Hindawi Publishing Corporation
Journal of Applied Mathematics and Stochastic Analysis
Volume 2007, Article ID 81934, 9 pages
doi:10.1155/2007/81934

Research Article

Some Local Asymptotic Laws for the Cauchy Process on the Line

A. Chukwuemeka Okoroafor

Received 22 September 2006; Revised 19 January 2007; Accepted 9 May 2007

This paper investigates the lim inf behavior of the sojourn time process and the escape rate process associated with the Cauchy process on the line. The monotone functions associated with the lower asymptotic growth rate of the sojourn time are characterized and the asymptotic size of the large values of the escape rate process is developed.

1. Introduction

Let $X(t) = \{X(t,\omega), t \geq 0\}$ be a Levy process on a probability space (Ω, f, P) with values in $\mathbb{R}^n$. We are interested in the sample path properties of the function $X(t) = X(t,\omega)$ for a fixed $\omega \in \Omega$. Let $X(t)$, $t \geq 0$, denote a Levy process and define

$$T(r) = \int_0^\tau I_{\{|X(s)| \leq r\}} ds, \tag{1.1}$$

where

$$\tau = \inf\{s > o : |X(s)| > 1\}. \tag{1.2}$$

Let $B(0,r)$ denote the ball in $\mathbb{R}^n$ of radius r centered at zero, then, $T(r)$ is the sojourn time of $X(t)$ in $B(0,r)$ up to time τ.

It is well known (see [1, 2]) that the sojourn time forms a useful tool in studying the local geometric properties of fractal sets determined by the sample paths of Levy processes in $\mathbb{R}^n$.

For example, the application of the density theorem of Taylor and Tricot [3] which remains one of the main tools for establishing packing measure and packing dimension results relies on the lower growth rate of $T_1(r)+T_2(r)$. Here T_1 and T_2 are independent copies of T.

Results in [4] show that the lower growth rate of $T_1(r)+T_2(r)$ is of higher order of magnitude than that of $T(r)$ for the symmetric stable processes of index $\alpha>1$ in $\mathbb{R}^n$. Pruitt and Taylor [5] further observed that they may differ by a factor $|\log r|(\log|\log r|)^{1/2}$.

The aim of this note is to investigate the lower asymptotic behaviour of $T(r)$ for the Cauchy process on the line. This may serve as a useful tool to characterize the geometric structure of the random set determined by the symmetric Cauchy processes and also we consider asymptotic result, which may be related to the sojourn time process.

2. Preliminaries

A symmetric Cauchy process on the line, which we will denote by $X(t)$, is a Levy process which is uniquely determined by its Fourier transform

$$\int_{-\infty}^{\infty} \exp[ixz]g(t,x)dx = \exp[-t|z|], \tag{2.1}$$

where $g(t,x) = (1/\pi)(t/(t^2+x^2))$. $t>0$, $x \in \mathbb{R}^1$ and satisfies the scaling property that $c^{-1}X(ct)$ is a version of the same process $X(t)$ for every $c>0$. $X(t)$ is recurrent, that is, $\{t : X(t) \in G\}$ is unbounded for an open interval G containing the origin. In this case, $T(r)$ is almost surely infinite as $r \to \infty$. Thus we instead consider the process

$$f(s) = f(s,\omega) = \inf\{|X(t)| : s \le t \le \tau\} \tag{2.2}$$

so that we have the relationship

$$\{\omega : f(s) > r\} \subseteq \{\omega : T(r) \le s\}. \tag{2.3}$$

The first passage time is, as usual, defined by

$$P(a) = P(a,\omega) = \inf\{t : |X(t)| > a\}, \tag{2.4}$$

whose distribution is obtained from that of

$$M(t) = M(t,\omega) = \sup_{0\le h\le t} |X(h)| \tag{2.5}$$

by the means of an obvious relationship

$$\{\omega : P(a) < r\} = \{\omega : M(r) > a\}, \quad a>0,\ r>0. \tag{2.6}$$

We will need the estimates for the distribution of the following events which we state as lemmas.

Lemma 2.1. *For the symmetric Cauchy process on the line $X(t)$,*

$$P\{|X(1)| > s\} \sim s^{-1} \quad \text{as } s \longrightarrow \infty. \tag{2.7}$$

Write $g_1(s) \sim g_2(s)$ *if* g_1 *and* g_2 *are asymptotic, that is,*

$$\lim_{s\to\infty} \frac{g_1(s)}{g_2(s)} = 1. \tag{2.8}$$

Proof. This is the consequence of [6, Lemma 2.2]. □

Lemma 2.2 [7]. *If* $\tau_E = \inf\{t > 0 : X(t) \in E\}$,

$$\Gamma_1 = S(\delta_1) = \{x \in \mathbb{R}^1 : |x| \le \delta_1\}, \quad \Gamma_2^c = \{S(\delta_2)\}^2, \quad \delta_1 \le \rho \le \delta_2, \tag{2.9}$$

$X(0) = x \in \mathbb{R}^1$ *with* $|x| = \rho$*, then for the symmetric Cauchy process on the line*

$$\frac{\log \rho/\delta_1}{\log \delta_2/\delta_1} \le P_x\{\tau_{\Gamma_2^c} < \tau_{\Gamma_1}\} \le c_1 \frac{\log \rho/\delta_1}{\log \delta_2/\delta_1}, \tag{2.10}$$

where P_x *is the conditional probability under the condition* $X(0) = x$*. Assume* $X(0) = 0$ *with probability one, and use the abbreviation* $P_0 = P$*. The* $c_1, c_2, \dots$ *will denote positive constants whose values are not important.*

Lemma 2.3 [8]. *Let* $\{E_n\}$ *be a sequence of events and suppose that*

(i) $\sum_{k=1}^{\infty} P(E_k) = \infty$ *then*

(ii) $\lim_n \inf[\sum_{k=1}^{n}\sum_{j=1}^{n} P(E_k \cap E_j)][\sum_{k=1}^{n}\sum_{j=1}^{n} P(E_k)P(E_j)]^{-1} \le c_2 \Rightarrow P\{E_n$ *occur i, o*$\} \ge c_2^{-1}$.

Assume also that a version of the process is dealt with, which is strong Markov.

3. The lower asymptotic behaviour of the sojourn time for the symmetric Cauchy process on the line

In [9], Ray obtained a function ψ for which

$$\limsup_{r\to 0} \frac{T(r)}{\psi(r)} = c_2, \tag{3.1}$$

for the symmetric Cauchy process on the line. Here we consider the lim inf behaviour of $T(r)$ and state the following.

Theorem 3.1. *Suppose* $\psi(r) = rh(r)$*, where* $h(r)$ *is a monotone increasing function.*

For a symmetric Cauchy process on the line

$$\liminf_{r\to 0} \frac{T(r)}{\psi(r)} = \begin{cases} 0 & \text{if } \int_{0+} \frac{h(x)}{x\ln(1/x)}dx = \infty, \\ \infty & \text{otherwise.} \end{cases} \tag{3.2}$$

Proof. Set $a_k = \rho^{-k}$, $\rho > 1$, it is easy to see that

$$\sum \frac{h(\rho^{-k})}{k} < \infty \quad \text{iff} \int_{0+} \frac{h(x)}{x\ln(1/x)}dx < \infty. \tag{3.3}$$

First suppose $\sum(h(a_k)/k) < \infty$. For any fixed λ, define $G_k = \{T(a_{k+1}) < \lambda\psi(a_k)\}$ and $M_k = \{P(a_{k+1}) < \lambda\psi(a_k)\}$ so that $G_k \subset M_{k.}$. Set N_k = number of returns of the process, started at x with $|x| = \hat{\rho}$, $a_{k+1}/2 \le \hat{\rho} \le a_{k+1}$, makes from $S(a_{k+1})$ to $S(a_{k+1}/2)$ before $\{S(1)\}^c$.

We set up sequences of stopping times:

$$\begin{aligned} \sigma_1 &= \min\left\{s > 0 : \frac{a_{k+1}}{2} \le |X(s)| \le a_{k+1}\right\}, \\ \sigma_2 &= \min\left\{s > \sigma_1 : |X(s)| \le \frac{a_{k+1}}{2} \text{ or } |X(s)| > 1\right\}. \end{aligned} \tag{3.4}$$

This continues until the process enters $\{S(1)\}^c$ at σ_{2N_k+2}. There exists contribution to $T(a_{k+1})$ from σ_{2i} to σ_{2i+1} which is greater than the first passage time for X out of the sphere of radius $a_{k+1}/2$.

Each time the process returns to $S(a_{k+1}/2)$ from $S(a_{k+1})$, since the process is recurrent, the event

$$M_{k/2} = \left\{P\left(\frac{a_{k+1}}{2}\right) < \lambda\psi(a_k)\right\} \tag{3.5}$$

occurs for the restarted process a.s. Thus for j returns, $M_{k/2}$ happens j times a.s. so that

$$G_k \subset \bigcup_{j=0}^{\infty} M_k \bigcap \{N_k = j\} \bigcap \{M_{k/2}, j \text{ times}\}, \tag{3.6}$$

since there exists c_3 such that

$$P\{M(1) \ge a\} \le c_3 P\{|X(1)| \ge a\} \tag{3.7}$$

(see [6, page 353]),

$$P(M_k) \le c_2\tilde{\rho}\lambda h(a_k) = \beta h(a_k), \tag{3.8}$$

where $\beta = c_3\tilde{\rho}\lambda$,

$$\begin{aligned} P\{N_k = j\} &\le c_4\left[1 - \frac{c_5}{\log 1/a_{k+1}}\right]^j \left[\frac{c_6}{\log 1/a_{k+1}}\right] \text{ by Lemma 2.2}, \\ P\left\{\frac{M_k}{2}, j \text{ times}\right\} &\le c_7[\beta h(a_k)]^j. \end{aligned} \tag{3.9}$$

But

$$P(G_k) \le c_8 P(M_k) \sum_{j=0}^{\infty} P\{N_k = j\} P\left\{\frac{M_k}{2}, j \text{ times}\right\} \tag{3.10}$$

so that

$$P(G_k) \le c_9 \frac{h(a_k)}{k}\left\{\frac{1}{1 - h(a_k)[1 - (c_{10}/k)]}\right\} \le c_{11}\frac{h(a)}{k}. \tag{3.11}$$

Thus

$$\sum P(G_k) \le c_{12} \sum \frac{h(a_k)}{k} < \infty \tag{3.12}$$

so $P(G_k, i \cdot 0) = 0$ by Borel Cantelli lemma.

Hence a.s. G_k happens for at most a finite number of k for each λ, so that we can find $r \in [a_{k+1}, a_k]$ for which

$$\lim_{r\to 0} \inf \frac{T(r)}{\psi(r)} = \infty \text{ a.s.} \tag{3.13}$$

if $\sum(h(\rho^{-k})/k) < \infty$.

In the opposite direction, set $a_k = \rho^{-k}$, $\rho > 1$ and for $\in = \in(\omega) > 0$. Suppose $\sum(h(a_k)/k) = \infty$, then for any fixed λ, choose ρ large enough so that $P(B_k)$ is close to 1 whenever

$$\begin{aligned} B_k &= |X(\lambda\psi(a_{k+1}))| > \in(a_k), \qquad \inf_k P(B_k) > 0, \\ C_k &= \{a_k > |X(\lambda\psi(a_k)) - X(\lambda\psi(a_{k+1}))| > (1-\in)a_k\}, \\ D_k &= \{\text{the process } X(t) \text{ started at } X(\lambda\psi(a_k)) \text{ enters } \{S(1)\}^c \text{ before } S((1-3\in)a_k)\}. \end{aligned} \tag{3.14}$$

Define $G_k = \{T_{(a_{k+1})} < \lambda\psi(a_k)\} \supset \{f(\lambda\psi(a_k)) > a_{k+1}\}$ by (2.3) $\supset B_k \cap C_k \cap D_k$. Then

$$\omega \in B_k \cap C_k \Longrightarrow (1-2\in)a_k < |X(\lambda\psi(a_k))| < (1+\in)a_k \tag{3.15}$$

so that $P(D_k \mid B_k \cap C_k) \sim 1/k$ by Lemma 2.2.

Since B_k and C_k are independent, we have by Lemma 2.2,

$$P(B_k \cap C_k \cap D) = P(D_k \mid B_k \cap C_k)P(B_k)P(C_k) \ge C_{12}P(D_k \mid B_k \cap C_k)P(C_k) = C_{13}\lambda\frac{h(a_k)}{k}, \tag{3.16}$$

where $P(C_k) \sim C_{14}h(a_k)$ by Lemma 2.1. Thus

$$\sum P(G_k) \ge C_{15} \sum \frac{h(a_k)}{k} = \infty. \tag{3.17}$$

If we set $E_k = B_k \cap C_k \cap D_k$, similar arguments as in [5, page 140] suffice to show that

$$P(E_k \cap E_j) \le P(C_k)P(T_{kj})P(C_j)P(D_j), \tag{3.18}$$

where T_{kj} = {the process $X(s)$ started at $X(\lambda\psi(a_k))$ enters $\{S(a_j)\}^c$ at a time t before entering $S((1-3\in)a_k)$} and $PT_{kj} \ge 1/(k-j)$ by Lemma 2.2 so that if $k \ge j+1$, conditions of Lemma 2.3 are satisfied.

Thus $P(E_k, i \cdot o) \ge C_1^{-1} > 0$. Therefore $P(E_k, i \cdot o) = 1$ by Blumenthal zero-one law.

Hence for each λ, E_k and therefore G_k happen infinitely often a.s., which in turn implies that $\lim_{r\to 0}\inf(T(r)/\psi(r)) = 0$ a.s. if $\sum(h(\rho^{-k})/k) = \infty$. □

4. The asymptotic size of the large values of $f(s)$ as $s \to 0$

The asymptotic size of the small values of $f(s)$ as $s \to 0$ was obtained by Takeuchi and Watanabe [10], where $f(s)$ is as in (2.2).

In this section we obtain the asymptotic size of the large values of $f(s)$ as $s \to 0$.

Our basic arguments will follow those in [11, Lemmas 3.3 and 3.4], although some modifications are necessary.

Theorem 4.1. *For the symmetric Cauchy process $X(t)$ on the line,*

$$\limsup_{r\to 0} \frac{f(s)}{\varphi(s)} = 0 \quad \text{a.s. or } \infty \tag{4.1}$$

according as $\sum(1/g(\rho^{-k})[k+|\log g(\rho^{-k})|])$ is finite or infinite, where $\varphi(s) = sg(s)$ and $g(s)$ is a monotone decreasing function and $f(s)$ is defined in (2.2).

Proof. Set $a_k = \rho^{-k}$, $\rho > 1$. Suppose

$$\sum \frac{1}{g(a_k)[k+|\log g(a_k)|]} < \infty. \tag{4.2}$$

For any fixed λ, and some $\in > 0$, define

$$E_k = \{f(a_k) > \lambda\varphi(a_k)\} \Longrightarrow \{f(a_k) > \lambda(1-\in)\varphi(a_k)\} \Longrightarrow A_k = \{|X(a_k)| > \lambda\varphi(a_k)\} \tag{4.3}$$

so that when $B_k = \{X(s)$ does not enter $S(\lambda(1-\in)\varphi(a_k))$ after a_k before $\tau\}$, $E_k \subset A_k \cap B_k$. Thus

$$P(E_k) \leq P(A_k \cap B_k) = P\left(B_k \cap \bigcup_{i=1}^{\infty} A_k^i\right) = P\left(\bigcup_{i=1}^{\infty}(B_k \cap A_k^i)\right), \tag{4.4}$$

where $A_k^i = \{2^i\lambda\varphi(a_k) \geq |X(a_k)| > 2^{i-1}\lambda\varphi(a_k)\}$, $i = 1,2,3,\ldots$ so that

$$P(E_k) \leq \sum_{i=1}^{n} P(A_k^i)P(B_k \mid A_k^i), \tag{4.5}$$

(where $B_k \cap A_k^i = \phi$ for $i \geq n$). Hence $P(E_k) \sim c_{16}/g(a_k)[k+|\log g(a_k)|]$ by Lemmas 2.1 and 2.2. and $\sum P(E_k) < \infty$. Thus $P(E_k, i\cdot o) = 0$ by Borel Cantelli lemma. Therefore there exists k_0 such that for $k > k_0$, $f(a_k) \leq \lambda\varphi(a_k)$ so that

$$\limsup_{k\to\infty} \frac{f(a_k)}{\varphi(a_k)} \leq \lambda. \tag{4.6}$$

But λ is any fixed number, hence as $k \to \infty$,

$$P\left\{\limsup_{k\to\infty} \frac{f(a_k)}{\varphi(a_k)} = 0\right\} > 0. \tag{4.7}$$

By the Blumenthal zero-one law, we have $\lim_{k\to 0}\sup(f(a_k)/\varphi(a_k)) = 0$ a.s. with

$$\sum \frac{1}{g(a_k)[k+|\log g(a_k)|]} < \infty. \tag{4.8}$$

By monotonicity of $f(s)$ and $\varphi(s)$, we have

$$\lim_{r\to 0}\sup \frac{f(s)}{\varphi(s)} = 0 \text{ a.s.} \tag{4.9}$$

if $\sum(1/g(a_k)[k+|\log g(a_k)|]) < \infty$.

In the opposite direction, set $a_k = \rho^{-k}$, $\rho > 1$ and suppose

$$\sum \frac{1}{g(a_k)[k+|\log g(a_k)|]} = \infty. \tag{4.10}$$

For any fixed λ, and some $\in > 0$, define

$$E_k = \{f(s) > \lambda\varphi(s) \text{ for some } s \in (a_{k+1}, a_k]\} \supseteq \{f(a_k) > \lambda\varphi(a_k)\} \supset A_k \cap B_k \cap c_k, \tag{4.11}$$

where

$$\begin{aligned} A_k &= |X(a_{k+1})| \leq \in \varphi(a_k), \\ B_k &= \{\varphi(a_k) > |X(a_k) - X(a_{k+1})| > (1- \in)\varphi(a_k)\} \end{aligned} \tag{4.12}$$

and $C_k = \{X(t)$ started at $X(a_k)$ enters $\{S(1)\}^c$ before $S((1-3\in)\varphi(a_k))\}$. Then

$$\omega \in A_k \cap B_k \Longrightarrow \{(1-2\in)\varphi(a_k) < |X(a_k)| < (1+\in)\varphi(a_k)\} \tag{4.13}$$

so that

$$\begin{aligned} P(C_k \mid A_k \cap B_k) &\sim \frac{c_{15}}{k+|\log g(a_k)|} \text{ by (Lemma 2.2)}, \\ P(B_k) &\sim \frac{C_{19}}{g(a)} \text{ by (Lemma 2.1)}. \end{aligned} \tag{4.14}$$

Set $D_k = A_k \cap B_k \cap C_k$. Since A_k and B_k are independent,

$$P(D_k) = P(C_k \mid A_k \cap B_k)P(A_k)P(B_k). \tag{4.15}$$

If ρ is chosen large enough so that $P(A_k)$ is close to one, we have

$$P(D_{ki}) \geq \frac{C_{20}}{g(a_k)[k+|\log g(a_k)|]}. \tag{4.16}$$

Thus $\sum P(D_k) = \infty$. Similar arguments in [11, Lemma 3.5] suffice to show that

$$P(D_k \cap D_j) \leq P(B_k)P(T_{kj})P(B_j)P(C_j), \tag{4.17}$$

where $T_{kj} = \{X(s)$ started at $X(a_k)$ enters $\{S(\in \varphi(a_j))\}^c$ at time t before $(1-3\in)\varphi(a_k)\}$ and for $k \geq j+1$,

$$P(T_{kj}) \sim \frac{C_{21}}{(k-j)+\log(g(a_j)/g(a_k))} \text{ by (Lemma 2.2)} \tag{4.18}$$

so that

$$\liminf_n \left[\sum_{k=1}^{n}\sum_{j=1}^{n} p(D_k \cap D_j)\right]\left[\sum_{k=1}^{n}\sum_{j=1}^{n} P(D_k)P(D_j)\right]^{-1} < \beta, \tag{4.19}$$

where $0 < \beta < \infty$. Thus by (Lemma 2.3),

$$P(D_k, i \cdot o) \geq C^{-1} > 0. \tag{4.20}$$

Therefore $P(D_k, i \cdot o) = 1$ by Blumenthal zero-one law.

Therefore we can find $s \in (a_{k+1}, a_k]$ for which

$$\limsup_{s\to 0} \frac{f(s)}{\varphi(s)} \geq \limsup_{k\to\infty} \frac{f(a_k)}{\varphi(a_k)} \geq \lambda. \tag{4.21}$$

Thus

$$\limsup_{s\to 0} \frac{f(s)}{\varphi(s)} = \infty \text{ a.s.} \tag{4.22}$$

if

$$\sum \frac{1}{g(a_k)[k+|\log g(a_k)|]} = \infty. \tag{4.23}$$

□

Acknowledgments

The author is very thankful to the referee and the principal editor, Eugene Dshalalow for their comments and suggestions that have led to significant improvement to this paper.

References

[1] S. J. Taylor, "The measure theory of random fractals," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 100, no. 3, pp. 383–406, 1986.

[2] Y. Xiao, "Random fractals and Markov processes," in *Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot—Part 2*, M. L. Lapidus and M. van Frankenhuijsen, Eds., vol. 72 of *Proceedings of Symposia in Pure Mathematics*, pp. 261–338, American Mathematical Society, Providence, RI, USA, 2004.

[3] S. J. Taylor and C. Tricot, "Packing measure, and its evaluation for a Brownian path," *Transactions of the American Mathematical Society*, vol. 288, no. 2, pp. 679–699, 1985.

[4] A. C. Okoroafor and O. O. Ugbebor, "Lower asymptotic behaviour of the sojourn time for a stable process," in *Contemporary Stochastic Analysis (Ibadan, 1989)*, G. O. S. Ekhaguere, Ed., pp. 109–126, World Scientific, River Edge, NJ, USA, 1991.

[5] W. E. Pruitt and S. J. Taylor, "Packing and covering indices for a general Lévy process," *Annals of Probability*, vol. 24, no. 2, pp. 971–986, 1996.

[6] B. Fristedt, "Sample functions of stochastic processes with stationary, independent increments," in *Advances in Probability and Related Topics, Vol. 3*, pp. 241–396, Marcel Dekker, New York, NY, USA, 1974.

[7] E. A. Perkins and S. J. Taylor, "Uniform measure results for the image of subsets under Brownian motion," *Probability Theory and Related Fields*, vol. 76, no. 3, pp. 257–289, 1987.

[8] S. Kochen and C. Stone, "A note on the Borel-Cantelli lemma," *Illinois Journal of Mathematics*, vol. 8, pp. 248–251, 1964.

[9] D. Ray, "Some local properties of Markov processes," in *Proceedings of the 5th Berkeley Symposium on Mathematical Statistics and Probability (Berkeley, Calif, 1965/1966), Vol. II: Contributions to Probability Theory—Part 2*, pp. 201–212, University of California Press, Berkeley, Calif, USA, 1967.

[10] J. Takeuchi and S. Watanabe, "Spitzer's test for the Cauchy process on the line," *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 3, pp. 204–210, 1964.

[11] A. C. Okoroafor and O. O. Ugbebor, "Upper rates of escape for stable process," in *Contemporary Stochastic Analysis (Ibadan, 1989)*, G. O. S. Ekhaguere, Ed., pp. 127–147, World Scientific, River Edge, NJ, USA, 1991.

A. Chukwuemeka Okoroafor: Department of Mathematics, Abia State University, Uturu, Nigeria
Email address: dracokoroafor@yahoo.com