ON THE VARIANCE OF THE NUMBER OF REAL ROOTS OF A RANDOM TRIGONOMETRIC POLYNOMIAL*

K. Farahmand

Department of Mathematical Statistics
University of Cape Town
Rondebosch 7700, South Africa

ABSTRACT

This paper provides an upper estimate for the variance of the number of real zeros of the random trigonometric polynomial $g_1 \cos\theta + g_2 \cos 2\theta + \ldots. + g_n \cos n\theta$. The coefficients g_i $(i = 1, 2, \ldots, n)$ are assumed independent and normally distributed with mean zero and variance one.

Key words: random trigonometric polynomial, number of real roots, variance.

AMS subject classification: 60H, 42.

1. INTRODUCTION

Let

$$T(\theta) \equiv T_n(\theta, \omega) = \sum_{i=1}^{n} g_i(\omega) \cos i\theta ,$$

where $g_1(\omega), g_2(\omega), \ldots, g_n(\omega)$ is a sequence of independent random variables defined on a probability space (Ω, A, P) each normally distributed with mathematical expectation zero and variance one. Denote by $N(\alpha,\beta)$ the number of real roots of the

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equation $T(\theta) = 0$ in the interval (α, β), where multiple roots are counted only once. Dunnage [3] showed that except for a set of functions of $T(\theta)$ of measure not larger than $(\log n)^{-1}$

$$N(0, 2\pi) = 2n/\sqrt{3} + O\{n^{11/13}(\log n)^{3/13}\}.$$

Later Sambandham and Renganathan [9] and Farahmand [4] generalized this result to the case where the coefficients g_i have a non zero mean. They show that for n sufficiently large the mathematical expectation of the number of real roots, EN, satisfies

$$EN(0, 2\pi) \sim (2/\sqrt{3})\,n.$$

The results for the dependent coefficients with constant correlation coefficient or otherwise are due to Renganathan and Sambandham [6] and Sambandham [7] and [8]. A comprehensive treatment of the zeros of random polynomial constitutes the greater part of a book by Bharucha-Reid and Sambandham [1] which gives a rigorous and interesting survey of earlier works in this field.

Qualls [5] resolved the only known variance of the number of real roots of a random trigonometric polynomial. Indeed he considered a different type of random polynomial,

$$\sum_{i=0}^{n}(a_i \cos i\theta + b_i \sin i\theta)$$

which has the property of being stationary and for which a special theorem has been developed by Cramer and Leadbetter [2]. Here we shall prove the following theorem:

Theorem. Let $g_1(\omega), g_2(\omega), \dots, g_n(\omega)$ be the independent random variables corresponding to a Gaussian distribution with mean zero. Then the variance of the number of real roots of $T(\theta)$ satisfies

$$\mathrm{Var}\ N(0, 2\pi) = O[n^{24/13}(\log n)^{16/13}].$$

2. OVERVIEW OF PROOF OF THE THEOREM AND SOME LEMMAS

In general we make use of a delicate analysis suggested by the work of Dunnage in [3] with which we assume the reader is familiar. We divide the interval $(0, 2\pi)$ into intervals $I_1, I_2, \ldots, I_s$, each of equal length δ. Then with each I_j $(j = 1, 2, \ldots, s)$, we associate the following two functions:

$N_j(\omega)$ = number of zeros of $T(\theta)$ in I_j, counted according to their multiplicity

and

$$N_j^*(\omega) = \begin{cases} N_j(\omega) & \text{if } N_j(\omega) \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Now if $T(a)\,T(b) \leq 0$ we shall say, being prompted by a graphical idea, that $T(\theta)$ has a single crossover (s.c.o.) in (a, b), and let

$$\mu_j(\omega) = \begin{cases} 1 & \text{if } T(\theta) \text{ has a (s.c.o.) in } I_j \\ 0 & \text{otherwise} \end{cases}$$

clearly

$$(2.1) \quad 0 \le N_j(\omega) - \mu_j(\omega) \le N_j^*(\omega) .$$

For the proof of the theorem we need the following lemmas.

Lemma 1. Provided that the interval of I, of length $\delta = o(1/n)$ does not overlap the ε - neighborhood of $0, \pi$ and 2π, where $\varepsilon \sim n^{-6/13} (\log n)^{-4/13}$, the probability that $T(\theta)$ has at least two zeros (counted according to their multiplicity) in I is $O(n^3, \delta^3)$.

Proof. This is lemma 11 of [3].

We denote by $N(\omega)$ the number of real zeros that $T(\theta)$ has in I and we define

$$N^*(\omega) = \begin{cases} N(\omega) & \text{if } N(\omega) \ge 2 \\ 0 & \text{otherwise} . \end{cases}$$

Lemma 2. For a constant A

$$E[N^*(\omega)]^2 < A n^3 \delta^3 \log n .$$

Proof. Suppose $T(\theta)$ has at least $k (\ge 2)$ zeros in I. Then if I is divided into 2^p equal parts where p is chosen as an integer satisfying $2^p < k < 2^{p+1}$ at least one part must contain two or more zeros, and by lemma 1, the probability of this occurring does not exceed

$$A\, 2^p\, n^3\, (\delta / 2^p)^3 = A\, n^3\, \delta^3\, 2^{-2p} < A\, n^3\, \delta^2 / k^2 .$$

Hence if q_k is the probability that $T(\theta)$ has at least k zeros in I, we have

$$q_k < A\, n^3\, \delta^3 / k^2 .$$

Now we find the mathematical expectation of N^{*2} as

$$E\,[N^{*2}] = \sum_{k=2}^{n} k^2 \operatorname{Prob}(n = k) \sum_{k=2}^{n} k^2 (q_k - q_{k+1})$$

$$= \sum_{k=2}^{n} k^2 q_k - \sum_{k=3}^{n+1} (k-1)^2 q_k$$

$$\leq 4\, q_2 + \sum_{k=3}^{n+1} (2k - 1)\, q_k < A\, n^3\, \delta^3 \log n$$

which completes the proof of lemma 2.

Now we define

$$\alpha_j = E\,(N_j) \quad \text{and} \quad m_j = E\,(\mu_j) .$$

Lemma 3.

$$\sum m_j = (N / \sqrt{3}) + O\,\{N^{11/13}\, (\log n^{3/13} \} .$$

Proof. This is lemma 16 of [3].

3. PROOF OF THE THEOREM.

First we consider the interval $(\varepsilon, \pi - \varepsilon)$. We have

$$(3.1) \quad \operatorname{Var} N(\varepsilon, \pi-\varepsilon) \le 4E\left\{\sum_j (N_j - \mu_j)\right\}^2 + 4E\left\{\sum_j (\mu_j - m_j)\right\}^2 + 4E\left\{\sum_j (m_j - \alpha_j)\right\}^2 .$$

From (2.1) and lemma 2 we have

$$(3.2) \quad E\left[\sum_j (N_j - \mu_j)^2\right] \le E\left[\sum_j N_j^*\right]^2 < s\,E\left[\sum_{j=1}^{s} (N_j^*)^2\right] \le \frac{\pi}{\delta}\sum_{j=1}^{s} E(N_j^{*\,2}) < A\,\pi\, s\, n^3 \delta^2 \log n .$$

So far $\delta = o\,(1/n)$ has been an arbitrary constant; now since the total number of δ - intervals is $(\pi - 2\varepsilon) / \delta$, we choose δ such that

$$(\pi - 2\varepsilon) / \delta = n^{15/13} (\log n)^{-3/13} .$$

So from (3.2) we have

$$(3.3) \quad E\sum_j \{ (N_j - \mu_j) \}^2 < A\, n^{24/13} (\log n)^{16/13} .$$

Also from lemma 3 and the fact that

$$\sum_j \alpha_j = n/\sqrt{3} + O\{n^{11/13} (\log n)^{3/13}\}$$

we have

$$(3.4) \quad E\left\{\sum_j (m_j - \alpha_j)\right\}^2 = E[n/\sqrt{3} + O\{n^{11/13} (\log n)^{3/13}\} - n/\sqrt{3}$$

$$+ O\{n^{11/13} (\log n)^{3/13}\}]^2 = O\{n^{22/13} (\log n)^{6/13}\}.$$

Hence from (3.1), (3.2), (3.3) and since from [3, page 81]

$$E\left[\sum_j (\mu_j - m_j)\right]^2 = O\{n^{22/13} (\log n)^{6/13}\}$$

we have

$$(3.5) \quad \operatorname{Var} N(\varepsilon, \pi - \varepsilon) = O(n^{24/13} (\log n)^{16/13}) .$$

To find the variance in the interval $(-\varepsilon, \varepsilon)$ let $\eta(r) = \eta(r, \omega)$ be the number of zeros of $T(\theta)$ in the circle $|z| \le r$. From [3, page 83] we know that outside an exceptional set of measure at most $\exp(-n^2/2) + (2\pi)^{1/2}/n$

$$\eta(\varepsilon) \le 1 + (2 \log n + 2 n \varepsilon) / \log 2 .$$

Since the number of real roots in the segment of the real axis joining points $\pm\varepsilon$ does not exceed the number in the circle $|z| \le \varepsilon$, we can obtain

$$(3.6) \quad N(-\varepsilon, \varepsilon) = O\{n^{7/13} (\log n)^{-4/13}\}$$

except for sample functions in an ω - set of measure not exceeding $\exp(-n^2/2)$ +

$(2\pi)^{1/2} / n$. Now let d be any integer of $O\{n^{7/13} (\log n)^{-4/13}\}$, then since the trigonometric polynomial has at most $2n$ zeros in $(0, 2\pi)$ from (3.6) we have

$$\text{(3.7)} \quad \operatorname{Var} N(-\varepsilon, \varepsilon) \le \sum_{i=0}^{2n} i^2 \operatorname{Prob}(N = i)$$

$$= \sum_{i \le d} i^2 \operatorname{Prob}(N = i) + \sum_{i > d}^{2n} i^2 \operatorname{Prob}(N = i)$$

$$< B\, n^{23/13} \operatorname{Prob}\{N < C\, n^{7/13} (\log n)^{-4/13}\}$$

$$+ 4n^2 \operatorname{Prob}\{N > C'\, n^{7/13} (\log n)^{-4/13}\}$$

$$< D\, n^{23/13} + 4n^2 \{\exp(-n^2/2) + (\sqrt{2/\pi}\,)/n\}$$

$$= O(n^{23/13}),$$

where B, C, C' and D are constants. Finally from (3.5) and (3.7) we have proof of the theorem.

Remark. Although in this paper we assumed that the coefficients $g_i(\omega)$, $i = 1, 2, \ldots, n$ are independent with means zero and variance one, we can show that our theorem for the case of dependent coefficients with mean zero or non-zero (finite or infinite) and any finite variance would remain valid. However a subsequent study could be directed to reduce the upper bound obtained in our theorem, or further, to establish an asymptotic formula for the variance.

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