{b_N}{a_n}\right) &= \sum_{n=1}^{N} \frac{a_n^2}{b_N^2}\left[\int_{-b_N/a_n}^{0} \frac{qx^2\,dx}{\pi(1+x^2)} + \int_{0}^{b_N/a_n} \frac{px^2\,dx}{\pi(1+x^2)}\right] \\
&< \frac{C}{b_N^2}\sum_{n=1}^{N} a_n^2\left[\int_{-b_N/a_n}^{0} dx + \int_{0}^{b_N/a_n} dx\right] \\
&< \frac{C}{b_N^2}\sum_{n=1}^{N} a_n^2\left[\frac{b_N}{a_n}\right] = \frac{C\sum_{n=1}^{N} a_n}{b_N} < \frac{C}{\lg N} \longrightarrow 0
\end{aligned} \tag{3.3}$$

as in the previous calculation.

As for our truncated expectation, using our work from the proof of Theorem 2.1, we have

$$\begin{aligned}
&\sum_{n=1}^{N} \frac{a_n}{b_N} EXI\left(|X| \leq \frac{b_N}{a_n}\right) \\
&\quad\sim \frac{p-q}{\pi}\sum_{n=1}^{N} \frac{a_n}{b_N}\lg\frac{b_N}{a_n} = \frac{p-q}{\pi b_N}\sum_{n=1}^{N} a_n[\lg(b_N) - \lg(a_n)] \\
&\quad= \frac{p-q}{\pi N^{\alpha+1}L(N)\lg N}\sum_{n=1}^{N} n^\alpha L(n)[\lg(N^{\alpha+1}L(N)\lg N) - \lg(n^\alpha L(n))] \\
&\quad= \frac{p-q}{\pi}\left[\frac{(\alpha+1)\sum_{n=1}^{N} n^\alpha L(n)}{N^{\alpha+1}L(N)} + \frac{\sum_{n=1}^{N} n^\alpha L(n)\lg(L(N))}{N^{\alpha+1}L(N)\lg N}\right. \\
&\qquad\left. + \frac{\sum_{n=1}^{N} n^\alpha L(n)\lg_2 N}{N^{\alpha+1}L(N)\lg N} - \frac{\alpha\sum_{n=1}^{N} n^\alpha L(n)\lg n}{N^{\alpha+1}L(N)\lg N} - \frac{\sum_{n=1}^{N} n^\alpha L(n)\lg(L(n))}{N^{\alpha+1}L(N)\lg N}\right].
\end{aligned} \tag{3.4}$$

The first term converges to one since

$$\frac{(\alpha+1)\sum_{n=1}^{N} n^{\alpha}L(n)}{N^{\alpha+1}L(N)} \longrightarrow \frac{\alpha+1}{\alpha+1} = 1. \tag{3.5}$$

The second term converges to zero since

$$\frac{\sum_{n=1}^{N} n^{\alpha}L(n)\lg\left(L(N)\right)}{N^{\alpha+1}L(N)\lg N} < \frac{C\lg\left(L(N)\right)}{\lg N} \longrightarrow 0 \tag{3.6}$$

using the fact that $L(\cdot)$ is slowly varying. Similarly the third term is bounded above by

$$\frac{C\lg_2 N}{\lg N} \longrightarrow 0. \tag{3.7}$$

However, the fourth term

$$\frac{-\alpha\sum_{n=1}^{N} n^{\alpha}L(n)\lg(n)}{N^{\alpha+1}L(N)\lg N} \longrightarrow \frac{-\alpha}{\alpha+1}. \tag{3.8}$$

Lastly, we have

$$\frac{\sum_{n=1}^{N} n^{\alpha}L(n)\lg\left(L(n)\right)}{N^{\alpha+1}L(N)\lg N} < \frac{C\lg\left(L(N)\right)}{\lg N} \longrightarrow 0. \tag{3.9}$$

Collecting all our terms we have

$$\sum_{n=1}^{N} \frac{a_n}{b_N} EXI\left(|X| \leq \frac{b_N}{a_n}\right) \longrightarrow \frac{p-q}{\pi}\left[1-\frac{\alpha}{\alpha+1}\right] = \frac{p-q}{\pi(\alpha+1)} \tag{3.10}$$

which completes this proof. □

4. Discussion

It is important to note here that there is not a comparable strong law to Theorem 3.1. We see that in the ensuing result.

THEOREM 4.1. *If $\{X_n,\ n \geq 1\}$ are i.i.d. asymmetrical Cauchy random variables, then for all $\alpha > -1$ and any slowly varying function $L(\cdot)$ one has*

$$\limsup_{N\to\infty} \left| \frac{\sum_{n=1}^{N} n^{\alpha}L(n)X_n}{N^{\alpha+1}L(N)\lg N} \right| = \infty \quad \textit{almost surely}. \tag{4.1}$$

Proof. We once again use our lemma, but now in the opposite direction. Here $a_n = n^{\alpha}L(n)$, $b_n = n^{\alpha+1}L(n)\lg n$, and $c_n = n\lg n$. If $M > 0$, then

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{|X| > Mc_n\} &= \sum_{n=1}^{\infty}\left[\int_{-\infty}^{-Mc_n}\frac{q\,dx}{\pi(1+x^2)}+\int_{Mc_n}^{\infty}\frac{p\,dx}{\pi(1+x^2)}\right]\\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[-q\arctan(Mc_n)+\frac{q\pi}{2}+\frac{p\pi}{2}-p\arctan(Mc_n)\right]\\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(p+q)\frac{\pi}{2}-(p+q)\arctan(Mc_n)\right]\\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\pi-2\arctan(Mc_n)\right]\geq C\sum_{n=1}^{\infty}\frac{1}{c_n}=C\sum_{n=1}^{\infty}\frac{1}{n\lg n}=\infty.
\end{aligned} \tag{4.2}$$

Thus

$$\limsup_{n\to\infty}\left|\frac{a_nX_n}{b_n}\right| = \infty \quad \text{almost surely,} \tag{4.3}$$

which implies that

$$\limsup_{N\to\infty}\left|\frac{\sum_{n=1}^{N} n^{\alpha}L(n)X_n}{N^{\alpha+1}L(N)\lg N}\right| = \infty \quad \text{almost surely,} \tag{4.4}$$

and thus the proof is complete. □

References

[1] Y. S. Chow and H. Teicher, *Probability Theory. Independence, Interchangeability, Martingales*, Springer Texts in Statistics, Springer, New York, NY, USA, 3rd edition, 1997.

André Adler: Department of Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA
Email address: adler@iit.edu