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RELATIVE STABILITY AND WEAK CONVERGENCE IN NON-DECREASING STOCHASTICALLY MONOTONE MARKOV CHAINS[1]

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ABSTRACT

Let $\{\xi_n\}$ be a non-decreasing stochastically monotone Markov chain whose transition probability $Q(\cdot,\cdot)$ has $Q(x,\{x\}) = \beta(x) > 0$ for some function $\beta(\cdot)$ that is non-decreasing with $\beta(x)\uparrow 1$ as $x \to +\infty$, and each $Q(x,\cdot)$ is non-atomic otherwise. A typical realization of $\{\xi_n\}$ is a Markov renewal process $\{(X_n, T_n)\}$, where $\xi_j = X_n$ for T_n consecutive values of j, T_n geometric on $\{1,2,...\}$ with parameter $\beta(X_n)$. Conditions are given for X_n to be relatively stable and for T_n to be weakly convergent.

Key words: Markov chain, stochastic monotonicity, Markov renewal process, relative stability, weak convergence.

AMS (MOS) subject classifications: 60G, 60K15.

1. INTRODUCTION

In this paper, R is the real line and $\mathfrak{R}$ the σ-field of Borel subsets of R. Let $\{\xi_n\}_0^\infty$ be a Markov chain with state space $\{R, \mathfrak{R}\}$, an initial distribution π and transition probability Q. The π and Q determine completely and uniquely a probability measure P on the countable product space $\{R^\infty, \mathfrak{R}^\infty\}$. When $\pi(\cdot) = \epsilon_x(\cdot)$ (the Dirac measure concentrated at x) we shall write P_x instead of P. The corresponding expectation operator is denoted then by E_x.

Throughout this paper it is assumed that the Q is subject to the following regularity conditions:

(*i*) for each $x \in R$ the support of $Q(x,\cdot)$ is in $[x,\infty)$:

(*ii*) the chain $\{\xi_n\}_0^\infty$ is stochastically monotone (Daley, [3]); in other words; for any $x_1 \leq x_2$, $Q(x_2, B_y) \leq Q(x_1, B_y)$ where $B_y = (-\infty, y]$; (1.1)

(*iii*) $$Q(x,\{y\}) = \begin{cases} 0 & x \neq y \\ \beta(x) > 0 & x = y \end{cases}$$

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Concerning the function $\beta(\cdot)$ we assume that $(x_1 \leq x_2)$

$$\beta(x_1) \leq \beta(x_2) \text{ and } \lim_{x \to +\infty} \beta(x) = 1 \tag{1.2}$$

From (1.1.i) it follows that

$$\xi_0 \leq \xi_1 \leq \cdots \tag{1.3}$$

Markov processes of this type are of considerable interest in reliability theory as models of the amount of deterioration of a mechanical device subject to shocks and wear during its service (Barlow and Proschan, [1]; Brown and Changanty, [2]).

Set

$$\begin{gathered} \tau_0 = sup\{k; \xi_k = \xi_0\},\ \tau_n = sup\{k; \xi_k = \xi_{\tau_{n-1}+1}\},\ T_0 = \tau_0 \\ T_n = \tau_n - \tau_{n-1},\ X_0 = \xi_0,\ X_n = \xi_{\tau_{n-1}+1},\ W_0 = X_0,\ W_n = X_n - X_{n-1}. \end{gathered} \tag{1.4}$$

From this one readily obtains that

$$P_x\{T_0 \geq i\} = \{\beta(x)\}^i \quad i = 0, 1, \ldots \tag{1.5}$$

2. AUXILIARY RESULTS

Here we list some basic properties of the bivariate sequence $\{(X_n, T_n)\}$ needed in the rest of this paper. Some simple calculations show that this sequence is a Markov renewal process with the transition probability

$$P\{X_n \in du, T_n = i \mid X_{n-1}\} = P(X_{n-1}, du)\{1 - \beta(u)\}\{\beta(u)\}^{i-1} \quad (a.s.) \tag{2.1}$$

where $i = 1, 2, \ldots$ and

$$P(x, B_y) = \begin{cases} 0 & y < x \\ \dfrac{Q(x, B_y) - \beta(x)}{1 - \beta(x)} & y \geq x. \end{cases} \tag{2.2}$$

The $P(x, B_y)$ is the transition probability of the Markov chain $\{X_n\}_0^\infty$. It is easy to verify that

$$P(x_2, By) \leq P(x_1, B_y) \text{ for all } x_1 \leq x_2. \tag{2.3}$$

From (2.1) we deduce

$$P\{X_n \in du, T_n = i\} = \{1 - \beta(u)\}\{\beta(u)\}^{i-1} P\{X_n \in du\} \tag{2.4}$$

which clearly implies (see (1.5)) that

$$P\{T_n = i \mid X_n\} = \{1-\beta(X_n)\}\{\beta(X_n)\}^{i-1} \ (a.s.)$$
$$= P_{X_n}\{T_0 = i-1\}. \tag{2.5}$$

In addition, since

$$P_x\{X_1 \in du, T_1 = i\} = P(x,du)\{1-\beta(u)\}\{\beta(u)\}^{i-1} \tag{2.6}$$

it follows that

$$P\{X_n \in du, T_n = i \mid X_{n-1}\} = P_{X_{n-1}}\{X_1 \in du, T_1 = j\} \ (a.s.). \tag{2.7}$$

Denote by $P^n(x, B_y)$ the n-step transition probability of $\{X_n\}_0^\infty$, since

$$X_0 < X_1 < \dots \tag{2.8}$$

we have that $P^{n+1}(x, B_y) \le P^n(x, B_y)$. On the other hand, the stochastic monotonicity and the Chapman-Kolmogorov equation yield:

$$P^n(x, B_y) \le (P(x, B_y))^n. \tag{2.9}$$

From this, (2.8) and the Borel-Cantelli lemma it follows that $X_n \to +\infty$ (*a.s.*) if $P(x, B_y) < 1$ for all $y < \infty$.

It is clear from (2.5) that T_n is conditionally geometric with parameter $\beta(X_n)$. In addition, since

$$P\{T_n \ge i \mid X_n\} = \{\beta(X_n)\}^{i-1} \ (a.s.) \tag{2.10}$$

it is apparent that $\{T_n\}$ is stochastically monotone and that $T_n \xrightarrow{d.} \infty$ as $n \to \infty$. Finally, for each $n = 0, 1, \dots$ we have:

$$P\{T_0 = i_0, \dots, T_n = i_n \mid X_0, \dots, X_n \mid = \prod_{j=1}^{n} P\{T_j = i_j \mid X_j\} \ (a.s.). \tag{2.11}$$

In other words, conditioned on a realization of $\{X_n\}$ the sequence of sojourn times $\{T_n\}$ becomes a family of independent *r.v.*'s such that the distribution of T_n depends only on X_n.

Consider

$$P_x\{X_n \in du, T_n = i\} = \int_x^u P\{X_n \in du, T_n = i \mid X_{n-1} = z\} P^{n-1}(x, dz)$$
$$= \{1-\beta(u)\}\{\beta(u)\}^{i-1} P_x\{X_n \in du\}$$

from which we deduce that

$$P_x\{T_n = i \mid X_n\} = \{1-\beta(X_n)\}\{\beta(X_n)\}^{i-1}\ (a.s.) \tag{2.12}$$

where the right hand side is independent of x.

3. REMARKS ON THE STRUCTURE OF $\{W_n\}$

In this section we investigate asymptotic structure of the sequence $\{W_n\}_0^\infty$ assuming that the following condition holds for all $y \geq 0$:

$$\lim_{x\to\infty} \frac{P_x\{\xi_1 > x+y\}}{P_x\{\xi_1 > x\}} = 1 - F(y) \tag{3.1}$$

where $F(\cdot)$ is a proper $d.f.$

Remark 3.1: The condition (3.1) is similar to one introduced by Gnedenko [4].

Denote by

$$\Phi_n(y \mid x) = P_x\{W_n \leq y\} \quad n = 1,2,\ldots \tag{3.2}$$

then clearly

$$\Phi_1(y \mid x) = P(x, B_{x+y}). \tag{3.3}$$

Some simple calculations yield:

$$\Phi_n(y \mid x) = E_x\{\Phi_1(y \mid X_{n-1})\}. \tag{3.4}$$

Taking into account (2.2) and condition (3.1), we have:

$$\lim_{x\to\infty} \Phi_1(y \mid x) = F(y). \tag{3.5}$$

This, (3.4) and the Lebesgue bounded convergence theorem imply that

$$\lim_{n\to\infty} \Phi_n(y \mid x) = F(y). \tag{3.6}$$

In other words, (at least) $W_n \xrightarrow{d} Y$, where Y is a $r.v.$ with the $d.f.$ $F(y)$. The following proposition generalizes this simple observation.

Proposition 3.1: *Assume that* (3.1) *holds and* $F(\cdot)$ *is continuous, then under* P_x, *for all* $k = 1,2,\ldots$

$$(W_{n+1},\ldots,W_{n+k}) \xrightarrow{d} (Y_1,\ldots,Y_k) \quad as\ n\to\infty \tag{3.7}$$

where $\{Y_i\}_1^\infty$ *is an i.i.d. sequence of r.v.'s with common d.f.* $F(\cdot)$.

Proof: The method of proof will be amply illustrated by the case $n = 2$. Given $\epsilon > 0$, we obtain

$$P_x\{W_{n+1} \le y_1, W_{n+2} \le y_2\} \tag{3.8}$$

$$= \int_x^{\infty} \int_z^{z+y_1} P(z, du)\Phi_1(y_2 \mid u)P^n(x, dz)$$

$$= E_x\{\int_{X_n}^{X_n + y_1} P(X_n, du)\Phi_1(y_2 \mid u)\}$$

Since by assumption $F(\cdot)$ is continuous the convergence in (3.5) is uniform. Consequently, for any $\epsilon > 0$ there exists x_0 such that

$$|\Phi_1(y_2 \mid u) - F(y_2)| < \epsilon \text{ for all } u > x_0 \text{ and any } y_2 \in R.$$

From this and (3.8) we then have:

$$P_x\{W_{n+1} \le y_1, W_{n+2} \le y_2\} \le P^n(x, B_{x_0})$$

$$+ E_x\{I_{\{X_n > x_0\}} \int_{X_n}^{X_n + y_1} \Phi_1(y_2 \mid u)P(X_n, du)\}$$

$$\le P^n(x, B_{x_0}) + (\epsilon + F(y_2))E_x\{\Phi_1(y_1 \mid X_n)I_{\{X_n > x_0\}}\}.$$

Consequently,

$$\overline{\lim_{n \to \infty}} P_x\{W_{n+1} \le y_1, W_{n+2} \le y_2\} \le (\epsilon + F(y_2))F(y_1).$$

In the same fashion, one can show that

$$\underline{\lim_{n \to \infty}} P_x\{W_{n+1} \le y_1, W_{n+2} \le y_2\} \ge F(y_1)(F(y_2) - \epsilon).$$

Since $\epsilon > 0$ is arbitrary, the assertion follows.

Remark 3.2: The last proposition indicates that, roughly speaking, the remote members of $\{W_n\}_0^{\infty}$ are *i.i.d. r.v.*'s.

Next, we show that the sequence $\{W_n\}_0^{\infty}$ is endowed with a mixing property, which means, loosely speaking, that its elements far apart are nearly independent. Denote by $\mathfrak{F}_n = \sigma\{W_0, \ldots, W_n\}$ and by $\mathfrak{F}^n = \sigma\{W_n, W_{n+1} \ldots\}$ then we have:

Proposition 3.2: *For each $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$*

$$\lim_{m\to\infty} P_x\left(\bigcap_{j=1}^{n}\{X_j \le y_j\}\bigcap_{i=1}^{k}\{W_{n+m+i} \le z_i\}\right)$$
$$= P_x\left(\bigcap_{j=1}^{n}\{X_j \le y_j\}\right)\prod_{i=1}^{k} F(z_i) \quad (x < y_1 < \ldots < y_n) \tag{3.9}$$

Proof: By invoking the Markov property of $\{X_n\}_0^\infty$ and the proposition 3.1, we obtain

$$P_x\left(\bigcap_{j=1}^{n}\{X_j \le y_j\}\bigcap_{i=1}^{k}\{W_{n+m+i} \le z_i\}\right)$$

$$= \int_{(x,y_n]} P_x\left(\bigcap_{j=1}^{n-1}\{X_j \le y_j\} \mid X_n = s\right)P_s\left(\bigcap_{i=1}^{k}\{W_{m+i} \le z_i\}\right)P^n(x,ds)$$

$$\to \int_{(x,y_n]} P_x\left(\bigcap_{j=1}^{n-1}\{X_j \le y_j\} \mid X_n = s\right)\prod_{i=1}^{k} F(z_i)P^n(x,ds)$$

as $m\to\infty$ which proves the assertion.

Corollary 3.1: *$\{W_n\}_0^\infty$ and $\{Y_j\}_1^\infty$ are independent families. Set*

$$\mathcal{T} = \bigcap_{n=0}^{\infty} \mathcal{F}^n.$$

It follows from the last proposition that $\mathcal{F}_n$ and $\mathcal{T}$ are independent σ-algebras for all $n = 0, 1, \ldots$. Therefore $\mathcal{F}_n \cap \mathcal{T}$ is a trivial σ-algebra (its elements are either sure or null events). By letting $n\to\infty$ we have that $\mathcal{F}_\infty \supset \mathcal{T}$ and that their intersection is a trivial σ-algebra. This clearly implies that the tail σ-algebra $\mathcal{T}$ is a trivial one.

4. RELATIVE STABILITY OF $\{X_n\}$

The sequence $\{X_n\}$ is said to be relatively stable if there exist constants $\{a_n\}$ such that $X_n/a_n\to 1$ in probability (Gnedenko and Kolmogorov, [5]). If the convergence is $(a.s.)$ the sequence is called $(a.s.)$ relatively stable (Resnik, [6]). The following proposition gives a sufficient condition for $(a.s.)$ relative stability of $\{X_n\}$.

Proposition 4.1: *Assume that*

$$\sup_x E_x\{W_1^2\} < \infty \tag{4.1}$$

then $X_n/n\to\alpha_1$ $(a.s.)$ where $\alpha_1 = E\{Y_1\}$.

Proof: From (3.4), (4.1) and Fubini's theorem we deduce that

$$E_x\{W_k{}^2\} = \int_0^\infty y[1-\Phi_k(y \mid x)]dy \tag{4.2}$$

$$= E_x\{\int_0^\infty y[1-\Phi_1(y \mid X_{k-1})]dy\}$$

$$= E_x(E_{X_{k-1}}\{W_1{}^2\}) \leq \sup_x E_x\{W_1{}^2\} < \infty.$$

Consequently,

$$\sup_{x,k} E_x\{W_k{}^2\} < \infty.$$

Next, since

$$\{ \frac{1}{n}\sum_1^n W_k \text{ converges}\} \in \mathfrak{T}$$

and $\mathfrak{T}$ is a trivial σ-algebra, to prove the proposition it suffices to show that

$$P_x\{ \mid \frac{1}{n}X_n - \alpha_1 \mid \, > \epsilon\} \to 0 \text{ as } n \to \infty. \tag{4.3}$$

Consider

$$Var\{\frac{1}{n}X_n\} = \frac{1}{n^2}\left(\sum_{k=1}^n Var\{W_k\} + \sum_{i=1}^{n-1} \sum_{j=i+1}^n Cov(W_i, W_j)\right).$$

It is clear from (4.1) and (4.2) that under P_x

$$\frac{1}{n^2} \sum_{k=1}^n Var\{W_k\} \to 0 \text{ as } n \to \infty.$$

On the other hand, due to propositions 3.1 and 3.2

$$\lim_{k\to\infty} Cov(W_k, W_{k+n}) = 0 \quad \lim_{k\to\infty} Cov(W_n, W_{n+k}) = 0 \tag{4.4}$$

for each $n = 0, 1, \ldots$. Thus, given $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that

$$\mid Cov(W_i, W_j) \mid \, < \epsilon \quad \mid Cov(W_n, W_{n+k}) \mid \, < \epsilon \tag{4.5}$$

if $min\{i, j\} > n_0$ and $k > n_0$. Now, take $n > 2n_0$, then

$$\mid \sum_{i=1}^{n-1} \sum_{j=i+1}^n Cov(W_i, W_j) \mid \, \leq \, \mid \sum_{i=1}^{n_0} \sum_{j=i+1}^{2n_0} Cov(W_i, W_j) \mid$$

$$+ \mid \sum_{i=1}^{n_0} \sum_{j=2n_0+1}^n Cov(W_i, W_j) \mid + \mid \sum_{i=n_0+1}^{n-1} \sum_{j=i+1}^n Cov(W_i, W_j) \mid$$

$$\leq \left| \sum_{i=1}^{n_0} \sum_{j=i+1}^{2n_0} Cov(W_i, W_j) \right| + \epsilon(n - 2n_0) + \epsilon(n - n_0 - 1)(n - n_0)/2.$$

Consequently, for any $\epsilon > 0$

$$\lim_{n\to\infty} Var\{\tfrac{1}{n}X_n\} < \epsilon/2.$$

Finally, since $E_x\{W_n\} \to \alpha_1$ as $n \to \infty$ it follows that

$$\sum_{k=1}^{n} E_x\{W_k\}/n \to \alpha_1 \text{ as } n \to \infty.$$

Therefore, for n sufficiently large

$$P_x\{ | \tfrac{1}{n}X_n - \alpha_1 | > \epsilon\} \leq P_x\{\tfrac{1}{n} | \sum_{i=1}^{n} [W_i - E_x\{W_i\}] | > \epsilon\}.$$

This and (4.1) prove the assertion.

Corollary 4.1: *From proposition 4.1 we readily deduce that for each* $\epsilon > 0$

$$P_x\{X_n \leq (\alpha_1 + \epsilon)n\} \to 1 \text{ and } P_x\{X_n \leq (\alpha_1 - \epsilon)n\} \to 0 \tag{4.6}$$

as $n \to \infty$. *Consequently*

$$\lim_{n\to\infty} P^n(x, B_{ny}) = \begin{cases} 1 & \text{if} \quad y \geq \alpha_1 \\ 0 & \text{if} \quad y < \alpha_1. \end{cases} \tag{4.7}$$

Denote by

$$T(y) = inf\{k; X_k > y\} \tag{4.8}$$

then for any $x < y$

$$\begin{aligned} P_x\{T(y) \leq n\} &= P_x\{X_n > y\} \\ &= 1 - P^n(x, B). \end{aligned} \tag{4.9}$$

Proposition 4.2: *Under* P_x

$$\tfrac{1}{y}T(y) \to \alpha_1^{-1} \textit{ in probability as } y \to +\infty.$$

Proof: Assume $\alpha_1 > 0$, then we have to show

$$P_x\{ | \tfrac{1}{y}T(y) - \alpha_1^{-1} | \leq \epsilon\} \to 1 \text{ as } y \to +\infty$$

for any $\epsilon > 0$. Choose $\epsilon \in (0, \alpha_1^{-1})$ then from (4.9) we deduce

$$P_x\{ | \tfrac{1}{y}T(y) - \alpha_1^{-1} | \leq \epsilon\} = P_x\{X_{[(\alpha_1^{-1} - \epsilon)y]} \leq y\} - P_x\{X_{[(\alpha_1^{-1} + \epsilon)y]} \leq y\}$$

$$= P^{[(\alpha_1{}^{-1} - \epsilon)y]}(x, B_y) - P^{[(\alpha_1{}^{-1} + \epsilon)y]}(x, B_y)$$

where, as usual, $[x]$ stands for the integer part of x. Since

$$\frac{y}{[(\alpha_1{}^{-1} - \epsilon)]} \geq \frac{\alpha_1}{1 - \epsilon\alpha_1} > \alpha_1$$

it follows from (4.7) that

$$\lim_{y \to \infty} P^{[(\alpha_1{}^{-1} - \epsilon)y]}(x, B_y) = 1.$$

Similarly, when $y \to \infty$

$$\frac{y}{[(\alpha_1{}^{-1} + \epsilon)y]} < \frac{y}{(\alpha_1{}^{-1} + \epsilon)y - 1} \sim \frac{\alpha_1}{1 + \epsilon\alpha_1} < \alpha_1.$$

This and (4.7) imply

$$\lim_{y \to \infty} P^{[(\alpha_1{}^{-1} + \epsilon)y]}(x, B_y) = 0$$

which proves the proposition.

5. WEAK CONVERGENCE OF $\{T_n\}$

In this section we show that a sequence of scale factors $\{d_n\}$ exists such that under P_x

$$d_n T_n \xrightarrow{d} Z \tag{5.1}$$

where the *r.v.* Z has an exponential distribution independent of x. But first, we need the following auxiliary result.

Proposition 5.1: *Assume that condition* (4.1) *holds, then*

$$\frac{1 - \beta(X_n)}{1 - \beta(n\alpha_1)} \xrightarrow{P} 1 \quad as\ n \to \infty \tag{5.2}$$

where $\alpha_1 = E\{Y_1\}$.

Proof: Set

$$\beta^{-1}(y) = inf\{x; \beta(x) > y\} \tag{5.3}$$

then for any $\epsilon \in (0,1)$

$$P_x\{|\frac{1 - \beta(X_n)}{1 - \beta(n\alpha_1)} - 1| \leq \epsilon\} = P_x\{\frac{X_n}{n\alpha_1} \leq \frac{\beta^{-1}(\beta(n\alpha_1)(1 - \epsilon) + \epsilon)}{n\alpha_1}\}$$

$$-P_x\{\frac{\beta^{-1}(\beta(n\alpha_1)(1+\epsilon)-\epsilon)}{n\alpha_1}\}.$$

Since

$$\beta^{-1}(\beta(n\alpha_1)(1-\epsilon)+\epsilon) > n\alpha_1$$

$$\beta^{-1}(\beta(n\alpha_1)(1+\epsilon)-\epsilon) < n\alpha_1$$

for all $n = 1, 2, \ldots$, the assertion now follows from proposition 4.1.

Proposition 5.2: *Suppose that* (4.1) *holds, then*

$$\lim_{n\to\infty} P_x\{[1-\beta(n\alpha_1)]T_n > u\} = e^{-u}. \tag{5.4}$$

Proof: Denote by

$$U_n = [1-\beta(X_n)]T_n. \tag{5.5}$$

Then taking into account (2.10), we have:

$$P_x\{U_n > u\} = E_x(P_x\{T_n > \frac{u}{1-\beta(X_n)} \mid X_n\}) \tag{5.6}$$

$$= E_x(\{\beta(X_n)\}^{[\frac{u}{1-\beta(X_n)}]}).$$

Set

$$R_n(u,w) = \{\beta(X_n(\omega))\}^{[\frac{u}{1-\beta(X_n(\omega))}]}.$$

Since the function

$$h(y) = exp\{\frac{1}{1-\beta(y)} ln\beta(y)\}$$

is non-decreasing on R, it follows that

$$R_n(u,\cdot) \le R_{n+1}(u,\cdot)$$

at least $(a.s.)$. From this we readily obtain

$$\lim_{n\to\infty} R_n(u,\omega) = e^{-u} \tag{5.7}$$

at least $(a.s.)$ P_x. Invoking now the monotone convergence theorem, we deduce from (5.6) that

$$U_n \xrightarrow{d} Z. \tag{5.8}$$

Finally, write

$$[1-\beta(n\alpha_1)]T_n = \frac{1-\beta(n\alpha_1)}{1-\beta(X_n)} U_n$$

then the proof of (5.4) follows (5.8), proposition 4.2 and a Slutsky's theorem.

Remark ***5.1:*** One can easily show that the sequence of *r.v.*'s $\{U_n\}_0^\infty$ has the following properties:

$$E_x\{U_n\} = 1 \quad Var\{U_n\} = E_x\{\beta(X_n)\}$$

$$E_x\{U_{n+1} \mid U_1, \ldots, U_n\} = 1.$$

Remark ***5.2:*** The result of the last proposition can be easily extended as follows: Set

$$V_n = [1 - \beta(n\alpha_1)\}T_n$$

then after some straight forward calculations (see Todorovic and Gani, [7]) one can show that for each $k = 1, 2, \ldots$

$$(V_{n+1}, \ldots, V_{n+k}) \xrightarrow{d} (Z_1, \ldots, Z_k)$$

under P_x, where $\{Z_k\}_1$ is an *i.i.d.* sequence of *r.v.*'s with common non-negative exponential distribution of x. The sequence also possesses a mixing property.

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