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THE PROBABILISTIC APPROACH TO THE ANALYSIS OF THE LIMITING BEHAVIOR OF AN INTEGRO-DIFFERENTIAL EQUATION DEPENDING ON A SMALL PARAMETER, AND ITS APPLICATION TO STOCHASTIC PROCESSES[1]

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ABSTRACT

Using connection between stochastic differential equation with Poisson measure term and its Kolmogorov's equation, we investigate the limiting behavior of the Cauchy problem solution of the integro-differential equation with coefficients depending on a small parameter. We also study the dependence of the limiting equation on the order of the parameter.

Key words: Stochastic process, Kolmogorov's averaging, integro-differential equation, Cauchy problem, limiting behavior, small parameters, white and Poisson noise.

AMS (MOS) subject classifications: 60H15, 60H20, 35R60.

It is well known that investigation of a nonlinear oscillating systems with a small stochastic white noise at the input, can be accomplished applying the averaging method for Kolmogorov's parabolic equation with coefficients depending on a small parameter [1]. If both white and Poisson types of noise are present, then the corresponding Kolmogorov's equation is integro-differential [2],

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and we shall extend here the averaging principle to such equations.

Let us study behavior, as $\epsilon \to 0$, of the following equation

$$\frac{\partial}{\partial t}U(t,x) + \epsilon^{k_1}(f(t,x), \nabla U(t,x)) + \frac{\epsilon^{k_2}}{2}Tr(g(t,x)g^*(t,x)\nabla^2 U(t,x)) \tag{1}$$

$$+ \int_{R^d} [U(t,x+\epsilon^{k_3}q(t,x,y)) - U(t,x) - \epsilon^{k_3}(q(t,x,y), \nabla U(t,x))]\Pi(dy) = 0,$$

$$(t,x) \in [0,T) \times R^d,$$

where $\epsilon > 0$ is a small parameter and k_1, k_2, k_3, are some positive numbers, and $\nabla U(t,x) = \left\{\frac{\partial U(t,x)}{\partial x_i}, i = 1,\ldots,d\right\}$, $\nabla^2 U(t,x) = \left\{\frac{\partial^2 U(t,x)}{\partial x_i \partial x_j}, i,j = 1,\ldots,d\right\}$. Here Π is a finite measure on Borel sets in R^d, $f(t,x)$, $q(t,x,y)$ are d-dimensional vectors, and $g(t,x)$ is a $d \times d$ square matrix.

Lemma: *If*

$$\lim_{s\to\infty} \frac{1}{s} \int_A^{s+A} b(t,x)dt = \bar{b}(x)$$

uniformly with respect to A for each x, the function $b(x)$ is continuous, and $b(t,x)$ is continuous in x uniformly with respect to (t,x) in arbitrary compact $|x| \leq C$, and stochastic process $\xi(t)$ is continuous, then

$$\lim_{\epsilon \to 0} \int_0^t b\left(\frac{\tau}{\epsilon}, \xi(\tau)\right)d\tau = \int_0^t \bar{b}(\xi(\tau))d\tau.$$

The proof is similar to that in [2].

Now, replacing t with t/ϵ^k in (1), where $k = min(k_1, k_2, k_3)$, and denoting $V_\epsilon(t,x) = U(t/\epsilon^k, x)$, we can derive the following equation:

$$\frac{\partial}{\partial t}V_\epsilon(t,x) + \epsilon^{k_1 - k}(f(t/\epsilon^k, x), \nabla V_\epsilon(t,x)) + \frac{\epsilon^{k_2 - k}}{2}Tr(g(t/\epsilon^k, x)g^*(t/\epsilon^k, x)\nabla^2 V_\epsilon(t,x))$$

$$+ \frac{1}{\epsilon^k} \int_{R^d} [V_\epsilon(t, x + \epsilon^{k_3}q(t/\epsilon^k, x, y)) - V_\epsilon(t,x) - \epsilon^{k_3}(q(t/\epsilon^k, x, y) \nabla V_\epsilon(t,x))]\Pi(dy) = 0,$$

$$(t,x) \in [0,T) \times R^d.$$

Theorem: *Let the following conditions hold:*

1) *the functions $f(t,x), g(t,x), q(t,x,y)$ are continuous in (t,x), bounded and twice continuously differentiable with respect to x, with derivatives also bounded;*

2) *uniformly with respect to A for each $x \in R^d$, $y \in R^d$ there exists the following three limits*

$$\lim_{s\to\infty} \frac{1}{s}\int_A^{s+A} f(t,x)dt = \bar{f}(x), \quad \lim_{s\to\infty} \frac{1}{s}\int_A^{s+A} g(t,x)g^*(t,x)dt = \bar{G}(x),$$

and

$$\lim_{s\to\infty} \frac{1}{s}\int_A^{s+A} q(t,x,y)q^*(t,x,y)dt = \bar{Q}(x,y).$$

3) *The functions $\bar{f}(x)$, $\bar{G}(x)$, $\bar{Q}(x,y)$ satisfy the Lipschitz condition in x, and the matrix*

$$\bar{B}(x) = \bar{G}(x) + \int_{R^d} Q(x,y)\Pi(dy)$$

is uniformly parabolic.

Then,

a) *if $k_1 = k_2 = 2k_3$ and $V_\epsilon(t,x)$ satisfies (2) and the "Cauchy" condition*

$$\lim_{t\uparrow T} V_\epsilon(t,x) = F(x), \quad F(x) \in C_b^2(R^d), \tag{3}$$

then $\lim_{\epsilon\to 0} V_\epsilon(t,x) = \bar{V}(t,x)$, where $\bar{V}(t,x)$ is a solution of the problem:

$$\frac{\partial}{\partial t}\bar{V}(t,x) + (\bar{f}(x), \nabla \bar{V}(t,x)) + \frac{1}{2}Tr(\bar{B}(x)\nabla^2\bar{V}(t,x)) = 0, \tag{4}$$

$$\lim_{t\uparrow T}\bar{V}(t,x) = F(x). \tag{5}$$

b) *If $k < k_1$, then V satisfies (4)-(5) but in this case there is no term containing $\bar{f}(x)$ in (4); Similarly, if $k < k_2$, then $\bar{B}(x)$ does not depend on $\bar{G}(x)$; and if $k < 2k_3$, then $\bar{B}(x)$ does not contain the term*

$$\int_{R^d} \bar{Q}(x,y)\Pi(dy).$$

Proof: Applying the results of [2-3] to the conditions of the theorem, it follows that the solution of the problem (2)-(3) exists for each ϵ, is unique and can be represented in the form

$$V_\epsilon(t,x) = E[F(\xi_\epsilon(t,x,T))],$$

where $\xi_\epsilon(t,x,T)$ is the solution of the stochastic equation

$$\xi_\epsilon(t,x,s) = x + \epsilon^{k_1 - k}\int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))d\tau$$
$$+ \epsilon^{\frac{1}{2}(k_2 - k)}\int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))dw(\tau)$$
$$+ \epsilon^{k_3}\int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau), y)\widetilde{\nu}\,(d\tau, dy),$$

where $w(t)$ is a d-dimensional Wiener process, $\nu(t,A)$ is a Poisson measure independent of w, A is a Borel set of R^d, and

$$\widetilde{\nu}\,(t,A) = \nu(t/\epsilon^k, A) - t\Pi(A)/\epsilon^k; \quad E\nu(t,A) = t\Pi(A).$$

Let

$$\zeta_\epsilon(t,x,s) = \epsilon^{\frac{1}{2}(k_2 - k)}\int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))dw(\tau)$$
$$+ \epsilon^{k_3}\int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau), y)\widetilde{\nu}_\epsilon(d\tau, dy).$$

Then we can obtain the following estimates:

$$E\,|\,\xi_\epsilon(t,x,s)\,|^2 \le C[x^2 + (\epsilon^{2(k_1 - k)} + \epsilon^{k_2 - k} + \epsilon^{2k_3 - k})\,|\,s - t\,|\,],$$
$$E\,|\,\zeta_\epsilon(t,x,s)\,|^2 \le C(\epsilon^{k_2 - k} + \epsilon^{2k_3 - k})\,|\,s - t\,|\,,$$

$$E\,|\,\xi_\epsilon(t,x,s_2) - \xi_\epsilon(t,x,s_1)\,|^2 \le C[\epsilon^{2(k_1 - k)}\,|\,s_2 - s_1\,|^2 + (\epsilon^{k_2 - k} + \epsilon^{2k_3 - k})\,|\,s_2 - s_1\,|\,],$$
$$E\,|\,\zeta_\epsilon(t,x,s_2) - \zeta_\epsilon(t,x,s_1)\,|^2 \le C(\epsilon^{k_2 - k} + \epsilon^{2k_3 - k})\,|\,s_2 - s_1\,|\,.$$

From these estimates we infer that the family of processes $(\xi_\epsilon(t,x,s), \zeta_\epsilon(t,x,s))$ satisfies the Skorokhod's compactness conditions [4]. Therefore, for any sequence $\epsilon \to 0$ there exists a subsequence $\epsilon_m \to 0$, $m = 1,2,\ldots$, and processes $\overline{\xi}\,(t,x,s)$, $\overline{\zeta}\,(t,x,s)$ such that $\xi_{\epsilon_m}(t,x,s) \to \xi(t,x,s)$, $\zeta_{\epsilon_m}(t,x,s) \to \zeta(t,x,s)$ in probability as $\epsilon_m \to 0$. From (6) we can also find that

$$\xi_{\epsilon_m}(t,x,s) = x + \epsilon_m^{k_1 - k}\int_t^s f(\tau/\epsilon_m^k, \xi_{\epsilon_m}(t,x,\tau))d\tau + \zeta_{\epsilon_m}(t,x,s). \tag{7}$$

Then, for any fixed $(t,x) \in [0,T]$ we have:

$$E \mid \xi_\epsilon(t,x,s_2) - \xi_\epsilon(t,x,s_1) \mid^4 \leq C[\epsilon^{4(k_1-k)} \mid s_2 - s_1 \mid^4 + E \mid \zeta_\epsilon(t,x,s_2) - \zeta_\epsilon(t,x,s_1) \mid^4],$$

$$E \mid \zeta_\epsilon(t,x,s_2) - \zeta_\epsilon(t,x,s_1) \mid^4 \leq C[(\epsilon^{2(k_2-k)} + \epsilon^{2(2k_3-k)}) \mid s_2 - s_1 \mid^2 + \epsilon^{4k_3 - 3k/2} \mid s_2 - s_1 \mid^{3/2} + \epsilon^{4k_3 - k} \mid s_2 - s_1 \mid].$$

Therefore,

$$E \mid \bar{\xi}(t,x,s_2) - \bar{\xi}(t,x,s_1) \mid^4 \leq C[\mid s_2 - s_1 \mid^4 + \mid s_2 - s_1 \mid^2],$$

$$E \mid \bar{\zeta}(t,x,s_2) - \bar{\zeta}(t,x,s_1) \mid^4 \leq C \mid s_2 - s_1 \mid^2,$$

and the processes $\bar{\xi}(t,x,s)$, $\bar{\zeta}(t,x,s)$ satisfy the Kolmogorov's continuity condition on s [5].

a) Let us consider the case $k_1 = k_2 = 2k_3$. Then from (7) we obtain:

$$\xi_\epsilon(t,x,s) = x + \int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))d\tau + \zeta_\epsilon(t,x,s). \tag{8}$$

From this point we shall omit the subindex m in ϵ_m for simplicity. Then for each fixed $(t,x) \in [0,T]$ the process

$$\zeta_\epsilon(t,x,s) = \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))dw(\tau) + \epsilon^{k/2} \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau), y)\widetilde{\nu}(d\tau, dy)$$

is a vector-valued martingale with matrix characteristic

$$\langle \zeta_\epsilon(t,x,s), \zeta_\epsilon(t,x,s) \rangle = \int_t^s g(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))g^*(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))d\tau + \int_t^s \int_{R^d} q(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau), y)q^*(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau), y)\Pi(dy)d\tau.$$

Using the above lemma, it is easy to show that

$$P - \lim_{\epsilon \to 0} \int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))d\tau = \int_t^s \bar{f}(\bar{\zeta}(t,x,\tau))d\tau, \tag{9}$$

and

$$P - \lim_{\epsilon \to 0} \langle \zeta_\epsilon(t,x,s), \zeta_\epsilon(t,x,s) \rangle = \int_t^s \bar{B}(\bar{\zeta}(t,x,\tau))d\tau. \tag{10}$$

Hence, from (8), (9), and (10) we obtain a continuous square integrable vector-valued martingale

$$\bar{\zeta}(t,x,s) = x + \int_t^s \bar{f}(\bar{\zeta}(t,x,\tau))d\tau + \bar{\zeta}(t,x,s),$$

with matrix characteristic

$$\langle\bar{\zeta}(t,x,s),\bar{\zeta}(t,x,s)\rangle = \int_t^s \bar{B}(\bar{\zeta}(t,x,\tau))d\tau.$$

It follows from [6] that there exists a d-dimensional Wiener process $w(t)$ such that

$$\bar{\zeta}(t,x,s) = \int_t^s \bar{\sigma}(\bar{\zeta}(t,x,\tau))d\bar{w}(\tau),$$

where

$$\bar{\sigma}(x)\bar{\sigma}^*(x) = \bar{B}(x).$$

Consequently, the process $\bar{\xi}(t,x,s)$ satisfies the equation which, according [2], has a unique solution:

$$\bar{\xi}(t,x,s) = x + \int_t^s \bar{f}(\bar{\xi}(t,x,\tau))d\tau + \int_t^s \bar{\sigma}(\bar{\xi}(t,x,\tau))d\bar{w}(\tau). \qquad (11)$$

The matrix $\bar{B}(x)$ is positive definite for all $x \in R^d$, satisfies Lipschitz conditions, and therefore matrix $\bar{\sigma}(x)$ satisfies Lipschitz condition as well. Then, using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{\epsilon_m \to 0} V_{\epsilon_m}(t,x) = \bar{V}(t,x) = E[F(\bar{\xi}(t,x,T))]$$

for any sequence $\epsilon_m \to 0$. But as it follows from [7] the function $\bar{V}(t,x)$ is a unique solution of the problem (4)-(5), which completes the proof of the part $a)$ of the theorem.

$b)$ When $k < k_1$, the boundedness of $f(t,x)$ implies that

$$E\,|\int_t^s f(\tau/\epsilon^k, \xi_\epsilon(t,x,\tau))d\tau\,| \le C$$

and therefore the second term in the right side of (6) converges to 0 with $\epsilon \to 0$ in probability. The matrix characteristic of the martingale $\zeta_\epsilon(t,x,s)$ in (7) has the form

$$\langle\zeta_\epsilon,\zeta_\epsilon\rangle = \epsilon^{k_2 - k}\int_t^s g(\tau/\epsilon^k,\xi_\epsilon(t,x,\tau))g^*(\tau/\epsilon^k,\xi_\epsilon(t,x,\tau))d\tau \qquad (12)$$

$$+\,\epsilon^{2k_3 - k}\int_t^s \int_{R^d} q(\tau/\epsilon^k,\xi_\epsilon(t,x,\tau),y)q^*(\tau/\epsilon^k,\xi_\epsilon(t,x,\tau),y)\Pi(dy)d\tau.$$

From the boundedness of g, q, similarly to the inference made above, we obtain that either first or second term in the right side of (12) converges to 0 (respectively to the $k < k_2$ or $k < 2k_3$ case) as $\epsilon \to 0$, which allows to complete the proof of the theorem as in part a).

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