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ON SECOND ORDER DISCONTINUOUS DIFFERENTIAL EQUATIONS IN BANACH SPACES[1]

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ABSTRACT

In this paper we study a second order semilinear initial value problem (IVP), where the linear operator in the differential equation is the infinitesimal generator of a strongly continuous cosine family in a Banach space E. We shall first prove existence, uniqueness and estimation results for weak solutions of the IVP with Carathéodory type of nonlinearity, by using a comparison method. The existence of the extremal mild solutions of the IVP is then studied when E is an ordered Banach space. We shall also discuss the dependence of these solutions on the data. A characteristic feature of the results concerning extremal solutions is that the nonlinearity is not assumed to be continuous in any of its arguments. Moreover, no compactness conditions are assumed. The obtained results are then applied to a second order partial differential equation of hyperbolic type.

Key words: Second order semilinear initial value problems, discontinuous nonlinearities, weak and mild solutions.

AMS (MOS) subject classifications: 34G20, 47H10, 47H15, 35D05 35L15.

1. INTRODUCTION

In this paper we consider the second order semilinear initial value problem

$$x'' = Ax + g(t, x, x'), \qquad x(0) = x_0, x'(0) = x_1, \tag{1.1}$$

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where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$ in a Banach space E and $g: J \times E^2 \to E$, $J = [0,T]$, $T > 0$. The existence of mild solutions of (1.1) is considered in [16] when g is continuous. Our purpose is to study the case when g is discontinuous.

We first prove existence, uniqueness and estimation results for weak solutions of the initial value problem (IVP) (1.1), by using a comparison method and assuming that g satisfies Carathéodory conditions. The existence of the extremal mild solutions of (1.1) is then studied when E is an ordered Banach space, and when g does not depend on x'. We shall also discuss the dependence of these solutions on the initial values and on g. A characteristic feature of the results concerning extremal solutions is that g is not assumed to be continuous in any of its arguments. Moreover, no compactness assumptions are imposed on g. The obtained results are then applied to a second order partial differential equation of hyperbolic type.

2. PRELIMINARIES

Given a Banach space E, we say that a family $\{C(t) \mid t \in \mathbb{R}\}$ in the space $L(E)$ of bounded linear linear operators on E is a *strongly continuous cosine family* if

(i) $C(0) = I$;

(ii) $t \mapsto C(t)x$ is strongly continuous for each fixed $x \in E$;

(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$.

The *strongly continuous sine family* $\{S(t) \mid t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$, is defined by

$$S(t)x = \int_0^t C(s)x\,ds, x \in E, \quad t \in \mathbb{R}. \tag{2.1}$$

Denote

$$E_1 = \{x \in E \mid C(\cdot)x \in C^1(\mathbb{R}, E)\} \text{ and } E_2 = \{x \in E \mid C(\cdot)x \in C^2(\mathbb{R}, E)\}. \tag{2.2}$$

It can be shown that $\bar{E}_2 = E$. Obviously, E_2 is a subspace of E_1. As for the properties of strongly continuous cosine and sine families, see [3, 4, 5, 15, 16].

The *infinitesimal generator* $A: E_2 \to E$ of a cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2}C(0)x. \tag{2.3}$$

Assume now that A is the infinitesimal generator of a given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$.

By a *strong solution* of the IVP (1.1) on the interval $J = [0,T]$ we mean a function $x: J \to E$ with absolutely continuous first derivative, whose second derivative $x''(t)$ exists and equals to $Ax(t) + g(t, x(t), x'(t))$ for almost all (*a.a.*) $t \in J$, and which satisfies the initial conditions $x(0) = x_0$, $x'(0) = x_1$. Given $(x_0, x_1) \in E_2 \times E$, we say that $x \in C^1(J,E)$ is a *weak solution* of (1.1) if there is $y \in C(J,E)$ such that

$$\begin{aligned} x(t) &= x_0 + \int_0^t y(s)ds, t \in J, \\ y(t) &= S(t)Ax_0 + C(t)x_1 + \int_0^t C(t-s)g(s, x_0 + \int_0^s y(\tau)d\tau, y(s))ds. \end{aligned} \tag{2.4}$$

By the reasoning used in the proofs of proposition 1.2 and theorem 1.3 in [13] (see also [16]), one can show that

(*a*) a strong solution x of (1.1) is also its weak solution if $g(\cdot, x(\cdot), x'(\cdot))$ is continuous;

(*b*) a weak solution x of (1.1) is also its strong solution if $g(\cdot, x(\cdot), x'(\cdot))$ is absolutely continuous and almost everywhere differentiable;

(*c*) if $x \in C^1(J,E)$ is a weak solution of the IVP (1.1), then it satisfies the integral equation

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g(s, x(s), x'(s))ds, \quad t \in J. \tag{2.5}$$

3. EXISTENCE, UNIQUENESS AND DEPENDENCE ON INITIAL VALUES

When $x \in C(J,E)$ we denote $|x| = t \mapsto \|x(t)\|$ and $\|x\|_0 = max\{\|x(t)\| \mid t \in J\}$. The considerations of this section are based on the following fixed point result (cf. [6, 18]),

Lemma 3.1: *Given $F: C(J,E) \to C(J,E)$, assume that*

$$| Fy - F\bar{y} | \leq Q(| y - \bar{y} |) \text{ for all } y, \bar{y} \in C(J,E), \tag{3.1}$$

where $Q: C(J,\mathbb{R}_+) \to C(J,\mathbb{R}_+)$ *is nondecreasing, the equation*

$$u_0 + Qu = u \tag{3.2}$$

has for each $u_0 \in \mathbb{R}_+$ *an upper solution* $v \in C(J,\mathbb{R}_+)$, *and* $Q^n v \to 0$ *uniformly on* J. *Then for each* $y_0 \in C(J,E)$ *the sequence* $(Fy_n)_{n=0}^{\infty}$ *converges uniformly on* J *to a unique fixed point* x *of* F.

In the following we shall assume that

($C0$) $\{C(t) \mid t \in \mathbb{R}\}$ is a strongly continuous cosine family, that A is its infinitesimal generator, and that $\{S(t) \mid t \in \mathbb{R}\}$ is the associated sine family.

Condition (*ii*), definition (2.1) and the uniform boundedness principle imply that

$$M = sup\{ \| C(t) \| \mid t \in J\} < \infty, \tag{3.3}$$

and that

$$\| S(t)x - S(\bar{t})x \| \leq M | t - \bar{t} | \, \| x \|, \quad x \in E, \, t, \bar{t} \in J. \tag{3.4}$$

Assume also that $g: J \times E^2 \to E$ satisfies the following conditions.

($g0$) $g(\cdot, x, y)$ is strongly measurable for all $x, y \in E$, and $g(\cdot, 0, 0)$ is Bochner integrable

($g1$)

$$\| g(t, x+h, y+k) - g(t, xy) \| \leq q(t, \| h \|, \| k \|)$$

for all $x, y, h, k \in E$ and for *a.a.* $t \in J$, where $q: J \times \mathbb{R}_+^2 \to \mathbb{R}_+$ is a Carathéodory function, $q(t, \cdot, \cdot)$ is nondecreasing for *a.a.* $t \in J$, the IVP

$$u'' = Mq(t, u, u'), \quad u(0) = u_0, \quad u'(0) = u_1 \tag{3.5}$$

with M given by (3.3), has for each $(u_0, u_1) \in \mathbb{R}_+^2$ an upper solution on J, and the zero-function is the only solution of (3.5) when $u_0 = u_1 = 0$.

Theorem 3.1: *If the hypotheses* ($C0$), ($g0$) *and* ($g1$) *hold, then for each* $(x_0, x_1) \in E_2 \times E$ *the IVP* (1.1) *has a unique weak solution* x *on* J. *Moreover,* x *is*

of the form $x(t) = x_0 + \int_0^t y(s)ds$, $t \in J$, *where* y *is the uniform limit of the sequence* $(y_n)_{n=0}^{\infty}$ *of the successive approximations*

$$y_{n+1}(t) = S(t)Ax_0 + C(t)x_1 + \int_0^t C(t-s)g(s, x_0 + \int_0^s y_n(\tau))d\tau,\ y_n(s))ds, \quad (3.6)$$

$t \in J$, $n \in \mathbb{N}$, *and with arbitrarily chosen* $y_0 \in C(J,E)$.

Proof: Let $(x_0, x_1) \in E_2 \times E$ be given. The function $s \mapsto q(s, v(s), v'(s))$ is for each $v \in C^1(J, \mathbb{R}_+)$ bounded above by $\frac{1}{M}u''$, where u is an upper solution of the IVP (3.5) with $u(0) = \| v \|_0$ and $u'(0) = \| v' \|_0$. The hypotheses given for q in condition $(g1)$ imply that the equation

$$Qw(t) = \int_0^t Mq(s, \int_0^s w(\tau)d\tau, w(s))ds, \quad t \in J \qquad (a)$$

defines a nondecreasing mapping $Q: C(J, \mathbb{R}_+) \to C(J, \mathbb{R}_+)$. Since $u(t) \equiv 0$ satisfies $u''(t) = q(t, u(t), u'(t))$ for *a.a.*. $t \in J$, then $q(t,0,0) = 0$ for *a.a.* $t \in J$. From $(g1)$ it then follows that also g is a Carathéodory function. Thus $g(\cdot, x(\cdot), y(\cdot))$ is strongly measurable in J for all $x, y \in C(J,E)$. From $(g1)$ it also follows that

$$\| g(t, x(t), y(t)) \| \leq \| g(t,0,0) \| + q(t, \| x \|_0, \| y \|_0), \quad t \in J,$$

whence $g(\cdot, x(\cdot), y(\cdot))$ is Bochner integrable. This implies that the equation

$$Fz(t) = S(t)Ax_0 + C(t)x_1 + \int_0^t C(t-s)g(s, x_0 + \int_0^s z(\tau)d\tau, z(s))ds \qquad (b)$$

defines a mapping $F: C(J,E) \to C(J,E)$. By using (3.3), $(g1)$, (a) and (b) it is easy to show that

$$| Fy - F\bar{y} | \leq Q | y - \bar{y} |, \quad y, \bar{y} \in C(J,E).$$

Condition $(g1)$ ensures that the operator equation

$$u_1 + Qv(t) = v(t), \quad t \in J \qquad (c)$$

has an upper solution for each $u_1 \in \mathbb{R}_+$. Moreover, for any such upper solution v the sequence $(Q^n v)_{n=0}^{\infty}$ converges by proposition 3.1 of [10] uniformly on J to the maximal solution of (3.5) with $u_0 = u_1 = 0$, i.e. to the 0-function. Thus all the hypotheses of lemma 3.1 are valid, whence the iteration sequence $(F^n y_0)_{n=0}^{\infty}$,

which equals to the sequence of the successive approximations (3.6), converges for each choice of $y_0 \in C(J,E)$ uniformly in J to a unique fixed point y of F. From the definition (b) of F it follows that y is the uniquely determined solution of the second integral equation (2.4), and hence $x(t) = x_0 + \int_0^t y(s)ds$, $t \in J$, is a unique weak solution of the IVP (1.1) on J. □

Remark 3.1: If the IVP (3.5) has for some positive value of M the zero function as the only solution when $u_0 = u_1 = 0$, the same does not necessarily hold for all positive M, as we see from the following example.

Choose $J = [0,1]$ and define $q: J \times \mathbb{R}_+^2, \mathbb{R}_+)$ by

$$q(t,r,s) = \begin{cases} 2t, & \text{for } s \geq t^2, t \in J, r \in \mathbb{R}, \\ \frac{2s}{t}, & \text{for } 0 \leq s < t^2,\ 0 < t \leq 1,\ r \in \mathbb{R}. \end{cases}$$

It is easy to show that $u(t) \equiv 0$ is the only solution of (3.5) when $M = \frac{1}{2}$ and $u_0 = u_1 = 0$, whereas $u(t) = \gamma t^3$, $t \in J$ is for each $\gamma \in [0, \frac{1}{3}]$ a solution of (3.5) when $M = 1$ and $u_0 = u_1 = 0$.

The dependence of the weak solution of the IVP (1.1) on the initial values x_0 and x_1 can be estimated by the minimal solutions of the comparison problem (3.5) in the following manner.

Theorem 3.2: *Let the hypotheses* $(C0)$, $(g0)$ *and* $(g1)$ *hold. Let* $x = x(\cdot, x_0, x_1)$ *denote the weak solution of the IVP* (1.1) *on* J, *and* $u = u(\cdot, u_0, u_1)$ *the minimal solution of the IVP* (3.5) *on* J. *Then for all* $x_0, \bar{x}_0 \in E_2$ *and* x_1, $\bar{x}_1 \in E$,

$$\| x(t) - \bar{x}(t) \| \leq u(t), \text{ and } \| x'(t) - \bar{x}'(t) \| \leq u'(t),\ t \in J, \tag{3.7}$$

where $x = x(\cdot, x_0, x_1)$, $\bar{x} = x(\cdot, \bar{x}_0, \bar{x}_1)$ *and* $u = u(\cdot, \| x_0 - \bar{x}_0 \|, MT \| A(x_0 - \bar{x}_0) \| + M \| x_1 - \bar{x}_1 \|)$.

Proof: Let $x_0, \bar{x}_0 \in E$ be given. The solutions $x = x(\cdot, x_0, x_1)$ and $\bar{x} = x(\cdot, \bar{x}_0, \bar{x}_1)$ exist by theorem 3.1, and

$$x(t) = x_0 + \int_0^t y(s)ds, \text{ and } \bar{x}(t) = \bar{x}_0 + \int_0^t \bar{y}(s)ds,$$

where y, $\bar{y}$ are the unique fixed points of the integral operators F, $\bar{F}: C(J,E) \to$

$C(J,E)$, defined by

$$Fz(t) = S(t)Ax_0 + C(t)x_1 + \int_0^t C(t-s)g(s, x_0 + \int_0^s z(\tau)d\tau, z(s))ds,$$

and

$$\bar{F}z(t) = S(t)A\bar{x}_0 + C(t)\bar{x}_1 + \int_0^t C(t-s)g(s, \bar{x}_0 + \int_0^s z(\tau)d\tau, z(s))ds.$$

Moreover,

$$u(t) = \| x_0 - \bar{x}_0 \| + \int_0^t v(s)ds, \quad t \in J,$$

where v is the least fixed point of the operator $G: C(J, \mathbb{R}_+) \to C(J, \mathbb{R}_+)$ given by

$$Gw(t) = MT \| A(x_0 - \bar{x}_0) \| + M \| x_1 - \bar{x}_1 \| + \int_0^t Mq(s, \| x_0 - \bar{x}_0 \| + \int_0^s w(\tau)d\tau, w(s))ds.$$

These definitions and condition $(g1)$ imply that

$$| Fy - \bar{F}\bar{y} | \leq G | y - \bar{y} | \text{ for } y, \bar{y} \in C(J,E). \tag{a}$$

Denoting $y_0(t) \equiv x_1$ and $\bar{y}(t) \equiv \bar{x}_1$, then equality holds in

$$| F^n y_0 - \bar{F}^n \bar{y}_0 | \leq G^n | y_0 - \bar{y}_0 | \tag{b}$$

when $n = 0$. Since G is nondecreasing, it follows from (a) and (b) that

$$| F^{n+1} y_0 - \bar{F}^{n+1} \bar{y}_0 | \leq G | F^n y_0 - \hat{F}^n \bar{y}_0 | \leq G^{n+1} | y_0 - \bar{y}_0 |,$$

whence (b) holds for all $n \in \mathbb{N}$. Theorem 3.1 above and proposition 3.1 of [10] imply that

$$y = \lim_{n\to\infty} F^n y_0, \ \bar{y} = \lim_{n\to\infty} \bar{F}^n \bar{y}_0 \text{ and } v = \lim_{n\to\infty} G^n | y_0 - \bar{y}_0 |.$$

From this and (b) it follows, when $n \to \infty$, that

$$\| y(t) - \bar{y}(t) \| \leq v(t) \text{ for each } t \in J.$$

This and the definitions of $y, \bar{y}$ and v imply that the estimates (3.7) hold. □

Remark 3.2: The hypotheses given for q in $(g1)$ ensure (cf. [8]) that the minimal solution u of the IVP (3.5) and its derivative u' are nondecreasing with respect to u_0 and u_1, and that both of them tend to zero uniformly over $t \in J$ as $u_0 \to 0$ and $u_1 \to 0$. This implies by (3.7) that under the hypotheses of theorem 3.2 the weak solution x of (1.1) and its derivative depend continuously on x_1. As a consequence of theorem 3.2 we then obtain.

Corollary 3.1: *Let the hypotheses $(C0)$ and $(g0)$ hold. Assume moreover that for all $x, y, h, k \in E$ and for a.a. $t \in J$,*

$$\| g(t, x+h, y+k) - g(t, x, y) \| \leq p(t) \| h \| + q(t) \| k \|,$$

where $p, q \in L^1(J, \mathbb{R}_+)$. Then the IVP (1.1) has for each $(x_0, x_1) \in E_2 \times E$ exactly one weak solution x. Moreover, x and x' depend continuously on x_1.

Proof: It is easy to see that condition $(g1)$ holds when $q(t, u, v) = p(t)u + q(t)v$. Thus the relations (3.7) hold. Moreover, the right hand sides of the inequalities (3.7) tend to 0 uniformly on J as $\bar{x}_0 = x_0$ and $\bar{x}_1 \to x_1$ in E, which implies the last conclusion of corollary. □

In particular, we have,

Corollary 3.2: *If the hypotheses $(C0)$ and $(g0)$ hold, and if $A_1, A_2 : J \to L(E)$ and $A_3 : J \to E$ are Bochner integrable, then the IVP*

$$x'' = Ax + A_1(t)x + A_2(t)x' + A_3(t), \quad x(0) = x_0,\ x'(0) = x_1$$

has for each $(x_0, x_1) \in E_2 \times E$ a unique weak solution x, which together with x', depend continuously on x_1. □

4. ON EXTREMAL MILD SOLUTIONS

In this section we shall consider the existence of extremal mild solutions of the IVP

$$x'' = Ax + g(t, x),\ x(0) = x_0,\ x'(0) = x_1, \tag{4.1}$$

between assumed upper and lower mild solutions, when E is an ordered Banach space with regular order cone and $g : J \times E \to E$.

Define a partial ordering in $C(J, E)$ by

$$x \leq y \text{ if and only if } x(t) \leq y(t) \text{ for each } t \in J.$$

If $\underline{x}, \bar{x} \in C(J,E)$, denote $[\underline{x}) = \{x \in C(J,E) \mid \underline{x} \leq x\}$, $(\bar{x}] = \{x \in C(J,E) \mid x \leq \bar{x}\}$ and $[\underline{x}, \bar{x}] = \{x \in C(J,E) \mid \underline{x} \leq x \leq \bar{x}\}$.

The following fixed point result is a basis to our considerations.

Lemma 4.1: *Let Y be a nonempty subset of $C(J,E)$, and let $G: Y \to Y$ be a nondecreasing mapping such that $G[Y]$ has a lower bound $\underline{x}$ in Y. If $x_* = supG[C]$ exists in Y, where C is the well-ordered chain in Y satisfying*

(C) $\underline{x} = minC$ and $\underline{x} < x \in C$ if and only if $x = supG\{y \in C \mid Gy < y\}$,

then $x_* = maxC = min\{y \in Y \mid Gy \leq y\}$, and x_* is the least fixed point of G. This holds in particular if the sequence $(Gy_n)_{n=0}^{\infty}$ converges uniformly on J to a function of Y whenever $(y_n)_{n=0}^{\infty}$ is a nondecreasing sequence in Y.

Proof: The first assertion is proved in [7, 8]. Because $G[C]$, with C given by (C), is also a well-ordered chain in Y, and because $C(J,E)$ is an ordered normed space with respect to pointwise ordering and the uniform norm, it is easy to see (cf. [8, 9]) that $supG[C]$ exist in Y if $G(y_n)_{n=0}^{\infty}$ converges uniformly on J to a function of Y whenever $(y_n)_{n=0}^{\infty}$ is a nondecreasing sequence in C. This implies the last assertion. □

We shall assume that E is ordered by a *regular order cone* K, i.e. all nondecreasing and order bounded sequences of K converge. This implies (cf. [12]) that K is also *normal*, i.e. there is $\gamma > 0$ such that

$$\| y \| \leq \gamma \| z \| \text{ whenever } y, z \in K \text{ and } y \leq z. \tag{4.2}$$

Assume now that condition $(C0)$ holds, and that

$(C1)$ $C(t)$ is order-preserving for all $t \in J$.

From $(C0)$ and $(C1)$ it follows by (2.1) that $S(t)$ is also order-preserving for each $t \in J$.

Given $x_0, x_1 \in E \times E$, we say that $x \in C(J,E)$ is a *lower mild solution* of the IVP (4.1) on J if

$$x(t) \leq C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g(s,x(s))ds, \; t \in J. \tag{4.3}$$

An upper mild solution of (4.1) is defined similarly, by reversing the inequality sign in (4.3). If equality holds in (4.3), we say that x is a *mild solution* of (4.1).

Compared with the notion of weak solution we now don't require the differentiability of the solution. In the case when $x_0 \in E_2$ and g is continuous it can be shown (cf. [13]) that (4.1) has the same weak and mild solutions.

Let us impose the following hypotheses on the mapping $g: J \times E \to E$.

($g2$) (1.1) has a lower mild solution $\underline{x}$ and an upper mild solution $\bar{x}$, such that $\underline{x} \leq \bar{x}$, and that the functions $g(\cdot, \underline{x}(\cdot))$ $g(\cdot, \bar{x}(\cdot))$ are Bochner integrable.

($g3$) $g(\cdot, x(\cdot))$ is strongly measurable whenever $x \in C(J,E)$.

($g4$) $g(t, \cdot)$ is nondecreasing for *a.a.* $t \in J$.

Theorem 4.1: *Let E be an ordered Banach space with regular order cone. If the hypotheses ($C0$), ($C1$) and ($g2$) – ($g4$) hold, then the IVP* (4.1) *has the extremal mild solutions between $\underline{x}$ and $\bar{x}$.*

Proof: By definitions $\underline{x}$, $\bar{x} \in C(J,E)$. If x belongs to the order interval $[\underline{x}, \bar{x}]$ of $C(J,E)$, it follows from ($g3$) and (2.1) that the mapping $s \mapsto S(t-s)g(s,x(s))$ is strongly measurable on $[0,t]$ for all $t \in J$. Conditions ($C1$) and ($g4$) imply that that for all $t \in J$ and for *a.a.* $s \in [0,t]$,

$$S(t-s)g(s,\underline{x}(s)) \leq S(t-s)g(s,x(s)) \leq S(t-s)g(s,\bar{x}(s)).$$

Applying this, the triangle inequality, (4.2) and (3.4), it follows that

$$\| S(t-s)g(s,x(s)) \| \leq MTN(s) \text{ for all } t \in J \text{ and for } a.a.\ s \in [0,t], \qquad (a)$$

where

$$N(t) = (1+\gamma)(\| g(s,\underline{x}(s)) \| + \| g(s,\bar{x}(s)) \|).$$

This and the hypotheses ($g2$) and ($g3$) imply that $s \mapsto S(t-s)g(s,x(s))$ is Bochner integrable on $[0,t]$ for each $t \in J$. Thus the equation

$$Gx(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g(s,x(s))ds, \quad t \in J, \qquad (4.4)$$

defines a mapping $Gx: J \to E$ for each $x \in [\underline{x}, \bar{x}]$. Moreover, if $0 \leq \bar{t} \leq t \leq T$ and $x \in [\underline{x}, \bar{x}]$, it follows from (2.1), (3.4), (4.4) and (a) that

$$\| Gx(t) - Gx(\bar{t}) \| \leq \| C(t)x_0 - C(\bar{t})x_0 \| + \| S(t)x_1 - S(\bar{t})x_1 \|$$

$$+ \int_0^{\bar{t}} \| S(t-s)g(s,x(s)) - S(\bar{t}-s)g(s,x(s)) \| ds + \int_{\bar{t}}^t \| S(t-s)g(s,x(s)) \| ds$$

$$\leq \| C(t)x_0 - C(\bar{t})x_0 \| + \| S(t)x_1 - S(\bar{t})x_1 \|$$

$$+ M \mid t - \bar{t} \mid \int_0^T N(s)ds + MT \int_{\bar{t}}^t N(s)ds.$$

Thus the family of functions Gx, $x \in [\underline{x}, \bar{x}]$, is equicontinuous. In view of conditions $(C1)$ and $(g4)$ it follows from (4.4) that $Gx \leq Gy$ whenever $x, y \in [\underline{x}, \bar{x}]$ and $x \leq y$. Moreover, it follows from (4.4) by the definition of mild upper and lower solutions of (4.1) that $\underline{x} \leq G\underline{x}$ and $G\bar{x} \leq \bar{x}$. Thus G is a nondecreasing mapping from $[\underline{x}, \bar{x}]$ to $[\underline{x}, \bar{x}]$.

Assume now that $(x_n)_{n=1}^{\infty}$ is a monotone sequence in $[\underline{x}, \bar{x},]$. From the monotonicity of G it follows that the sequence $(Gx_n)_{n=1}^{\infty}$ is also monotone. Since the order cone K of E is regular, it follows that

$$y(t) = \lim_{n \to \infty} Gx_n(t) \tag{c}$$

exists for all $t \in J$. Because the functions Gx_n are equicontinuous, the convergence in (c) is uniform on J. Obviously, $y \in [\underline{x}, \bar{x}]$.

The above proof shows that the mapping $G: [\underline{x}, \bar{x}] \to [\underline{x}, \bar{x}]$ satisfies the hypotheses of lemma 4.1, whence G has the least fixed point x_*. This and the definition (4.4) of G imply that x_* is a solution of the integral equation (4.3). Hence, x_* is a mild solution of the IVP (4.1).

If x is any mild solution of (4.1) and $\underline{x} \leq x \leq \bar{x}$, then $x \in [\underline{x}, \bar{x}]$, whence x is a fixed point of G. Since x_* is the least one, it follows that $x_* \leq x$. Thus x_* is the minimal solution of the IVP (4.1) in $[\underline{x}, \bar{x}]$.

The existence of the maximal mild solution of the IVP (1.1) is proved similarly, by using the dual result to lemma 4.1. □

Proposition 4.1: *If $g: J \times E \to E$ in theorem* 4.1 *is a Carathéodory function, then the sequence $(y_n)_{n=0}^{\infty}$ of the successive approximations*

$$y_{n+1}(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g(s, y_n(s))ds, n \in \mathbb{N}, t \in J \tag{4.5}$$

converges uniformly on $J = [t_0, T]$ to the minimal (resp. the maximal) mild solution of the IVP (4.1) *between $\underline{x}$ and $\bar{x}$, if $y_0 = \underline{x}$ (resp. $y_0 = \bar{x}$).*

Proof: Since g is a Carathéodory function, then $g(\cdot, x(\cdot))$ is strongly measurable on J for each $x \in [\underline{x}, \bar{x}]$. From the proof of theorem 4.1 it follows that the equation (4.4) defines a nondecreasing mapping $G:[\underline{x}, \bar{x}] \to [\underline{x}, \bar{x}]$. Choosing $y_0 = \underline{x}$ and denoting $y_n = G^n y_0$, we obtain a nonincreasing sequence $(y_n)_{n=0}^{\infty}$ in $[\underline{x}, \bar{x}]$, which equals to the sequence of the successive approximations defined in (4.5). The proof of theorem 4.1 implies that the sequence $(Gy_n)_{n=0}^{\infty}$ converges uniformly on J to a continuous function $x_* \in [\underline{x}, \bar{x}]$. From the definition of y_n it follows that

$$x_*(t) = \lim_{n\to\infty} Gy_n(t) = \lim_{n\to\infty} y_n(t), \; t \in J. \tag{a}$$

By using (4.4), (a), and the dominated convergence theorem, and noticing that $g(t, \cdot)$ is continuous for *a.a.* $t \in J$, it is easy to show that

$$Gx_*(t) = \lim_{n\to\infty} Gy_n(t), \; t \in J.$$

Thus x_* is a fixed point of G. Since $y_0 = \underline{x}$ is a lower bound of $G[\underline{x}, \bar{x}]$, and since G is nondecreasing, it is easy to see that x_* is the least fixed point of G, and hence, by the proof of theorem 4.1, the minimal mild solution of (1.1) in $[\underline{x}, \bar{x}]$. □

Lemma 4.2 *Let the hypotheses* $(C0)$ *and* $(C1)$ *hold, and assume that* $g: J \times E \to E$ *satisfies condition* $(g4)$ *and conditions*

$(g5)$ $g(\cdot, z)$ *is strongly measurable for each* $z \in E$ *and* $g(\cdot, 0) \in L^1(J, E)$.

$(g6)$ $\| g(t,x) - g(t,y) \| \leq q(t, \| x - y \|)$ *for all* $x, y \in E$ *and for a.a.* $t \in J$, *where* $q: J \times \mathbb{R}_+ \to \mathbb{R}_+$ *is a Carathéodory function,* $q(t, \cdot)$ *is nondecreasing for a.a.* $t \in J$, *the IVP*

$$u' = MTq(t,u), \quad u(0) = u_0, \tag{4.6}$$

with M *given by* (3.3), *has for each* $u_0 \in \mathbb{R}_+$ *an upper solution on* J, *and the zero-function is the only solution of* (4.6) *when* $u_0 = 0$.

Then the IVP (4.1) *has for each* $(x_0, x_1) \in E \times E$ *a unique mild solution* x *on* J *which depends continuously on* (x_0, x_1). *Moreover,*

a) $x \leq \bar{x}$ *for each upper mild solution* $\bar{x}$ *of* (4.1),

b) $\underline{x} \leq x$ *for each lower mild solution* $\underline{x}$ *of* (4.1).

Proof: Let $x_0, x_1 \in E$ be given. The existence and uniqueness of the mild solution $x = x(\cdot, x_0, x_1)$ of (4.1) can be proved as in theorem 3.1. Following the proof of theorem 3.2 it is easy to show that

$$\| x(t,x_0,x_1) - x(t,\bar{x}_0,\bar{x}_1) \| \leq u(t, M \| x_0 - \bar{x}_0 \| + MT \| x_1 - \bar{x}_1 \|), \; t \in J, \quad (a)$$

where $u = u(\cdot, u_0)$ denotes the minimal solution of the IVP (4.6). Moreover, it can be shown (cf. [8], proposition 2.1.9) that u is nondecreasing with respect to $u_0 \in \mathbb{R}_+$ and tends to 0 uniformly over $t \in J$ as $u_0 \to 0$. This and (a) imply that $x(\cdot, x_0, x_1)$ depends continuously on (x_0, x_1).

Let $\underline{x}$ be a lower mild solution of (4.1) on J. The hypotheses given for q in condition $(g6)$ imply (cf. the proof of theorem 3.1) that the relation

$$Qu(t) = \int_0^t MTq(s,u(s))ds, \quad t \in J \qquad (a)$$

defines a nondecreasing operator $Q: C(J,\mathbb{R}_+) \to C(J,\mathbb{R}_+)$, that the operator equation

$$u_0 + Qv = v$$

has for each $u_0 \in \mathbb{R}_+$ an upper solution v, and that the sequence $(Q^n v)_{n=0}^{\infty}$ converges uniformly on J to the fixed point of Q, i.e. to the 0-function. From (4.4), $(g6)$ and (a) it follows that

$$| Gy - G\bar{y} | \leq Q | y - \bar{y} |, \text{ for } y, \bar{y} \in C(J,E).$$

The above proof implies by lemma 3.1 that the sequence $(G^n \underline{x})_{n=0}^{\infty}$ converges uniformly on J to a unique fixed point of G. By definition (4.4) of G this fixed point is the mild solution x of the IVP (4.1). Since conditions $(C1)$ and $(g4)$ imply that G is nondecreasing, and since $\underline{x} \leq G\underline{x}$ by $(g3)$ and $(g4)$, the sequence $(G^n \underline{x})_{n=0}^{\infty}$ is nondecreasing. Thus $\underline{x} \leq lim_n G^n \underline{x} = x$. The proof of assertion $b)$ is similar. □

Proposition 4.2: *Let E be an ordered Banach space with regular order cone. Assume that $g: J \times E \to E$ satisfies conditions $(g3)$ and $(g4)$, and that*

$$g_1(t,x) \leq g(t,x) \leq g_2(t,x) \text{ for all } x \in E \text{ and for a.a. } t \in J, \qquad (4.7)$$

where $g_i: J \times E \to E$, $i = 1,2$ satisfy conditions $(g4)$, $(g5)$ and $(g6)$. Then the IVP (4.1) has for each choice of $x_0 \in E$ the extremal mild solutions.

Proof: Let $x_0, x_1 \in E$ be given. Lemma 4.2 implies that the integral equation

$$x(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g_i(t,x(s))ds \qquad (a)$$

has a unique mild solution $\underline{x}$ when $i=1$ and $\bar{x}$ when $i=2$. From (a) and (4.7) it follows that $\underline{x}$ is a lower solution of (a) with $i=2$. Thus $\underline{x} \leq \bar{x}$ by lemma 4.2. Applying, $(g4)$, the triangle inequality, (4.7) and (4.2), it follows that if $x \in [\underline{x}, \bar{x}]$, then

$$\| g(t,x(t)) \| \leq N(t) \text{ for } a.a.\ \ t \in J, \qquad (a)$$

where

$$N(t) = (1+\gamma)(\| g_1(t,\underline{x}(t)) \| + \| g_2(t,\bar{x}(t)) \|).$$

This and $(g3)$ imply that the functions $g(\cdot, \underline{x}(\cdot))$ and $g(\cdot, \bar{x}(\cdot))$ are Bochner integrable. Moreover,

$$\underline{x}(t) \leq C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)g(s,\underline{x}(s))ds \text{ for } a.a.\ \ t \in J,$$

whence $\underline{x}$ is a mild lower solution of (4.1). Similarly, it can be shown that $\bar{x}$ is a mild upper solution. Thus g satisfies condition $(g2)$. By theorem 4.1 the IVP (4.1) has the minimal mild solution x_* and the maximal mild solution x^* between $\underline{x}$ and $\bar{x}$.

If x is a mild solution of (4.1) on J, it follows from (4.3), (4.7) and (a) that x is a mild upper solution of (a) for $i=1$ and a mild lower solution of (a) for $i=2$, whence the results $a)$ and $b)$ of lemma 4.2 imply that $\underline{x} \leq x \leq \bar{x}$. Hence all the mild solutions of (4.1) on J lie between $\underline{x}$ and $\bar{x}$, whence x_* is the least and x^* the greatest of all the mild solutions of (4.1) on J. □

5. ON THE EXISTENCE OF MINIMAL OR MAXIMAL MILD SOLUTION

In the case when the order of E is fully regular, condition $(g2)$ can be replaced by the existence of either upper or lower mild solution of the IVP (4.1), and condition

$(g7)$ $\| g(t,x) \| \leq h(t, \| x \|)$ for all $x \in E$ and for $a.a.\ \ t \in J$, where $h: J \times \mathbb{R}_+ \to \mathbb{R}_+$ is a standard function (cf. [15]), $h(t,\cdot)$ is nondecreasing for $a.a.\ \ t \in J$, and the IVP

$$u' = MTh(t,u),\ \ u(t_0) = u_0, \tag{5.1}$$

with M given by (3.3), has for each $u_0 \in \mathbb{R}_+$ an upper solution.

Theorem 5.1: *Given an ordered Banach space E with fully regular order cone, assume that $g: J \times E \to E$ satisfies conditions $(g3)$, $(g4)$ and $(g7)$ and that $(C0)$ and $(C1)$ hold.*

a) If (4.1) *has a lower mild solution $\underline{x}$, then* (4.1) *has the minimal mild solution in $[\underline{x})$.*

b) If (4.1) *has an upper mild solution $\bar{x}$, then* (4.1) *has the maximal mild solution in $(\bar{x}]$.*

Proof: *a)* Assume that $\underline{x}$ is a lower mild solution of the IVP (4.1). Denote $u_0 = M \| x_0 \| + MT \| x_1 \| + \| \underline{x} \|_0$, and let $\bar{u}$ be an upper solution of the IVP (5.1). Since the zero function is a lower solution of (5.1), it follows that (5.1) has the minimal solution u_* on J between 0 and $\bar{u}$, and that

$$u_*(t) = u_0 + \int_0^t MTh(s, u_*(s))ds, \quad t \in J. \tag{a}$$

Denote $Y = \{y \in C(J,E) \mid \underline{x} \leq y \text{ and } |\, y \,| \leq u_*\}$. Y is nonempty because $\underline{x} \in Y$. If $y \in Y$, it is easy to see that the equation (4.4) defines a mapping $Gy: J \to E$. From (3.3), (3.4) and (g7) it follows that

$$\| Gy(t) \| \leq \| C(t)x_0 \| + \| S(t)x_1 \| + \int_0^t \| S(t-s)g(s,y(s)) \| ds$$

$$\leq M \| x_0 \| + MT \| x_1 \| + MT \int_0^t \| g(s,y(s)) \| ds$$

$$\leq u_0 + \int_0^t MTh(s, \| y(s) \|)ds \leq u_0 + \int_0^t MTh(s,u_*(s))ds = u_*(t)$$

for each $t \in J$. The definition of a mild lower solution of (4.1) implies that $\underline{x} \leq G\underline{x}$. These properties, together with conditions $(g4)$ and $(C1)$ imply that the equation (4.4) defines a nondecreasing mapping $G: Y \to Y$. Moreover, if $y \in Y$ and $0 \leq \bar{t} \leq t \leq T$, then

$$\| Gy(t) - Gy(\bar{t}\,) \| \leq \| C(t)x_0 - C(\bar{t}\,)x_0 \| + \| S(t)x_1 - S(\bar{t}\,)x_1 \|$$
$$+ M \mid t - \bar{t} \mid \int_0^T h(s, u_*(s))ds + MT \int_{\bar{t}}^t h(s, u_*(s))ds. \qquad (b)$$

As in the proof of theorem 4.1 it can be shown that if $(x_n)_{n=0}^{\infty}$ is a nondecreasing sequence in Y, then $(Gx_n)_{n=0}^{\infty}$ converges uniformly on J to a function $y \in Y$. Since $\underline{x}$ is the least element of Y, then G has by lemma 4.1 the least fixed point x_* in Y. By definition, x_* is a mild solution of the IVP (1.1) in Y.

Assume now that $x: J \to E$ is a mild solution of (4.1). By choosing $u_0 = M \| x_0 \| + MT \| x_1 \| + \| \underline{x} \|_0 + \| x \|_0$ in the above proof we may assume that $x \in Y$. By the definition (4.4) of G, $x = Gx$. To prove that $x_* \leq x$, let C be the well-ordered chain satisfying condition (C) of lemma 4.1. To show that $y \leq x$ for each $y \in C$ make a counter-hypothesis: there is the least element $y \in C$ such that $y \not\leq x$. Because $\underline{x} \leq x$, then $\underline{x} < y$. If $z \in C$ and $z < y$ then $z \leq x$. Since G is nondecreasing, then $Gz \leq Gx = x$. Thus $y = supG\{z \in C \mid z < y\} \leq x$, contradicting with the choice of y. Consequently $x_* = maxC \leq x$, which proves that x_* is the least of all the mild solutions of the IVP (4.1) in $[\underline{x}\,)$.

The proof of the case b) is similar. □

Corollary 5.1: *Let the hypotheses of theorem* 5.1 *hold, and let* g *be a Carathéodory function.*

a) If the IVP (4.1) *has a lower mild solution* $\underline{x}$, *then the sequence* $(y_n)_{n=0}^{\infty}$, *defined by* (4.5) *with* $y_0 = \underline{x}$ *is nondecreasing and converges on* J *uniformly to the minimal mild solution of the IVP* (4.1) *in* $[\underline{x}\,)$.

b) If the IVP (4.1) *has an upper mild solution* $\bar{x}$, *then the sequence* $(y_n)_{n=0}^{\infty}$, *defined by* (4.5) *with* $y_0 = \bar{x}$ *is nonincreasing and converges on* J *uniformly to the maximal mild solution of the IVP* (4.1) *in* $(\bar{x}\,]$.

6. DEPENDENCE ON THE DATA

Next we shall consider the dependence of the extremal mild solutions of the IVP (4.1) on the initial values x_0 and x_1 and on the function g. Let $\Gamma = \{(t,s) \mid 0 \leq s \leq t \leq T\}$ and $T(t,s): \Gamma \to L(E)$ satisfying

(T_0) $T(t,t) = I$ (I is the identity operator in E);

(T_1) $T(t,s)T(s,r) = T(t,r)$, for $0 \le r \le s \le t \le T$.

Proposition 6.1: *Let E be an ordered Banach space with regular order cone, let $T:\Gamma \to L(E)$ satisfy conditions $(T0)$ and $(T1)$, and let $g, \widehat{g}: J \times E \to E$ satisfy the hypotheses given for g in proposition 4.2. If $x_j, z_j \in E$, $x_j \le z_j$, $j = 0,1$, and if*

$$g(t,x) \le \widehat{g}(t,x) \text{ for all } x \in E \text{ and for a.a. } t \in J,$$

then all the mild solutions of x the IVP (4.1) and the IVP

$$x' = A(t)x + \widehat{g}(t,x),\ x(0) = z_0, \tag{6.1}$$

satisfy $x_* \le x \le z^*$, where x_* is the minimal mild solution of (4.1) and z^* is the maximal mild solution of (6.1), and $A(t)x = \lim_{h \to 0+} \frac{T(t+h,t)x - x}{h}$.

Proof: By the proof of proposition 4.2 there exist $\underline{x}$, $\bar{x} \in C(J,E)$ such that all the mild solutions of (4.1) and (6.1) belong to the order interval $[\underline{x}, \bar{x}]$. Let x be any mild solution of (6.1). The hypotheses imply that x is a mild upper solution of (4.1). Because of this and (4.4) we have $Gx \le x$, whence $x_* = min\{y \in [\underline{x}, \bar{x}] \mid Gy \le y\} \le x$.

The above proof and theorem 4.1 imply that $x_* \le x$ whenever x is a mild solution of (4.1) or (6.1). The dual reasoning shows that $x \le z^*$ for any solution x of (4.1) or (6.1). □

As special case of the above result we obtain

Corollary 6.1: *If the hypotheses of proposition 4.2 hold, then the extremal mild solutions of the IVP (4.1) are nondecreasing with respect to x_0, x_1 and g.*

Similarly, it can be shown

Corollary 6.2: *If the hypotheses of theorem 5.1 a) hold, then the minimal mild solution of the IVP (4.1) in $[\underline{x})$ is nondecreasing with respect to x_0, x_1 and g.*

Corollary 6.3: *Let E be an ordered Banach space with regular order cone. Assume that $g: J \times E \to E$ satisfies conditions $(g3)$ and $(g4)$, that $T: J \to L(E)$ satisfies $(T0)$ and $(T1)$, and that for all $x \in E$ and for a.a. $t \in J$*

$$C_1(t) \le g(t,x) \le C_2(t),$$

where the functions C_1, $C_2: J \to E$ are Bochner integrable. Then the IVP (4.1) *has for each choice of $x_0 \in E$ the extremal mild solutions, both of which are nondecreasing in x_0, x_1 and g.*

Proof: It is easy to see that the hypotheses of proposition 4.2 hold when $g_i(t,x) = C_i(t)$, $i = 1,2$, whence the assertions follow from proposition 4.2 and corollary 6.1. □

7. APPLICATION TO A PARTIAL DIFFERENTIAL EQUATION

Consider the IVP of the second order hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial \xi^2} + f(\xi, t, u), \quad u(\xi, 0) = x_0(\xi), \quad \frac{\partial}{\partial t} u(\xi, 0) = x_1(\xi), \tag{7.1}$$

where k is a given positive constant. If $J = [0,T]$, $T > 0$ and $f: \mathbb{R} \times J \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable, it is easy to see that a twice continuously differentiable function $u: \mathbb{R} \times J \to \mathbb{R}$ is a solution of the IVP (7.1) if and only if it satisfies the integral equation

$$u(\xi,t) = \tfrac{1}{2}(x_0(\xi + kt) + x_0(\xi - kt)) + \frac{1}{2k} \int_{\xi - kt}^{\xi + kt} x_1(z)\,dz$$

$$+ \frac{1}{2k} \int_0^t \int_{\xi - kt}^{\xi + kt} f(z, s, u(z,s))\,dz\,ds. \tag{7.2}$$

On the other hand, (7.2) may have solutions also when f is not even continuous. Given $p \in [1, \infty)$ we shall now study existence of solutions of (7.2) in the set $\mathfrak{U}_p$ of those measurable functions $u: \mathbb{R} \times J \to \mathbb{R}$ for which $u(\cdot, t) \in L^p(\mathbb{R})$ for each $t \in J$ and $\lim_{t \to t_0} \int_{-\infty}^{\infty} |(u(\xi,t) - u(\xi,t_0)|^p d\xi = 0$ for each $t_0 \in J$. We say that $u \in \mathfrak{U}_p$ is a *mild solution* of the IVP (7.1) if (7.2) holds for all $t \in J$ and for *a.a.* $\xi \in \mathbb{R}$.

Choose $E = L^p(\mathbb{R})$, and define for each $x \in E$

$$C(t)x(\xi) = \tfrac{1}{2}(x(\xi + kt) + x(\xi - kt)), t \in \mathbb{R}, \ \xi \in \mathbb{R}. \tag{7.3}$$

To show that (7.3) defines a strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$ of operators in $L(L^p(\mathbb{R}))$, note first that properties $C(0) = I$ and $C(t+s) + C(t-s) = 2C(t)C(s)$ are trivially verified. The equation $T(t)x(\xi) = x(\xi + kt)$ defines a strongly continuous semigroup $T: \mathbb{R} \to L(L^p(\mathbb{R}))$ (cf.

[2, 11, 19]) and $C(t)=\frac{1}{2}(T(t)+T(-t))$, which implies strong continuity of the mapping $t\mapsto C(t)x$. The corresponding sine family, defined in (2.1), can be given by

$$S(t)x(\xi)=\frac{1}{2k}\int_{\xi-kt}^{\xi+kt}x(z)dz. \tag{7.4}$$

Assume first that $f:\mathbb{R}\times J\times\mathbb{R}\to\mathbb{R}$ has the following properties:

- $(f0)$ $f(\cdot,\cdot,z)$ is measurable on $\mathbb{R}\times J$ for each $z\in\mathbb{R}$ and $t\mapsto f(\cdot,t,0)$ is a Bochner integrable mapping from J to $L^p(\mathbb{R})$.
- $(f1)$ There is $q\in L^p_+(\mathbb{R})$ such that $|f(\xi,t,y)-f(\xi,t,z)|\leq q(t)|y-z|$ for all $t\in J$, y, $z\in\mathbb{R}$ and *a.a.* $\xi\in\mathbb{R}$.

If $x_0\in L^p(\mathbb{R})$ and $u\in\mathfrak{U}_p$, it follows that the right hand side of (7.2) is defined. In view of (7.3) and (7.4) the equation (7.2) can be rewritten as (cf. [1])

$$u(\xi,t)=C(t)x_0(\xi)+S(t)x_1(\xi)+\int_0^t S(t-s)f(\xi,s,u(\xi,s))ds. \tag{7.5}$$

If $u\in\mathfrak{U}_p$ is a mild solution of (7.1), i.e. a solution of (7.5), then denoting

$$x(t)=u(\cdot,t),\ \text{and}\ g(t,y)=f(\cdot,t,y(\cdot)),\quad t\in J,\ y\in L^p(\mathbb{R}), \tag{7.6}$$

then $x\in C(J,L^p(\mathbb{R}))$, and x is a solution of the integral equation

$$x(t)=C(t)x_0+S(t)x_1\int_0^t S(t-s)g(s,x(s))ds,\quad t\in J. \tag{7.7}$$

Conversely, if $x\in C(J,L^p(\mathbb{R}))$ is a solution of (7.7), there is $u\in\mathfrak{U}_p$ such that $x(t)=u(\cdot,t)$ for all $t\in J$, and u is a solution of (7.5), and thus a mild solution of (7.1) (cf. [11]).

Conditions $(f0)$ and $(f1)$ imply that (7.6) defines a mapping $g:J\times L^p(\mathbb{R})\to L^p(\mathbb{R})$ which satisfies conditions $(g5)$ and $(g6)$. Thus the hypotheses of lemma 4.2 hold, whence the integral equation (7.7) has for each choice of x_0, $x_1\in L^p(\mathbb{R})$ a unique solution x in $C(J,L^p(\mathbb{R}))$, which depends continuously on x_0 and on x_1. According to the above stated correspondence between solutions of (7.7) in $C(J,L^p(\mathbb{R}))$ and mild solutions of (7.1) and identification of *a.e.* equal functions we then obtain

Proposition 7.1: *If conditions* $(f0)$ *and* $(f1)$ *hold, then the IVP* (7.1)

has for each $x_0 \in L^p(\mathbb{R})$ *a unique mild solution* $u = u(\cdot, \cdot; x_0, x_1)$ *in* $\mathcal{U}_p$, *which depends continuously on* x_0 *and on* x_1 *in the sense that* $\| u(\cdot, t; \bar{x}_0, \bar{x}_1) - u(\cdot, t; x_0, x_1) \|_p \to 0$, *uniformly over* $t \in J$ *as* $\bar{x}_0 \to x_0$ *and* $\bar{x}_1 \to x_1$ *in* $L^p(\mathbb{R})$.

Assume next that f has the following properties.

$(f2)$ f is a standard function from $(\mathbb{R} \times J) \times \mathbb{R}$ to $\mathbb{R}$ in the sense defined in [15].

$(f3)$ $f(\xi, t, \cdot)$ is nondecreasing for all $(\xi, t) \in J \times \mathbb{R}$.

$(f4)$ There exist $f_1, f_2 : \mathbb{R} \times J \times \mathbb{R} \to \mathbb{R}$ which satisfy conditions $(f0)$, $(f1)$ and $(f3)$ such that $f_1(\xi, t, v) \le f(\xi, t, v) \le f_2(\xi, t, v)$ for all $(\xi, t, v) \in \mathbb{R} \times J \times \mathbb{R}$.

Assume that $L^p(\mathbb{R})$ is partially ordered by the cone $L^p_+(\mathbb{R})$ of *a.e.*. nonnegative-valued elements of $L^p(\mathbb{R})$. This and (7.3) imply that $C(t)$ is order-preserving for each $t \in \mathbb{R}$. In view of conditions $(f2) - (f4)$ and definition (7.6) of g it is easy to see that conditions $(g3)$ and $(g4)$ hold, and that $g_1(t, y) \le g(t, y) \le g_2(t, y)$ for all $t \in J$ and $y \in L^p(\mathbb{R})$, where g_i, $i = 1, 2$, is defined by (7.6) with $f = f_i$. Noticing also that the order cone $L^p_+(\mathbb{R})$ of $L^p(\mathbb{R})$ is regular, then all the hypotheses of proposition 4.2 hold, whence (7.7) has for each $x_0 \in L^p(\mathbb{R})$ extremal solutions x_* and x^*, and they are nondecreasing in x_0 and in x_1. According to the above discussion there exist $u_*, u^* \in \mathcal{U}_p$ such that $x_*(t) = u_*(\cdot, t)$ and $x^*(t) = u^*(\cdot, t)$ for all $t \in J$ so that u_* and u^* are mild solutions of the IVP (7.1). Moreover, defining a partial ordering in $\mathcal{U}_p$ by

$$u \le v \text{ if } u(\xi, t) \le v(\xi, t) \text{ for all } t \in J \text{ and for } a.a.\ \xi \in \mathbb{R}, \tag{7.8}$$

it is easy to see that if $x, y \in C(J, L^p(\mathbb{R}))$ and if u, v are their representatives in $\mathcal{U}_p$, respectively, then $x \le y$ if and only if $u \le v$. Thus u_* and u^* are the extremal mild solutions of (7.1) and they are nondecreasing in x_0 and in x_1. Thus we have proved the following result.

Proposition 7.2: *If conditions* $(f2) - (f4)$ *hold, then the IVP* (7.1) *has for each choice of* x_0, $x_1 \in L^p(\mathbb{R})$ *the extremal mild solutions in* $\mathcal{U}_p$, *and they are nondecreasing in* x_0 *and in* x_1.

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