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OSCILLATORY PROPERTIES AND ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF A CLASS OF OPERATOR-DIFFERENTIAL EQUATIONS[1]

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ABSTRACT

In the present paper an operator-differential equation is investigated. Sufficient conditions for the presence of Kneser's properties are found.

Key words: Oscillatory properties, asymptotic behavior, operators-differential equations.

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1. INTRODUCTION

In the present paper sufficient conditions are obtained for the presence of Kneser's properties for the operator-differential equation considered. Conditions are also found which guarantee the existence of nonoscillating solutions, and some of their asymptotic properties are investigated. Sufficient conditions for finding the number of the zeros of a given solution of this equation in a finite closed interval are given.

Analogous results for ordinary differential equations are obtained in [1]. The consideration of an operator-differential equation allows us by means of a single approach to investigate the properties of the solutions of a number of little investigated classes of differential equations.

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2. PRELIMINARY NOTES

Consider the operator-differential equation

$$x^{(n)}(t) + p(t) \cdot (Ax)(t) = 0 \tag{1}$$

where $n \geq 1$ is an integer, A is an operator with certain properties, and p is a nonnegative, locally integrable function in $\mathbb{R}_+ = [0, \infty)$.

Introduce the following notation:

$C([a, b]; \mathbb{R})$ – the set of all continuous functions u: $[a, b] \to \mathbb{R}$.

$AC([a, b]; \mathbb{R})$ – the set of all absolutely continuous functions u: $[a, b] \to \mathbb{R}$.

$AC^k(\mathbb{R}_+, \mathbb{R})$ – the set of all functions u: $\mathbb{R}_+ \to \mathbb{R}$ which are locally absolutely continuous, together with their derivatives up to order k inclusive.

$L(I, \mathbb{R})$, $I \subset \mathbb{R}$ – the set of all functions u: $I \to \mathbb{R}$ which are Lebesgue integrable.

$L_{loc}([a, \infty); \mathbb{R})$ – the set of all Lebesgue integrable functions u: $[a, \infty) \to \mathbb{R}$ in each finite closed interval $[a, b] \subset [a, \infty)$.

Definition 1. The function x: $\mathbb{R}_+ \to \mathbb{R}$ is said to be a *solution* of equation (1) if $x \in AC^{n-1}(\mathbb{R}_+, \mathbb{R})$ and x satisfies equation (1) almost everywhere.

Definition 2. A given function u: $\mathbb{R}_+ \to \mathbb{R}$ is said to *eventually* enjoy the property P if there exists a point $t_{p,u} \geq 0$ such that for $t \geq t_{p,u}$ the property P is valid.

Definition 3. The solution x of equation (1) is said to be *regular* if $sup|x(t)| > 0$ eventually.

Definition 4. The regular solution x of equation (1) is said to *oscillate* if it has infinitely many zeros, and to be *nonoscillating* otherwise.

Introduce the following conditions:

$H1$: $p \in L_{loc}(\mathbb{R}_+, \mathbb{R})$, $mes\{s \geq t;\ p(s) \neq 0\} > 0$ for $t \geq 0$.

$H2$: A: $AC^{n-1}(\mathbb{R}_+, \mathbb{R}) \to L_{loc}(\mathbb{R}_+, \mathbb{R})$.

$H3$: If the function $x \in AC^{n-1}(\mathbb{R}_+, \mathbb{R})$ is eventually nonzero and with a constant sign, then the function $Ax \in L_{loc}(\mathbb{R}_+, \mathbb{R})$ is also eventually nonzero and with a constant sign, and they have the same sign.

$H4$: If the functions x_1, $x_2 \in AC^{n-1}(\mathbb{R}_+, \mathbb{R})$, $x_1(t) \leq x_2(t)$ for $t \in \mathbb{R}_+$, then $(Ax_1)(t) \leq (Ax_2)(t)$ for $t \geq t_0 > 0$.

$H5$: If $x(t) \equiv 0$, then $(Ax)(t) \equiv 0$ eventually.

$H6$: The operator A is linear.

Introduce the following notation:

If s_0 is a zero of the function v: $\mathbb{R}_+ \to \mathbb{R}$ with multiplicity n_0, and $m \in \mathbb{N}$, then

$$\lambda_m(v,s_0) = \begin{cases} n_0 & \text{for } n_0 \leq m \\ m & \text{for } n_0 > m \end{cases} \tag{2}$$

By $\mu_m(v,s_0)$ ($\mu'_m(v,s_0)$) denote the number of indices ($i = 1, \ldots, m-1$) for which $v^{(i)}(s_0) = 0$.

If the function v in the interval $I \subset \mathbb{R}_+$ has a finite number of distinct zeros S_i $(1 \leq i \leq k)$, then $\lambda_m(v, I) = \sum_{i=1}^{k} \lambda_m(v, S_i)$.

Then $\nu_m(v;a,b) = \lambda_m(v;\ (a, b]) + \mu_m(v,a)$

or $\qquad \nu_m(v;a,b) = \lambda_m(v;\ [a, b]) + \mu'_m(v,a)$.

Let $t_0 \in \mathbb{R}_+$. Denote by E_{t_0} the set of all numbers t_1 $(t_1 > t_0)$ for which there exists a solution x of equation (1) such that $x(t_0) = x(t_1) = 0$ and $x(t) \geq 0$ for $t \in [t_0, t_1]$.

Introduce the notation $\tau_n(t_0, p) = sup\ E_{t_0}$.

Lemma 1. *Let the following conditions hold:*

1. *Conditions* $H1 - H3$ *are met.*
2. *There exists a solution* x *of equation* (1) *such that* $x(t) > 0$ *for* $t \geq t_0 > 0$.

Then there exist numbers $t_1 \in [t_0, \infty)$ *and* $l \in \{0, \ldots, n\}$ *such that* $n + l$ *is an odd number and*

$$\begin{aligned} &x^{(i)}(t) > 0 \text{ for } t \geq t_1,\ 0 \leq i \leq l-1 \\ &(-1)^{l+i}\, x^{(i)}(t) > 0 \text{ for } t \geq t_1,\ l \leq i \leq n-1 \\ &(-1)^{n+l}\, x^{(n)}(t) \geq 0 \text{ for } t \geq t_1. \end{aligned} \tag{3}$$

Remark 1. Lemma 1 follows from the Lemma of Kiguradze [1] and conditions $H2$ and $H3$.

3. MAIN RESULTS

Theorem 1. *Let the following conditions hold:*

1. *Conditions $H1 - H3$ are met.*

2. *There exists a regular solution x of equation* (1) *in the interval $[t_0, t^0] \subset \mathbb{R}_+$ such that*

$$x(t_0) = x(t^0) = 0, \ x(t) \geq 0 \ for \ t \in [t_0, t^0].$$

Then
$$\lambda_n(x; [t_0, t^0]) \leq n.$$

Proof: Suppose that the solution x of equation (1) changes its sign m times ($m \geq 0$ an integer) in the interval $[t_0, t^0]$, i.e., there exist intervals I_j $(0 \leq j \leq m)$ such that $\bigcup_{j=0}^{m} I_j = [t_0, t^0]$ and $x^{(n)}(t)$ does not change its sign in each of them.

1. Let the function $x^{(i)}(t)$ $(1 \leq i \leq n-1)$ in the interval $[t_0, t^0]$ have a finite number of zeros.

If $x^{(i-1)}(t_0) \neq 0$, then

$$\lambda_{n-i}(x^{(i)}; [t_0, t^0]) \geq \lambda_{n-i+1}(x^{(i-1)}; (t_0, t^0]) - 1$$

and

$$\mu_{n-i}(x^{(i)}; t_0) = \mu_{n-i+1}(x^{(i-1)}; t_0).$$

If $x^{(i-1)}(t_0) = 0$, then

$$\lambda_{n-i}(x^{(i)}; (t_0, t^0]) \geq \lambda_{n-i+1}(x^{(i-1)}; (t_0, t^0])$$

and

$$\mu_{n-i}(x^{(i)}; t_0) = \mu_{n-i+1}(x^{(i-1)}; t_0) - 1.$$

Then

$$\nu_{n-i}(x^{(i)}; t_0, t^0) \geq \nu_{n-i+1}(x^{(i-1)}; t_0, t^0) - 1, \ 1 \leq i \leq n-1$$
$$\nu_n(x; t_0, t^0) \leq \nu_1(x^{(n-1)}; t_0, t^0) + n - 1$$

or

$$\lambda_n(x; [t_0, t^0]) \leq \lambda_1(x^{(n-1)}; [t_0, t^0]) + \mu_1'(x^{(n-1)}; t_0) + n - 1. \tag{4}$$

But

$$\lambda_1(x^{(n-1)}; [t_0, t^0]) = 1 \text{ and } \mu_1'(x^{(n-1)}; t_0) = 0.$$

Then from (4) it follows that $\lambda_n(x; [t_0, t^0]) \leq n$.

2. Let an integer i exist $(1 \leq i \leq n-1)$ such that $x^{(i)}$ in the interval $[t_0, t^0]$ has infinitely many zeros. From condition $H1$ it follows that for each closed interval there exist a finite number of intervals T_{ij} (which can be also points – for instance T_{0j}) such that $x^{(i)}(t) \equiv 0$ for $t \in T_{ij}$ and there exists an ϵ-

neighborhood T^{ϵ}_{ij} of the interval T_{ij} such that for $t \in T^{\epsilon}_{ij} \backslash T_{ij}$, $x^{(i)}(t) \neq 0$.

Moreover, if T_{ij} and T_{kl} are two subintervals which do not degenerate into points, then either $T_{ij} = T_{kl}$ or $T_{ij} \cap T_{kl} = \emptyset$.

Introduce the following notation:

$$\bar{\lambda}_{n-i}(x^{(i)}, T_{ij}) = \begin{cases} \lambda_{n-i}(x^{(i)}, t_*) & \text{for } \{t_*\} = T_{ij} \\ n-i & \text{for } T_{ij} \neq \{t_*\} \end{cases}$$

$$\bar{\nu}_{n-i}(x^{(i)}; t_0, t^0) = \sum_{j=1}^{k} \bar{\lambda}_{n-i}(x^{(i)}, T_{ij}) + \mu_{n-i}(x^{(i)}, t_0)$$

where $\{T_{i1}, T_{i2}, \ldots, T_{ik}\}$ is the set of all intervals such that $T_{ij} \cap (t_0, t^0] \neq \emptyset$, $1 \leq j \leq k$.

Just as in Case 1 we obtain that

$$\bar{\nu}_n(x; t_0, t^0) \leq \bar{\nu}_1(x^{(n-1)}; t_0, t^0) + n - 1,$$

i.e.,

$$\lambda_n(x; [t_0, t^0]) \leq n. \qquad \square$$

Theorem 2: *Let the following conditions hold:*

1. *Conditions* $H1 - H4$ *are met.*
2. *There exist numbers* l, k *and* c_0 ($l \in \{1, \ldots, n\}$, $k \in \{0, \ldots, l-1\}$, $c_0 \in (0, \infty)$) *and*

a function $x \in C([t_1, \infty); \mathbb{R}_+)$ *such that for* $t \geq t_1 > 0$ *the following inequality holds:*

$$x(t) \geq c_0(t-t_1)^k - \frac{1}{(l-1)!(n-l-1)!} \int_{t_1}^{t} (t-s)^{l-1} \int_s^{\infty} (\xi - s)^{n-l-1} p(\xi) \cdot (Ax)(\xi) d\xi ds.$$

Then there exists a solution of the equation

$$x^{(n)}(t) + (-1)^{n-l-1} p(t) \cdot (Ax)(t) = 0 \tag{5}$$

which satisfies conditions (3) *and*

$$x(t_1) = x'(t_1) = \ldots = x^{(k-1)}(t_1) = 0.$$

Proof: Let U be the set of all functions $u \in C([t_1, \infty); \mathbb{R}_+)$ such that

$$c_0(t-t_1)^k \leq u(t) \leq x(t) \text{ for } t \geq t_1.$$

Define the operator S: $U \to U$ by the formula

$$(Su)(t) = c_0(t-t_1)^k + \frac{1}{(l-1)!(n-l-1)!} \int_{t_1}^{t} (t-s)^{l-1} \int_s^{\infty} (\xi - s)^{n-l-1} p(\xi) \cdot (Au)(\xi) d\xi ds.$$

Consider the sequence of functions $\{v_j\}_{j=1}^{\infty}$ defined as follows:

$$v_1(t) = c_0(t-t_1)^k$$
$$v_{j+1} = Sv_j,\ j = 1,2,...$$

From the definitions of the sequence $\{v_j\}_{j=1}^{\infty}$ and of the operator and from condition $H4$ it follows that

$$v_{j+1}(t) \geq v_j(t) \text{ for } t \geq t_1,\ j = 1,2,...$$

But

$$|(Sv_j)'(t)| \leq (Sx)'(t) \text{ for } t \geq t_1.$$

Hence the sequence $\{v_j\}_{j=1}^{\infty}$ is uniformly convergent in each finite closed interval of $[t_1, \infty)$. Let $\lim_{j\to\infty} v_j = v$. Then v is a fixed point for the operator S and v is the solution sought of equation (5) for which inequalities (3) are valid and

$$v(t_1) = v'(t_1) = ... = v^{(k-1)}(t_1) = 0. \qquad \square$$

Theorem 3: *Let the following conditions hold:*

1. *Conditions H1 – H3, H5 and H6 are met.*
2. $\tau_n(t_0, p) < +\infty$.

Then there exists a solution x of equation (1) *for which the following assertions are valid:*

1. *The solution x in the interval $[t_0, \tau_n(t_0, p))$ has n zeros.*
2. $x(t_0) = x(\tau_n(t_0, p)) = 0$, $x(t) \geq 0$ *for* $t \in [t_0, \tau_n(t_0, p)]$.

Proof: From the definition of the set E_{t_0} and condition 2 of Theorem 1 it follows that there exists a solution x of equation (1) such that $x(t_0) = x(t^0) = 0$, $x(t) \geq 0$ for $t \in [t_0, t^0]$, where $t^0 = \tau(t_0, p) < \infty$.
Among all these solutions of equation (1) we choose this solution x for which $\lambda_n(x; [t_0, t^0])$ has the greatest value. From Theorem 1 it follows that $\lambda_n(x, [t_0, t^0]) \leq n$.

Suppose that $\lambda_n(x; [t_0, t^0]) < n$. Let $t_0 < t_1 < ... < t_k < t^0$ be zeros of the so chosen solutions x of multiplicity $n_0, n_1, ..., n_k, n^0$ respectively. Here n_i $(i = 1, ..., k)$ are even numbers.

Let $\{v_m\}_{m=1}^{\infty}$ be regular solutions of equation (1) such that

$$v_m^{(j-1)}(t_i) = 0,\ j = 1, ..., n_i,\ i = 0, ..., k$$
$$v_m^{(j-1)}(t^0 + \tfrac{1}{m}) = 0,\ j = 1, ..., n^0.$$

Since $t^0 = \tau_n(t_0, p) = supE_{t_0}$, then it follows that v_m changes its sign in the interval $(t_0, t^0 + \frac{1}{m})$.

If the solution v_m changes its sign at the point $t^* \neq t_i$, then v_m has in the interval $[t_0,\ t^0 + \frac{1}{m}]$ at least $\lambda_n(x;\ [t_0,\ t^0]) + 1$ zeros. If the solution v_m changes its sign at the point t_i, then t_i is a zero of multiplicity $n_i + 1$ of the function v_m.

Consequently,

$$\lambda_n(v_m;\ [t_0,\ t^0 + \tfrac{1}{m}]) \geq \lambda_n(x;\ [t_0,\ t^0]) + 1.$$

Without loss of generality we can assume that $\sum_{j=0}^{m} \left| v_m^{(j-1)}(t_0) \right| = 1$ and $\{v_m\}_{m=1}^{\infty}$ is a uniformly convergent sequence in each finite closed interval of $\mathbb{R}_+$. Let $\lim\limits_{m\to\infty} = v_0$. Then

$$\lambda_n(v_0;\ [t_0,\ t^0]) \geq \lambda_n(v_m;\ [t_0,\ t^0 + \tfrac{1}{m}]) \geq \lambda_n(x;\ [t_0,\ t^0]) + 1. \tag{6}$$

From (6) it follows that x and v are linearly independent solutions of equation (1). From the fact that the solution x was chosen so that the number $\lambda_n(x;\ [t_0,\ t^0])$ is maximal, it follows that v_0 changes its sign in the interval $[t_0,\ t^0]$.

Let $x_\epsilon = x - \epsilon v_0$. Since the zeros of x are also zeros of v_0, then for sufficiently small $\epsilon > 0$ we obtain that

$$x_\epsilon(t) \geq 0 \text{ for } t \in [t_0,\ t^0]. \tag{7}$$

Denote by ϵ_0 the greatest among all ϵ for which inequality (7) is valid. But

$$\lambda_n(x_{\epsilon_0};\ [t_0,\ t^0]) > \lambda_n(x;\ [t_0,\ t^0])$$

which contradicts the fact that $\lambda_n(x,\ [t_0,\ t^0])$ is the maximal number chosen. □

Theorem 4. *Let the following conditions hold:*

1. *Conditions* $H1 - H5$ *are met.*
2. $\tau_n(t_0,\ p) < \infty$.

Then each solution x *of equation* (1) *such that* $x(t_0) = 0$ *has a zero in the interval* $(t_0,\ \tau_n(t_0,\ p)]$.

Proof: Suppose that there exists a solution x of equation (1) such that $x(t_0) = 0$ and $x(t) > 0$ for $t \in (t_0,\ t^0]$, where $t^0 = \tau_n(t_0,\ p) < \infty$.

From Theorem 3 it follows that there exists a solution v of equation (1) which in the interval $[t_0,\ t^0]$ has n zeros, $v(t_0) = v(t^0) = 0$ and $v(t) \geq 0$ for $t \in [t_0,\ t^0]$.

Moreover, t^0 is a zero of the function $v(t)$ of odd multiplicity. Hence we can choose $t_1 > \tau_n(t_0,\ p)$ and $\epsilon_0 > 0$ such that

$$x(t_1) + \epsilon_0 v(t_1) = 0$$
$$x(t) + \epsilon_0 v(t) > 0, \text{ for } t \in (t_0, t^0)$$

i.e., $t_1 \in E_{t_0}$, which contradicts the fact that $\tau_n(t_0, p) = supE_{t_0}$. □

Theorem 5. *Let the following conditions hold:*

1. *Conditions $H1 - H5$ are met.*

2. *Each regular solution x of equation (1) for even n oscillates, and for odd n either oscillates or $\lim_{t\to\infty} |x^{(i)}(t)| = 0$ monotonically decreasing $(i = 0, 1, ..., n-1)$.*

Then $\tau_n(t_0, p) < \infty$ for $t_0 \in \mathbb{R}_+$.

Proof: Let each regular solution x of equation (1) for n an even number oscillates, and for n an odd number either oscillates or $\lim_{t\to\infty} |x^{(i)}(t)| = 0$ monotonically decreasing, but $\tau_n(t_0, p) = +\infty$. This implies the existence of a sequence $\{t_k\}_{k=1}^{\infty}$ and of solutions of equation (1) such that

$$t_0 < t_1 < t_2 < \ldots < t_k < \ldots, \ \lim_{k\to\infty} t_k = \infty$$
$$x_k(t_0) = x_k(t_k) = 0, \ x_k(t) > 0 \text{ for } t \in (t_0, t_k).$$

Without loss of generality assume that $\sum_{i=0}^{n-1} |x_k^{(i)}(t_0)| = 1$ and the sequence of solutions $\{x_k\}_{k=0}^{\infty}$ is uniformly convergent in each finite closed interval $[a, b] \subset \mathbb{R}_+$. Let $\lim_{k\to\infty} x_k = x$. Then x is a nonoscillating solution of equation (1). If n is an even number, we obtain a contradiction with condition 2 of Theorem 5. Let n be an odd number. From the fact that $\lim_{t\to\infty} |x^{(i)}(t)| = 0$ it follows that

$$(-1)^i x^{(i)}(t) x(t) > 0 \text{ for } t \in \mathbb{R}_+; \ i = 0, \ldots, n-1$$

i.e.,

$$(-1)^i x^{(i)} t > 0 \text{ for } t \in \mathbb{R}_+; \ i = 0, \ldots, n-1$$

which contradicts the fact that $x(t_0) = 0$. □

Theorem 6. *Let the following conditions hold:*

1. *Conditions $H1 - H5$ are met.*

2. *Each solution x of equation (1)which vanishes at least once oscillates.*

Then equation (1) has no solution satisfying condition (3) for $l = n-1$.

Proof: Let equation (1) have a solution satisfying condition (3) for $l = n-1$. Then from equation (1) we obtain that

$$x(t) \geq x^{(n-2)}(t_1)\frac{(t-t_1)^{n-2}}{(n-2)!} + \frac{1}{(n-2)!} \int_{t_1}^{t} (t-s)^{n-2} \int_s^{\infty} p(\xi) \cdot (Ax)(\xi) d\xi ds. \quad (8)$$

From (8) and from Theorem (2) it follows that there exists a nonoscillating solution x of equation (1) such that $x(t_1) = 0$, which contradicts condition 2 of Theorem 6. □

Theorem 7. *Let the following conditions hold:*

1. *Condition H1 – H3 are met.*

2. *For each* $l \in \{1, \ldots, n-1\}$ *such that* $l+n$ *is an odd number, equation* (1) *has no solution satisfying condition* (3).

Then each regular solution x *of equation* (1) *for* n *even oscillates, and for* n *odd either oscillates or* $\lim_{t \to \infty} |x^{(i)}(t)| = 0$, $i = 0, \ldots, n-1$.

Proof: From Lemma 1 and condition 2 of Theorem 7 it follows that if n is an even number, equation (1) has no nonoscillating solution, and if n is an odd number, each nonoscillating solution of equation (1) satisfies condition (3) for $l = 0$ and $(-1)^i x^{(i)}(t)x(t) > 0$, $i = 0, \ldots, n-1$., i.e., $\lim_{t \to \infty} |x^{(i)}(t)| = 0$. □

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