ERROR BOUNDS FOR TWO EVEN DEGREE TRIDIAGONAL SPLINES

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Abstract.

We study a $C^{(1)}$ parabolic and a $C^{(2)}$ quartic spline which are determined by solution of a tridiagonal matrix and which interpolate subinterval midpoints. In contrast to the cubic $C^{(2)}$ spline, both of these algorithms converge to any continuous function as the length of the largest subinterval goes to zero, regardless of "mesh ratios". For parabolic splines, this convergence property was discovered by Marsden [1974]. The quartic spline introduced here achieves this convergence by choosing the second derivative zero at the breakpoints. Many of Marsden's bounds are substantially tightened here. We show that for functions of two or fewer coninuous derivatives the quartic spline is shown to give yet better bounds. Several of the bounds given here are optimal.

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1. Introduction.

M. Marsden [1974, 1978] gave results concerning the approximation of functions by even degree splines, including the simple parabolic splines. If the breakpoints are the same as the interpolated points, then the resulting spline is ill-behaved, as can be seen by simple examples (deBoor [1978]). On the other hand, if we take the interpolated points midway between breakpoints, the parabolic splines are rather nicely behaved.

Many of Marsden's error bounds are improved in the present paper. At the same time, we introduce a class of quartic splines for which very similar arguments give better bounds. In order to be more explicit, we introduce some definitions and notation. Let f be a continuous function of period b - a and let us denote by $\{x_i\}^k{}_{i=0}$ the

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partition

$$a = x_0 < z_1 < x_1 < z_2 < \ldots < x_{k-1} < z_k < x_k = b$$

where $z_i = (x_i + x_{i+1}) / 2$. Denote

(1.1) $h_i = x_i - x_{i-1}$, $h = \max_i h_i$,

$h_0 = h_k$,

$\| f \| = \sup \{ |f(x)| : a \leq x \leq b \}$,

$\omega(f,d) = \sup_{|x-y|<d} | f(x) - f(y) |$.

A function s(x) is defined to be a periodic quadratic spline interpolant associated with f and $\{x_i\}_{i=0}^{k}$ if

(1.2) a) s(x) is a parabola when restricted to any interval $[x_{i-1}, x_i]$,

b) $s(x) \in C^{(1)}[a,b]$,

c) $s(z_i) = f(z_i)$, $i = 1, 2, \ldots, k$,

d) $s(a) = s(b)$, $s'(a) = s'(b)$.

Denote $e(x) = f(x) - s(x)$, $s_i = s(x_i)$, and $e_i = e(x_i)$. The following theorem was obtained by Marsden.

Theorem 1 (Marsden). Let $\{x_i\}_{i=0}^{k}$ be a partition of [a,b], f(x) be a continuous function of period b - a and s(x) be the periodic quadratic spline interpolant assocated with f and $\{x_i\}_{i=0}^{k}$. Then

(1.3) $\| s_i \| \leq 2 \| f_i \|$, $\| s \| \leq 2 \| f \|$,

$\| e_i \| \leq 2\, \omega(f, \frac{h}{2})$,

$\| e \| \leq 3\, \omega(f, \frac{h}{2})$.

The constant 2 which appears in the first of the above equations cannot, in general, be decreased.

Theorem 1 is of interest as it indicates that the parabolic spline, in contrast to the cubic spline, gives good approximations to continuous functions, provided only that subintervals are sufficiently small and that $\omega(f, \frac{h}{2})$ is small.

Keeping the same notation as above, we define a quartic spline S(x) as follows:

(1.4) a) S(x) is a quartic polynomial on each interval (x_{i-1}, x_i) ,

b) $S(x) \in C^{(2)}[a,b]$,

c) $S''(x_i) = 0$,

d) $S(z_i) = f(z_i)$, $i = 1, 2, \ldots, k$,

e) $S(a) = S(b)$, $S'(a) = S'(b)$.

For this quartic spline we will prove the following:

Theorem $1'$. Let $\{x_i\}_{i=0}^{k}$ be a partition of [a,b], f(x) be a continuous function of period b - a and S(x) be the periodic quartic spline interpolant associated with f and $\{x_i\}_{i=0}^{k}$. Then

(1.5)
$$\| S_i \| \leq \frac{8}{5} \| f(z_i) \| ,$$
$$\| E_i \| \leq \frac{8}{5} \omega(f, \frac{h}{2}) , \quad \| E_i \| \leq \frac{4}{5} \omega(f, h)$$
$$\| E \| \leq \frac{13}{8} \omega(f, \frac{h}{2}) .$$

The constant 8/5 which appears in the first of the above equations cannot, in general, be decreased.

As Marsden predicted, many of the other bounds he gives can be improved.

Theorem 2 Let f and f' be continuous functions of period b - a. Then for the parabolic spline s(x) we have

(1.6)
$$\| e(x) \| \leq c_{0,1}\, h \, \| f' \| ,$$

where $a_0 = \frac{4 - \sqrt{13}}{6}$ and $c_{0,1} = 1 + a_0 - 8a_0^2 + 4a_0^3 \approx 1.0323$. The analogous constant from Marsden was 5/4.

To further improve this bound, we can employ the quartic spline S(x).

Theorem $2'$. Let f and f' be contiuous functions of period b - a. Then for the quartic spline S(x) we have

(1.7)
$$|E(x_i)| \leq \frac{8}{5} \frac{h_i^2}{h_i + h_{i-1}} \| f' \| .$$

If we choose even spaced knots, we obtain

(1.8)
$$\| E(x) \| \leq C_{0,1}\, h \, \| f' \|$$

where $C_{0,1} \approx .846056$.

Theorem 3. Let f, f', and f'' be continuous functions of period b - a. Then for the parabolic spline s(x)

(1.9)
$$\| e(x) \| \leq \frac{1}{6} h^2 \| f'' \| ,$$

(1.10)
$$\| e_i' \| \leq \frac{9}{16} h \| f'' \| ,$$

(1.11)
$$\| e' \| \leq \frac{17}{16} h \| f'' \|$$

where $e_i' = f'(x_i) - s'(x_i)$. The constant 9/16 in (1.10) was due to Marsden. Equations (1.9) and (1.11) represent substantial improvements over Marsden's results. For comparison, Marsden's constant for (1.9) was 5/8 while in (1.11) the value was 2. If we make the additional assumption that the partition consists of equally spaced intervals, then we can improve (1.11) to

(1.12) $$\| e' \| \leq .7431\, h \| f'' \| .$$

Concerning the quartic S(x) we can show

Theorem 3'. Let f, f', and f'' be continuous functions of period b - a. Then

(1.13) $$\| E(x) \| \leq \frac{37}{260} h^2 \| f'' \| ,$$

(1.14) $$| E_i' | \leq \frac{137}{320} \frac{h_i^2 + h_{i-1}^2}{h_i + h_{i-1}} \| f'' \| ,$$

(1.15) $$\| E' \| \leq \frac{51}{65} h \| f'' \| .$$

Again each of these bounds represents an improvement over the parabolic spline.

Finally concerning the parabolic spline s(x), we can show

Theorem 4. Let f, f', f'', and f''' be continuous functions of period b - a. Then

(1.16) $$\| e_i \| \leq \frac{1}{24} h^3 \| f''' \| ,$$

(1.17) $$\| e_i' \| \leq \frac{1}{6} h^2 \| f''' \| ,$$

(1.18) $$\| e \| \leq \frac{1}{24} h^3 \| f''' \| ,$$

(1.19) $$\| e' \| \leq \frac{7}{24} h^2 \| f''' \| ,$$

(1.20) $$| e''(x) | \leq \left\{ \frac{h_i}{2} + \frac{h^3}{3\, h_i^2} \right\} \| f''' \| , \ x_{i-1} \leq x \leq x_i .$$

Marsden's analogous constants for (1.16) to (1.20) are 1/8 , 1/3, 17/96, and 11/24 respectively. In each case a substantial improvement is indicated. Furthermore, (1.16) and (1.18) are best possible. In fact we also have the pointwise bound for arbitrarily spaced breakpoints which is exact for equally spaced knots,

(1.21) $$| e(x) | \leq | Q_3(t) | \ h^3 \| f''' \| , \ x_i \leq x_{i+1} ,$$

where $t = \frac{x - x_i}{x_{i+1} - x_i}$ and

$$Q_3(t) = \frac{1}{24} - \frac{t^2}{4} + \frac{t^3}{6}$$

is the "Euler spline" of degree three.

The techniques used in proofs of Theorems 2, $2'$, 3 and $3'$ are similar to those used by Hall and Meyer [1976] to give optimal bounds for derivatives of cubic spline and in previous joint work with A. K. Varma [1989]. For a given partition subinterval $[x_i, x_{i+1}]$, we write

$$(1.22)\quad | f^{(i)}(x) - s^{(i)}(x) | \le | f^{(i)} - L^{(i)}(x) | + | L^{(i)}(x) - s^{(i)}(x) |$$

(or S in place of s) where L is a polynomial interplation of f. We then proceed by obtaining pointwise estimates of the quantities on the right hand side of (1.22) . Due to the similarity of most of these derivations, we will show only a few in detail, confining ourselves in most cases to mentioning the necessary choice (or choices) of L to derive a given bound.

2. Existence of the $C^{(2)}$ Quartic Spline Interpolant S(x) .

We wish to show the unique construction of the quartic spline defined by equations (1.4 a)-e). Define $z_{i+1} = \frac{x_{i+1} + x_i}{2}$, $h_i = x_{i+1} - x_i$, and $t = \frac{x - x_i}{h_i}$. We can write the unique quartic polynomial $P_i(x)$ interpolating $f(x_i)$, $f''(x_i)$, $f(z_{i+1})$, $f(x_{i+1})$, and $f''(x_{i+1})$ as

$$(2.1)\quad P_i(x) = f(x_i)\, A_0(t) + f''(x_i)\, C_0(t)\, h_i^2 + f(z_{i+1})\, A_{1/2}(t) + f(x_{i+1})\, A_1(t) + f''(x_{i+1})\, C_1(t)\, h_i^2$$

where

$$A_0(t) = 1 - \frac{13}{5} t + \frac{16}{5} t^3 - \frac{8}{5} t^4 ,$$

$$C_0(t) = -\frac{4}{30} t + \frac{t^2}{2} - \frac{17}{30} t^3 + \frac{t^4}{5} ,$$

$$A_{1/2}(t) = \frac{16}{5} t - \frac{32}{5} t^3 + \frac{16}{5} t^4 ,$$

$$A_1(t) = -\frac{3}{5} t + \frac{16}{5} t^3 - \frac{8}{5} t^4 ,$$

$$C_1(t) = \frac{t}{30} - \frac{7}{30} t^3 + \frac{t^4}{5} .$$

We set $S''(x_i) = S''(x_{i+1}) = 0$ and $S(z_{i+1}) = f(z_{i+1})$. Then the quartic S on $[x_i, x_{i+1}]$ is given by

(2.2) $$S(x) = S(x_i)\, A_0(t) \;+\; f(z_{i+1})\, A_{1/2}(t) \;+\; S(x_{i+1})\, A_1(t)\ .$$

Then

$$S'(x_i+) = S(x_i)\, \frac{1}{h_i}\, (-\tfrac{13}{5}) \;+\; f(z_{i+1})\, \frac{1}{h_i}\, (\tfrac{16}{5}) \;+\; S(x_{i+1})\, \frac{1}{h_i}\, (-\tfrac{3}{5})$$

and

$$S'(x_i-) = S(x_{i-1})\, \frac{1}{h_{i-1}}\, (-\tfrac{3}{5}) \;+\; f(z_i)\, \frac{1}{h_{i-1}}\, (-\tfrac{16}{5}) \;+\; S(x_i)\, \frac{1}{h_{i-1}}\, (\tfrac{13}{5})\ .$$

Setting $S'(x_i+) = S'(x_i-)$, we have on rearranging and multiplying by $5\, h_i\, h_{i+1}$

(2.3) $$3\, h_i\, S(x_{i-1}) \;+\; 13\, (\, h_i + h_{i-1}\,)\, S(x_i) \;+\; 3\, h_{i-1}\, S(x_{i+1}) \;=\; 16\, h_{i-1}\, f(z_{i+1}) \;+\; 16\, h_i\, f(z_i)\ .$$

This is the tridiagonal system to be solved in order to obtain a twice continuous piecewise quartic interpolating interval midpoints and having second derivatives zero at the breakpoints x_i. Since the resulting system of equations is diagonally dominant, it is clear that for the periodic case there exists a unique spline satisfying Eqs. 1.4).

3. Proof of Theorem 1′

From (2.3), It follows easily that

(3.1) $$\|\, S(x_i)\, \| \;\leq\; \tfrac{8}{5}\, \|\, f(z_i)\, \|$$

where $\|\quad\|$ denotes the supremum over the values i . It remains to show that this bound is best. Essentially, we would like to produce a sequence of values $S(x_i)$ of equal size and alternating sign. In the limit as $h_{i-1} >> h_i$, (2.3) reduces to

$$13\, S(x_i) \;+\; S(x_{i+1}) \;=\; 16\, f(z_{i+1})\ .$$

By choosing $f(z_i) = (-1)^{i+1}$, for $0 \leq i \leq n$, we get arbitrarily close to (3.1) . In fact, let $h_i = t\, h_{i-1}$ where $t < 1$. We then have for $1 \leq i \leq n - 1$

(3.2) $$3t\, S(x_{i-1}) \;+\; 13\, (1 + t)\, S(x_i) \;+\; 3\, S(x_{i+1}) \;=\; 16\, (1 - t)\ .$$

Using (3.1), it follows that $\mathrm{sign}(S(x_i)) = (-1)^i$, $1 \leq i \leq n - 1$. Therefore

$$|S(x_i)| \;\geq\; \frac{16\,(1-t)}{13\,(1+t)}\;,\;\; 2 \leq i \leq n-2\,.$$

Using this inequality in (3.2), we have

$$\begin{aligned} 13\,(1+t)\,|S(x_i)| \;&\geq\; 16\,(1-t) \;+\; \frac{16\,(1-t)}{13\,(1+t)}\,3(1+t) \\ &\geq\; 16\,(1-t)\,[\,1+\tfrac{3}{13}\,]\;,\;\; 3 \leq i \leq n-3 \end{aligned}$$

and hence

$$|S(x_i)| \;\geq\; \frac{16\,(1-t)}{13\,(1+t)}\,[\,1+\tfrac{3}{13}\,]\;,\;\; 3 \leq i \leq n-3\;.$$

Inductively, for $k \leq i \leq n-k$, we have

$$|S(x_i)| \;\geq\; \frac{16\,(1-t)}{13\,(1+t)}\,[\,1+\tfrac{3}{13}+\ldots(\tfrac{3}{13})^{k-2}\,]\;,\;\; k \leq i \leq n-k\,,$$

which in the limit as $t \to 0$ and $n, k \to \infty$ gives

$$|S(x_i)| \;\geq\; \frac{16}{13}\left(\frac{13}{10}\right) = \frac{8}{5}\,,$$

completing the demonstration that 8/5 is in fact the best possible constant in (3.1).

Also from (2.3), we have

$$\begin{aligned} &3\,h_i\,[\,S(x_{i-1}) - f(x_{i-1})\,] + 13\,(\,h_i + h_{i-1}\,)\,[\,S(x_i) - f(x_i)\,] \\ &\qquad + 3\,h_{i-1}\,[\,S(x_i) - f(x_i)\,] \\ &\quad = 3\,h_i\,[\,f(z_i) - f(x_{i-1})\,] + 13\,h_i\,[\,f(z_i) - f(x_i)\,] \\ &\qquad + 13\,h_{i-1}\,[\,f(z_{i+1}) - f(x_i)\,] + 3\,h_{i-1}\,[\,f(z_{i+1}) - f(x_{i+1})\,] \end{aligned}$$

If we assume that $|S(x_i) - f(x_i)| \geq |S(x_{i-1}) - f(x_{i-1})|$ and $|S(x_i) - f(x_i)| \geq |S(x_{i+1}) - f(x_{i+1})|$, then we have

$$\begin{aligned} &10\,(\,h_i + h_{i-1}\,)\,|S(x_i) - f(x_i)| \\ &\quad\leq\; 3\,h_i\,\omega(f;\,h/2) \;+\; 13\,h_i\,\omega(f;\,h/2) \\ &\qquad + 13\,h_{i-1}\,\omega(f;\,h/2) \;+\; 3\,h_{i-1}\,\omega(f;\,h/2) \\ &\quad=\; 16\,(\,h_{i-1} + h_i\,)\,\omega(f;\,h/2) \end{aligned}$$

and hence

$$\| S(x_i) - f(x_i) \| \leq \frac{8}{5} \omega(f; h/2) .$$

We next wish to bound $| f(x) - S(x) |$. As

$$A_0(t) + A_{1/2}(t) + A_1(t) = 1$$

and $S(x) = S(x_i) A_0(t) + f(z_{i+1}) A_{1/2}(t) + S(x_{i+1}) A_1(t)$, we have

$$S(x) - f(x) = A_0(t) [S(x_i) - f(x)] + A_{1/2}(t) [f(z_{i+1}) - f(x)] + A_1(t) [S(x_{i+1}) - f(x)] .$$

Assume that $x_i \leq x \leq z_{i+1}$. We have

$$S(x) - f(x) = A_0(t) [S_i - f_i + f_i - f(x)] + A_{1/2}(t) [f(z_{i+1}) - f(x)] + A_1(t) [S_{i+1} - f_{i+1} + f(z_{i+1}) - f(z_{i+1}) + f_{i+1} - f(x)]$$

and thus

$$\begin{aligned} |S(x) - f(x)| &= |A_0(t)| \{ |S_i - f_i| + |f_i - f(x)| \} \\ &\quad + |A_{1/2}(t)| \{ |f(z_{i+1}) - f(x)| \} \\ &\quad + |A_1(t)| \{ |S_{i+1} - f_{i+1}| + |f_{i+1} - f(z_{i+1})| + |f(z_{i+1}) - f(x)| \} \\ &\leq |A_0(t)| \{ \tfrac{13}{5} \omega(f; h/2) \} \\ &\quad + |A_{1/2}(t)| \{ \omega(f; h/2) \} \\ &\quad + |A_1(t)| \{ \tfrac{18}{5} \omega(f; h/2) \} \\ &\leq \omega(f; h/2) \{ \tfrac{13}{5} A_0(t) + A_{1/2}(t) - \tfrac{18}{5} A_1(t) \} \\ &\leq \omega(f; h/2) \tfrac{1}{5}\{ 13 - 7t - 48 t^3 + 24 t^4 \} \end{aligned}$$

Replacing t by 1 - t, symmetry gives the same arguments for $z_{i+1} \leq x \leq x_{i+1}$. By taking the maximum for $0 \leq t \leq 1/2$, it follows that

$$|f(x) - S(x)| \leq \frac{13}{5} \omega (f; h/2) .$$

This completes the demonstration of Theorem $1'$.

4. Proof of Theorem 4

We will give the proof of Theorem 4 in some detail. The remaining proofs are quite similar and we will confine ourselves to giving the appropriate interpolation to be used in (1.22). We first show that if f, f', f'', and f''' are continuous and of period b - a , then

(4.1) $$\|f - s\| \leq \frac{h^3}{24} \| f''' \|$$

where s is the parabolic and periodic once differentiable spline interpolating subinterval midpoints. Furthermore, "1/24" cannot be improved.

For a given partition and subinterval $[x_i, x_{i+1}]$, we write

(4.2) $$| f(x) - s(x) | \leq | f(x) - L(x) | + | L(x) - s(x) |$$

where L(x) is the Lagrange parabola saitsfying

(4.3) $$L(x_i) = f(x_i) , \quad L(z_{i+1}) = f(z_{i+1}) , \quad L(x_{i+1}) = f(x_{i+1})$$

where $z_{i+1} = \frac{x_{i+1} - x_i}{2}$. L(x) may be uniquely expressed as

(4.4) $$L(x) = f(x_i) A_0(t) + f(z_{i+1}) A_{1/2}(t) + f(x_{i+1}) A_1(t)$$

where $t = \frac{x - x_i}{h_i}$, and

$$A_0(t) = (1 - 2t) (1 - t) ,$$

$$A_{1/2} = 4 t (1 - t) ,$$

$$A_1(t) = t (1 - 2t) .$$

Proceeding by using the well-known Cauchy error one obtains

(4.5) $$| f(x) - L(x) | \leq h_i^3 \frac{|t (t - 1/2) (t - 1)|}{6} \| f''' \| .$$

In order to bound L(x) - s(x) , we use the fact that

$$s(x) = s(x_i) A_0(t) + f(z_{i+1}) A_1(t) + s(x_{i+1}) A_2(t) .$$

and hence

$$(4.6)\quad |\, L(x) - s(x)\,| = |\, [\, f(x_i) - s(x_i)\,]\; A_0(t) + [\, f(x_{i+1}) - s(x_{i+1})\,]\; A_2(t)\,|$$
$$\leq \|\, e_i \|\; \{\, |\, A_0(t)\,| + |\, A_2(t)\,|\,\}$$
$$= \|\, e_i \,\|\; |\, 1 - 2t\,|\,.$$

We now turn to bound $\|\, e_i \,\| = \max_{1 \leq j \leq k}\{\, |\, e_j\,|\,\}$ where $e_j = f(x_j) - s(x_j)$. We resort to the tridiagonal system given by Marsden ,

$$(4.7)\quad - h_i\, s_{i-1} - 3\,(\, h_i + h_{i-1}\,)\; s_i - h_{i-1}\, s_{i+1}$$
$$= -4\, h_i\, f(z_i) + 4\, h_{i-1}\, f(z_{i+1})\,,$$

from which it follows that

$$(4.8)\quad h_i\, e_{i-1} + 3\,(\, h_i + h_{i-1}\,)\; e_i + h_{i-1}\, e_{i+1}$$
$$= h_i\, f_{i-1} - 4\, h_i\, f(z_i) + 3\,(h_i + h_{i-1}\,)\, f_i$$
$$- 4\, h_{i-1}\, f(z_{i+1}) + h_{i-1}\, f_{i+1}$$
$$=: B_O\,(f)\,.$$

$B_O(f)$, so defined, is a linear functional identically zero for polynomials of degree two or less. We thus have by the Peano theorem, (see P. J. Davis [1963])

$$(4.9)\qquad B_O(f) = \int_{x_{i-1}}^{x_{i+1}} K(y)\, f'''(y)\, dy\,/\,2!$$

where

$$K(y) = B_O\,[\,(x - y)_+{}^2\,]$$
$$= h_{i-1}\,[\,(x_{i+1} - y)_+{}^2\,] - 4\, h_{i-1}\,[\,(z_{i+1} - y)_+{}^2\,]$$
$$+ 3\,(h_i + h_{i-1})\,[\,(x_i - y)_+{}^2\,]$$
$$- 4\, h_i\,[\,(z_i - y)_+{}^2\,] + h_i\,[\,(x_{i-1} - y)_+{}^2\,]$$
$$= h_{i-1}\,[\, h_i - (y - x_i)\,]^2\,,\qquad \frac{h_i}{2} \leq y - x_i \leq h_i$$
$$h_{i-1}\,[\, h_i - (y - x_i)\,]^2 - 4\, h_{i-1}\,[\,\frac{h_i}{2} - (y - x_i)\,]^2\,,$$
$$0 \leq y - x_i \leq \frac{h_i}{2}$$

$$- \; h_i \, [\, h_{i-1} + (y-x_i) \,]^2 \; + \; 4 \, h_i \, [\, \tfrac{h_{i-1}}{2} + (y-x_i) \,]^2 \;, \qquad \tfrac{-h_{i-1}}{2} \le y-x_i \le 0$$

$$- \; h_i \, [\, h_{i-1} + (y-x_i) \,]^2 \,, \quad -h_i \le y-x_i \le \tfrac{-h_{i-1}}{2} \,.$$

Then we have

$$(4.10) \qquad | \, h_i \, e_{i-1} \; + \; 3 \, (\, h_i + h_{i-1} \,) \; e_i \; + \; h_{i-1} \, e_{i+1} |$$

$$\le \int_{x_{i-1}}^{x_{i+1}} | \, K(y) | \; dy \; \| \, f''' \, \| \, / \, 2!$$

$$= (h_{i-1} \, h_{i-1}{}^3 + h_i \, h_{i-1}{}^3) \; \| \, f''' \, \| \, / \, 12 \,.$$

If we take i so that $| \, e_i \, |$ is maximal , then we have

$$(4.11) \qquad 2 \, (\, h_i + h_{i-1} \,) \; \| \, e_i \, \|$$

$$\le (h_{i-1} \, h_{i-1}{}^3 + h_i \, h_{i-1}{}^3) \; \| \, f''' \, \| \, / \, 12$$

and hence

$$(4.12) \qquad \| \, e_i \, \| \; \le \; \frac{h_{i-1} \, h_{i-1}{}^3 + h_i \, h_{i-1}{}^3}{24 \, (\, h_i + h_{i-1} \,)} \; \| \, f''' \, \|$$

$$\le \frac{1}{24} \, h^3 \; \| \, f''' \, \| \,.$$

Combining (4.5) and (4.12) , we have

$$(4.13) \qquad | \, f(x) - s(x) \, | \; \le \; | \, f(x) - L(x) \, | \; + \; \| \, e_i \, \| \; | \, 1 - 2t \, |$$

$$\le \left\{ \frac{| \, t \, (\, 1/2 - t \,) \, (\, 1 - t \,) \, |}{6} \; + \; \frac{|1 - 2t|}{24} \right\} h^3 \; \| \, f''' \, \|$$

$$\le \; | \, Q_3(t) \, | \, h^3 \; \| \, f''' \, \|$$

where $Q_3(t)$ is the Euler polynomial of degree three. Equation (4.13) is precisely (1.21). From it follow also (1.18), (4.1), and (1.16).

To see that (1.21) cannot be improved, consider the Euler polynomial $Q_n(x)$

constructed by integrating a constant n times so that the nth integration is odd for n odd and even for n even. On the unit interval, the first few Euler polynomials are

(4.14) $$Q_0(x) = 1 ,$$
$$Q_1(x) = x - 1/2 ,$$
$$Q_2(x) = x^2/2 - x/2 ,$$
$$Q_3(x) = x^3/6 - x^2/4 + 1/24 .$$

If we extend the Euler polynomials to the real line by setting

(4.15) $$Q_n(x) = (-1)^j \, Q_n(t+j) \, , \, 0 \leq t \leq 1$$

and j integer, then $Q_n(x)$ is n - 1 times continuously differentiable and piecewise n times continuously differentiable. We call Q_n the "Euler spline". It is of period 2 and has nth derivative of plus or minus one. Q_n is thus a member of the class of functions with nth derivative piecewise differentiable. The third derivative of Q_3 can be represented as the pointwise limit of the third derivative of a sequence $\{f_i\}$ of three times continuously differentiable functions which converge uniformly to Q_3. Furthermore the f_i can be defined as piecewise lines in such a way that the f_i have third derivative bounded in absolute value by one and so that the third derivative of the f_i converges pointwise to the third derivative of Q_3, i.e, to one on (2j, 2j+1) and minus one on (2j-1, 2j) .

Restricting $Q_3(x)$ to any interval [0, 2k] , consider the once continuously differentiable spline s parabolic in each interval [i, i+1] , and satisfying

(4.16) $$s(z_{i+1}) = Q_3(z_{i+1}) = 0$$
$$s(0) = s(2k) \, ; \, s'(0) = s'(2k) \, .$$

The spline s thus defined is identically zero and is the unique periodic quadratic spline interpolating Q_3 and each of the $\{f_i\}$ on [0, 2k] . It is not hard to see that the maximum error occurs at the integer knots and is 1/24. In fact we have shown that

$Q_3(x)$ is a pointwise exact bound.

We next demonstrate Equation (1.17) ,

$$(1.17) \qquad \| e_i' \| \leq \frac{1}{6} h^2 \| f'' \| ,$$

where $\| e_i' \| = \max_{0 \leq i \leq k} |f'(x_i) - s'(x_i)|$. From Marsden we have the tridiagonal system of equations

$$(4.17) \qquad h_{i-1} s_{i-1}' + 3 (h_i + h_{i-1}) s_i' + h_i s_{i+1}' = 8 f(z_{i+1}) - 8 f(z_i) ,$$

from which we obtain the functional B_1 defined by

$$(4.18) \qquad h_{i-1} e_{i-1}' + 3 (h_i + h_{i-1}) e_i' + h_i e_{i+1}'$$
$$= h_{i-1} f_{i-1}' + 8 f(z_i) + 3 (h_i + h_{i-1}) f_i' - 8 f(z_{i+1}) + h_i f_{i+1}'$$
$$=: B_1(f) ,$$

which is identically zero for all polynomials of degree two or less. Hence

$$(4.19) \qquad | B_1(f) | \leq \int_{x_{i-1}}^{x_{i+1}} | B_1[(x-y)_+{}^2]| \, dy \ \| f'''(y) \| / 2!$$

where

$$B_1[(x-y)_+{}^2] =$$
$$2 h_i (x_{i+1} - y)_+ - 8 (z_{i+1} - y)_+^2 + 6 (h_i + h_{i-1}) (y-x_i)_+ + 8 (y-z_i)_+^2 + 2 h_{i-1} (y-x_{i-1})_+$$
$$= \begin{cases} 2 h_i^2 - 2 h_i (y-x_i) , & \frac{h_i}{2} \leq y-x_i \leq h_i \\ 6 h_i (y-x_i) - 8 (y-x_i)^2 , & 0 \leq y-x_i \leq \frac{h_i}{2} \\ - 6 h_{i-1} (y-x_i) - 8 (y-x_i)^2 , & \frac{-h_{i-1}}{2} \leq y-x_i \leq 0 \\ 2 h_{i-1}^2 - 2 h_{i-1} (y-x_i) , & -h_i \leq y-x_i \leq \frac{-h_{i-1}}{2} . \end{cases}$$

where the last four expressions are the explicit form of B_1 on subintervals of $[x_{i-1},x_{i+1}]$. Conveniently, the above kernel is positive. Evaluation of the integral in (4.19) is thus straightforward, leading to

(4.20) $| h_{i-1} e_{i-1}' + 3 (h_i + h_{i-1}) e_i' + h_i e_{i+1}'| \leq \frac{h_i^3 + h_{i-1}^3}{3} \| f''' \|$.

Assuming that j is such that $|e_i'|$ attains its maximum, we then have

(1.17) $$\| e_i' \| \leq \frac{h_i^3 + h_{i-1}^3}{6 (h_j + h_{j-1})} \| f''' \| \leq (1/6) h^2 \| f''' \| .$$

In order to extend the bound (1.17) to the entire interval, choose any subinterval $[x_i, x_{i+1}]$ of the given partition and consider the line J' interpolating f_i' and f_{i+1}' . J' may be represented as

(4.21) $$J'(x) = (1 - t) f_i' + t f_{i+1}' .$$

By the triangle inequality, we have

(4.22) $$| f'(x) - s'(x) | \leq | f'(x) - J'(x) | + | J'(x) - s'(x) | .$$

As f' is twice continuously differentiable, we have the well-known inequality

(4.23) $$| f'(x) - J'(x) | \leq h_i^2 t (1 - t) \| f''' \| / 2! .$$

As both J' and s' are lines on $[x_i, x_{i+1}]$, we have

(4.24) $$\begin{aligned} | J'(x) - s'(x) | &\leq | [f_i' - s_i'] (1 - t) + [f_{i+1}' - s_{i+1}'] | \\ &\leq \| e_i' \| \{ |1 - t| + |t| \} \\ &\leq \| e_i' \| \\ &\leq (1/6) h^2 \| f''' \| . \end{aligned}$$

Adding (4.23) and (4.24) gives the desired formula

(4.25) $$\begin{aligned} | f'(x) - s'(x) | &\leq [1/6 + t(1 - t)/2] h^2 \| f''' \| \\ &\leq (7/24) h^2 \| f''' \| . \end{aligned}$$

Several further refinements of this argument are possible and may be given in future work.

We next wish to bound $f''(x) - s''(x)$. Choosing the arbitrary partition subinterval $[x_i, x_{i+1}]$, we consider the parabola L matching f at x_i , z_{i+1}, and x_{i+1}. By the triangle inequality, we have

$$(4.26) \qquad | f''(x) - s''(x) | \le | f''(x) - L''(x) | + | L''(x) - s''(x) |$$

$$\le | f''(x) - L''(x) | + \frac{1}{h_i^2} | e_i A_o''(t) + e_{i+1} A_2''(t) |$$

$$\le |f''(x) - L''(x)| + \frac{1}{h_i^2} \| e_i \| \{| A_o''(t) | + | A_2''(t) | \}$$

$$\le | f''(x) - L''(x) | + \frac{1}{h_i^2} \frac{h^3}{24} \| f''' \| [4 + 4]$$

$$\le | f''(x) - L''(x) | + \frac{h^3}{3 h_i^2} \| f''' \|$$

where $A_0(t)$, $A_1(t)$, and $A_2(t)$ are the fundamental functions of Lagrange interpolation.

The bound on $| f''(x) - L''(x) |$ is obtained by a Peano theorem technique similar to that used by Birkhoff and Priver [1967] (or see P. J. Davis [1963]). We have for $0 \le t \le 1$ and $f \in C'''[x_i, x_{i+1}]$, that

$$(4.27) \qquad | f''(x) - L''(x) | \le h_i \int_0^1 |K^{(2,0)}(t,z)| \, dz \; \frac{\| f''' \|}{2!}$$

where

$$K(t,z) = (t - z)_+^2 - A_0(t) \; (0 - z)_+^2$$

$$- A_1(t) \; (\tfrac{1}{2} - z)_+^2 - A_2(t) \; (1 - z)_+^2$$

$$= (t - z)_+^2 - (\tfrac{1}{2} - z)_+^2 \, 4t(1 - t) - (1 - z)_+^2 \, 2t(t - \tfrac{1}{2})$$

and for $0 \le t \le 1/2$,

$$\frac{K^{(2,0)}(t,z)}{2!} = \begin{cases} 2 z^2 & t \ge z \\ -1 + 2z^2 & t < z \, , \, z \le 1/2 \\ - 2 (1 + z)^2 & t < z \, , \, z \ge 1/2 \end{cases}$$

and where the notation $K^{(i,j)}$ denotes the mixed ith partial derivative with respect to the first variable and jth partial with respect to the second variable . Of the three terms for $K^{(2,0)}$, the first is positive and the last two are negative. Evaluation of the integral of 4.27) is straightforward, giving for $0 \leq t \leq 1/2$,

$$(4.28) \qquad | f''(x) - L''(x) | \leq h_i \{ \tfrac{4t^3}{3} - t + \tfrac{1}{2} \} \| f''' \| .$$

For other estimates of this type see Sard (1963) for theory or Varma and Howell (1983) for application to the family of Listone polynomials.

Using 4.28) in 4.26) gives for $0 \leq t \leq 1/2$,

$$(4.29) \qquad | f''(x) - s''(x) | \leq \left\{ h_i [\tfrac{4t^3}{3} - t + \tfrac{1}{2}] \ \frac{h^3}{3\, h_i^2} \right\} \| f''' \|$$

$$\leq \{ \frac{h_i}{2} + \frac{h^3}{3\, h_i^2} \} \| f''' \|$$

which by symmetry holds also for $1/2 \leq t \leq 1$.

Using the linear interpolation of f_i' and f_{i+1}' and the usual triangular inequality, we may obtain the alternate estimate

$$(4.30) \qquad | f''(x) - s''(x) | \leq \left\{ h_i [t(1 - t)] + \frac{h^2}{3\, h_i} \right\} \| f''' \|$$

which when h is larger than h_i may offer a lower estimate of the error. As the proof is very similar to those already given, we omit the details. The remaining assertions of Theorems 2 and 3 rely on the use in (1.22) of the unique parabola L which satisifies (4.3). The analogous results of Theorems $2'$ and Theorem $3'$ rely on the quartics L interpolating the same points as in the parabolic case, with the additional interpolatory conditions $J''= 0$ for $t = 0$ and $t = 1$ for each subinterval.

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