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EXISTENCE THEOREM FOR NONCONVEX STOCHASTIC INCLUSIONS

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ABSTRACT

An existence theorem for stochastic inclusions $x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\tilde{\nu}(d\tau, dz)$ with nonanticipative nonconvex-valued right-hand sides is proved.

Key words: Stochastic inclusions, existence solutions, solution set.

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1. Introduction

Existence theorem and weak compactness of the solution set to stochastic inclusion

$$x_t - x_s \in \int_s^t F_\tau(x_\tau)d\tau + \int_s^t G_\tau(x_\tau)dw_\tau + \int_s^t \int_{\mathbb{R}^n} H_{\tau,z}(x_\tau)\tilde{\nu}(d\tau, dz),$$

denoted by $SI(F,G,H)$, with predictable convex-valued right-hand sides have been considered in the author's paper [4]. These results were obtained by fixed points methods. Applying the successive approximation method we shall prove here an existence theorem for $SI(F,G,H)$ with nonanticipative nonconvex-valued multivalued processes F, G and H. To begin with, we recall the basic definitions dealing with set-valued stochastic integrals and stochastic inclusions presented in [5].

Let a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be given, where a family $(\mathcal{F}_t)_{t \geq 0}$, of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$, is assumed to be increasing: $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s \leq t$. Let $\mathbb{R}_+ = [0, \infty)$ and $\mathcal{B}_+$ be the Borel σ-algebra on $\mathbb{R}_+$. We consider set-valued stochastic processes $(\mathcal{F}_t)_{t \geq 0}$, $(\mathcal{G}_t)_{t \geq 0}$ and $(\mathcal{H}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ taking on values in the space $Comp(\mathbb{R}^n)$ of all nonempty compact subsets of n-dimensional Euclidean space $\mathbb{R}^n$. They are assumed to be nonanticipative and such that $\int_0^\infty \| \mathcal{F}_t \|^p dt < \infty$, $p \geq 1$, $\int_0^\infty \| \mathcal{G}_t \|^2 dt < \infty$ and $\int_0^\infty \int_{\mathbb{R}^n} \| \mathcal{H}_{t,z} \|^2 dtq(dz) < \infty$, a.s., where q is a measure on a Borel σ-algebra $\mathcal{B}^n$ of $\mathbb{R}^n$ and $\| A \| := sup\{ |a| : a \in A\}$, $A \in Comp(\mathbb{R}^n)$. The space $Comp(\mathbb{R}^n)$ is considered with the Hausdorff metric h defined in the usual way, i.e., $h(A,B) = max\{\bar{h}(A,B), \bar{h}(B,A)\}$, for $A, B \in Comp(\mathbb{R}^N)$, where $\bar{h}(A,B) = \{dist(a,B): a \in A\}$ and $\bar{h}(B,A) = \{dist(b,A): b \in B\}$.

2. Basic Definitions and Notations

Throughout the paper, we shall assume that a filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfies the following usual hypotheses:

(i) $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and

(ii) $\mathcal{F}_t = \bigcup_{u > t} \mathcal{F}_u$, all t, $0 \leq t < \infty$; that is, the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right continuous.

As usual, we shall consider a set $\mathbb{R}_+ \times \Omega$ as a measurable space with the product σ-algebra $\mathcal{B}_+ \otimes \mathcal{F}$.

An n-dimensional stochastic process x is understood as a function x: $\mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ with $\mathcal{F}$-measurable sections x_t, for $t \geq 0$, and it is denoted by $(x_t)_{t \geq 0}$. It is measurable if x is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable. The process $(x_t)_{t \geq 0}$ is $\mathcal{F}_t$-adapted or adapted if x_t is $\mathcal{F}_t$-measurable for $t \geq 0$. Every measurable and adapted process is called nonanticipative. In what follows, the Banach spaces $L^p(\Omega, \mathcal{F}_t, P, \mathbb{R}^n)$ and $L^p(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ with the usual norm $\|\cdot\|$ are denoted by $L_n^p(\mathcal{F}_t)$ and $L_n^p(\mathcal{F})$, respectively.

Let $\mathcal{M}^2(\mathcal{F}_t)$ denote the family of all (equivalence classes of) n-dimensional nonanticipative processes $(f_t)_{t \geq 0}$ such that $\int_0^\infty |f_t|^2 dt < \infty$, a.s. We shall also consider a subspace $\mathcal{L}_n^2$ of $\mathcal{M}^2(\mathcal{F}_t)$ defined by $\mathcal{L}_n^2 = \{(f_t)_{t \geq 0} \in \mathcal{M}^2(\mathcal{F}_t) : E\int_0^\infty |f_t|^2 dt < \infty\}$ with the norm $\|\cdot\|_{\mathcal{L}_n^2}$ defined in the usual way. The Banach spaces $L^p(\mathbb{R}_+, \mathcal{B}_+, dt, \mathbb{R}_+)$, $p \geq 1$ and $L^2(\mathbb{R}_+ \times \mathbb{R}^n, \mathcal{B}_+ \otimes \mathcal{B}^n, dt \times q, \mathbb{R}_+)$, with the usual norms $|\cdot|_p$ and $\|\cdot\|_2$ will be denoted by $L^p(\mathcal{B}_+)$ and $L^2(\mathcal{B}_+ \times \mathcal{B}^n)$, respectively. Finally, by $M_n(\mathcal{F}_t)$ we denote a space of all (equivalence classes of) n-dimensional $\mathcal{F}_t$-measurable mappings.

Throughout the paper, by $(w_t)_{t \geq 0}$ we mean a one-dimensional $\mathcal{F}_t$-Brownian motion starting at 0, i.e., such that $P(w_0 = 0) = 1$. By $\nu(t, A)$ we denote a $\mathcal{F}_t$-Poisson measure (see [1]) on $\mathbb{R}_+ \times \mathcal{B}^n$ and then define an $\mathcal{F}_t$-centered Poisson measure $\tilde{\nu}(t, A)$, $t \geq 0$, $A \in \mathcal{B}^n$, by taking $\tilde{\nu}(t, A) = \nu(t, A) - tq(A)$, $t \geq 0$, $A \in \mathcal{B}^n$, where q is a measure on $\mathcal{B}^n$ such that $E\nu(t, B) = tq(B)$ and $q(B) < \infty$ for $B \in \mathcal{B}_0^n$.

By $\mathcal{M}^2(\mathcal{F}_t, q)$, we shall denote the family of all (equivalence classes of) $\mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n$-measurable and $\mathcal{F}_t$-adapted functions $h: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\int_0^\infty \int_{\mathbb{R}^n} |h_{t,z}|^2 dtq(dz) < \infty$ a.s. Recall that a function $h: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is said to be $\mathcal{F}_t$-adapted or adapted if for every $x \in \mathbb{R}^n$ and $t \geq 0$, $h(t, \cdot, x)$ is $\mathcal{F}_t$-measurable. Elements of $\mathcal{M}^2(\mathcal{F}_t, q)$ will be denoted by $h = (h_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$. Finally, let $\mathcal{W}_n^2 = \{h \in \mathcal{M}^2(\mathcal{F}_t, q) : \|h\|_{\mathcal{W}_n^2}^2 < \infty\}$ where $\|h\|_{\mathcal{W}_n^2}^2 = E\int_0^\infty \int_{\mathbb{R}^n} |h_{t,z}|^2 dtq(dz)$.

Given $g \in \mathcal{M}^2(\mathcal{F}_t)$ and $h \in \mathcal{M}^2(\mathcal{F}_t, q)$, by $(\int_0^t g_\tau dw_\tau)_{t \geq 0}$ and $(\int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz))_{t \geq 0}$, we denote their stochastic integrals with respect to an $\mathcal{F}_t$-Brownian motion $(w_t)_{t \geq 0}$ and an $\mathcal{F}_t$-centered Poisson measure $\tilde{\nu}(t, A)$, $t \geq 0$, $A \in \mathcal{B}^n$, respectively. These integrals, understood as n-dimensional stochastic processes, have quite similar properties (see [1]).

Let us denote by D the family of all n-dimensional $\mathcal{F}_t$-adapted cádlág (see [6]) processes $(x_t)_{t \geq 0}$ such that $E sup_{t \geq 0} |x_t|^2 < \infty$. The space D is considered as a normed space with the norm $\| \xi \|_{\ell} = \| sup_{t \geq 0} | \xi_t | \|_{L^2}$ for $\xi = (\xi_t)_{t \geq 0} \in D$, where $\| \cdot \|_{L^2}$ is a norm of $L^2(\Omega, \mathcal{F}, P, \mathbb{R})$. It can be verified that $(D, \| \cdot \|_{\ell})$ is a Banach space.

Given $0 \leq \alpha < \beta < \infty$ and $(x_t)_{t \geq 0} \in D$, let $x^{\alpha, \beta} = (x_t^{\alpha, \beta})_{t \geq 0}$ be such that $x_t^{\alpha, \beta} = x_\alpha$ and $x_t^{\alpha, \beta} = x_\beta$ for $0 \leq t \leq \alpha$ and $t \geq \beta$, respectively, and $x_t^{\alpha, \beta} = x_t$ for $\alpha \leq t \leq \beta$. It is clear that $D^{\alpha, \beta} := \{x^{\alpha, \beta} : x \in D\}$ is a linear subspace of D, closed in the $\| \cdot \|_{\ell}$-norm topology. Then, $(D^{\alpha, \beta}, \| \cdot \|_{\ell})$ is also a Banach space.

Given a measure space $(X, \mathcal{B}, m)$, a set-valued function $\mathcal{R}: X \to Cl(\mathbb{R}^n)$ is said to be $\mathcal{B}$-measurable if $\{x \in X : \mathcal{R}(x) \cap C \neq \emptyset\} \in \mathcal{B}$ for every closed set $C \subset \mathbb{R}^n$. For such a multifunction, we define subtrajectory integrals as a set $\mathcal{S}(\mathcal{R}) = \{g \in L^p(X, \mathcal{B}, m, \mathbb{R}^n) : g(x) \in \mathcal{R}(x) \text{ a.e.}\}$. It is clear that for nonemptiness of $\mathcal{S}(\mathcal{R})$ we must assume more then $\mathcal{B}$-measurability of $\mathcal{R}$. In what follows, we shall assume that $\mathcal{B}$-measurable set-valued function $\mathcal{R}: X \to Cl(\mathbb{R}^n)$ is p-integrable bounded, $p \geq 1$, i.e., that a real-valued mapping: $X \ni x \to \| \mathcal{R}(x) \| \in \mathbb{R}_+$ belongs to $L^p(X, \mathcal{B}, m, \mathbb{R}_+)$. It can be verified (see [2], Th. 3.2) that a $\mathcal{B}$-measurable set-valued mapping $\mathcal{R}: X \to Cl(\mathbb{R}^n)$ is p-integrable bounded, $p \geq 1$, if and only if $\mathcal{S}(\mathcal{R})$ is nonempty and bounded in $L^p(X, \mathcal{B}, m, \mathbb{R}^n)$. Finally, it is easy to see that $\mathcal{S}(\mathcal{R})$ is decomposable, i.e., such that $\mathbb{I}_A f_1 + \mathbb{I}_{X/A} f_2 \in \mathcal{S}(\mathcal{R})$ for $A \in \mathcal{B}$ and $f_1, f_2 \in \mathcal{S}(\mathcal{R})$.

We have the following general result dealing with the properties of subtrajectory integrals (see [2], [3]).

Proposition 1. *Let $\mathcal{R}: X \to Cl(\mathbb{R}^n)$ be $\mathcal{B}$-measurable and p-integrable bounded, $p \geq 1$. Then, $\mathcal{S}(\mathcal{R})$ is a nonempty bounded and closed subset of $L^p(X, \mathcal{B}, m, \mathbb{R}^n)$. Moreover, if $\mathcal{R}$ takes on convex values then $\mathcal{S}(\mathcal{R})$ is convex and weakly compact in $L^p(X, \mathcal{B}, m, \mathbb{R})$.* □

Let $\mathcal{G} = (\mathcal{G}_t)_{t > 0}$ be a set-valued stochastic process with values in $Cl(\mathbb{R}^n)$, i.e., a family of $\mathcal{F}$-measurable set-valued mappings $\mathcal{G}_t: \Omega \to Cl(\mathbb{R}^n)$, $t \geq 0$. We call $\mathcal{G}$ measurable if it is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable. Similarly, $\mathcal{G}$ is said to be $\mathcal{F}_t$-adapted or adapted if $\mathcal{G}_t$ is $\mathcal{F}_t$-measurable for each $t \geq 0$. A measurable and adapted set-valued stochastic process is called nonanticipative.

In what follows, we shall also consider $\mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n$-measurable set-valued mappings $\mathcal{R}: \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \to Cl(\mathbb{R}^n)$. They will be denoted as families $(\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ and called measurable set-valued stochastic processes depending on a parameter $z \in \mathbb{R}^n$. The process $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$ is said to be $\mathcal{F}_t$-*adapted* or *adapted* if $\mathcal{R}_{t,z}$ is $\mathcal{F}_t$-measurable for each $t \geq 0$ and $z \in \mathbb{R}^n$. We call it nonanticipative if it is measurable and adapted.

Denote by $\mathcal{M}^2_{s-v}(\mathcal{F}_t)$ and $\mathcal{M}^2_{s-v}(\mathcal{F}_t, q)$ families of all nonanticipative set-valued processes $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ and $\mathcal{R} = (\mathcal{R}_{t,z})_{t \geq 0, z \in \mathbb{R}^n}$, respectively, such that $\int_0^\infty \| \mathcal{G}_t \|^2 dt < \infty$ and $\int_0^\infty \int_{\mathbb{R}^n} \| \mathcal{R}_{t,z} \|^2 dt q(dz) < \infty$, a.s. Immediately, from Kuratowski and Ryll-Nardzewski measurable selection theorem (see [3]) it follows that for every $F, \mathcal{G} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$ and $\mathcal{R} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t, q)$ their subtrajectory integrals

$$\mathcal{S}(F) := \{f \in \mathcal{M}^2(\mathcal{F}_t) : f_t(\omega) \in F_t(\omega), dt \times P - \text{a.e.}\},$$

$$\mathcal{S}(\mathcal{G}) := \{g \in \mathcal{M}^2(\mathcal{F}_t) : g_t(\omega) \in \mathcal{G}_t(\omega), dt \times P\text{-a.e.}\} \text{ and}$$

$$\mathcal{S}_q(\mathcal{R}) := \{h \in \mathcal{M}^2(\mathcal{F}_t, q) : h_{t,z}(\omega) \in \mathcal{R}_{t,z}(\omega), dt \times P \times q\text{-a.e.}\}$$

are nonempty. Indeed, let $\Sigma = \{Z \in \mathcal{B}_+ \otimes \mathcal{F} : Z_t \in \mathcal{F}_t, \text{ each } t \geq 0\}$, where Z_t denotes a section of Z determined by $t \geq 0$. It is a σ-algebra on $\mathbb{R}_+ \times \Omega$ and a function $f: \mathbb{R}_+ \times \Omega \to \mathbb{R}^n$ (a multifunction $F: \mathbb{R}_+ \times \Omega \to Cl(\mathbb{R}^n)$) is nonanticipative if and only if it is Σ-measurable. Therefore, by Kuratowski and Ryll-Nardzewski measurable selection theorem every nonanticipative set-valued function admits a nonanticipative selector. It is clear that for $F \in \mathcal{M}^2_{s-v}(\mathcal{F}_t)$ such selector belongs to $\mathcal{M}^2(\mathcal{F}_t)$. Similarly, define on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ a σ-algebra

$$\widetilde{\Sigma} = \{Z \in \mathcal{B}_+ \otimes \mathcal{F} \otimes \mathcal{B}^n\colon Z^u_t \in \mathcal{F}_t, \text{ each } t \geq 0 \text{ and } u \in \mathbb{R}^n\},$$

where $Z^u_t = (Z^u)_t$ and Z^u is a section of Z determined by $u \in \mathbb{R}^n$.

Given the set-valued processes

$$F = (F_t)_{t \geq 0} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t),\ \mathcal{G} = (\mathcal{G}_t)_{t \geq 0} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t) \text{ and}$$
$$\mathcal{H} = (\mathcal{H}_{t,z})_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}^2_{s-v}(\mathcal{F}_t, q)$$

by their stochastic integrals we mean families

$$\Big(\int_0^t F_\tau d\tau\Big)_{t \geq 0},\ \Big(\int_0^t \mathcal{G}_\tau dw_\tau\Big)_{t \geq 0}, \text{ and } \Big(\int_0^t \int_{\mathbb{R}^n} \mathcal{H}_{\tau,z} \widetilde{\nu}(d\tau, dz)\Big)_{t \geq 0}$$

of subsets defined by

$$\int_0^t F_\tau d\tau = \Big\{\int_0^t f_\tau \tau\colon f \in \mathcal{S}(F)\Big\},$$
$$\int_0^t \mathcal{G}_\tau dw_\tau = \Big\{\int_0^t \mathcal{G}_\tau dw_\tau\colon g \in \mathcal{S}(\mathcal{G})\Big\} \text{ and}$$
$$\Big\{\int_0^t \int_{\mathbb{R}^n} \mathcal{H}_{\tau,z} \widetilde{\nu}(d\tau, dz) = \Big\{\int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \widetilde{\nu}(d\tau, dz)\colon h \in \mathcal{S}_q(\mathcal{H})\Big\}.$$

Given $0 \leq \alpha < \beta < \infty$ we also define

$$\int_\alpha^\beta F_s ds\colon = \Big\{\int_\alpha^\beta f_s ds\colon f \in \mathcal{S}(F)\Big\},$$
$$\int_\alpha^\beta \mathcal{G}_s dw_s\colon = \Big\{\int_\alpha^\beta g_s dw_s\colon g \in \mathcal{S}(\mathcal{G})\Big\} \text{ and}$$
$$\int_\alpha^\beta \int_{\mathbb{R}^n} \mathcal{H}_{s,z} \widetilde{\nu}(ds, dz)\colon = \Big\{\int_\alpha^\beta \int_{\mathbb{R}^n} h_{s,z} \widetilde{\nu}(ds, dz)\colon h \in \mathcal{S}_q(\mathcal{H})\Big\}.$$

3. Stochastic Inclusions

Let $F = \{(F_t(x))_{t \geq 0}\colon x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0}\colon x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n}\colon x \in \mathbb{R}^n\}$. Assume F, G and H are such that $(F_t(x))_{t \geq 0} \in \mathcal{M}^p_{s-\nu}(\mathcal{F}_t)$, $(G_t(x))_{t \geq 0} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$ and $(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n} \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t, q)$ each $x \in \mathbb{R}^n$.

By a stochastic inclusion, denoted by $SI(F,G,H)$, corresponding to given above F, G and H we mean a relation

$$x_t - x_s \in \int_s^t F_\tau(x_\tau) d\tau + \int_s^t G_\tau(x_\tau) dw_\tau + \int_s^t \int_{\mathbb{R}} H_{\tau,z}(x_\tau) \widetilde{\nu}(d\tau, dz)$$

that is to be satisfied for every $0 \leq s < t < \infty$ by a stochastic process $x = (x_t)_{t \geq 0} \in D$ such that $F \circ x \in \mathcal{M}^p_{s-\nu}(\mathcal{F}_t)$, $G \circ x \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t)$ and $H \circ x \in \mathcal{M}^2_{s-\nu}(\mathcal{F}_t, q)$, where $F \circ x = (F_t(x_t))_{t \geq 0}$, $G \circ x = (G_t(x_t))_{t \geq 0}$ and $H \circ x = (H_{t,z}(x_t))_{t \geq 0, z \in \mathbb{R}^n}$. Every stochastic process $(x_t)_{t \geq 0} \in D$, satisfying conditions mentioned above is said to be a global solution to $SI(F,G,H)$.

A stochastic process $(x_t)_{t \geq 0} \in D$ is a local solution to $SI(F,G,H)$ on $[\alpha,\beta]$ if and only if $x^{\alpha,\beta}$ is a global solution to $SI(F^{\alpha\beta},G^{\alpha\beta},H^{\alpha\beta})$, where $F^{\alpha\beta} = \mathbb{I}_{[\alpha,b]}F$, $G^{\alpha\beta} = \mathbb{I}_{[\alpha,\beta]}G$ and $H^{\alpha\beta} = \mathbb{I}_{[\alpha,\beta]}H$.

A stochastic process $(x_t)_{t \geq 0} \in D$ is called a global (local on $[\alpha,\beta]$, resp.) solution to an initial value problem for stochastic inclusion $SI(F,G,H)$ with an initial condition $y \in L^2(\Omega, \mathfrak{F}_0, \mathbb{R}^n)$ $(y \in \mathfrak{F}_\alpha, \mathbb{R}^n)$, resp.) if $(x_t)_{t \geq 0}$ is a global (local on $[\alpha,\beta]$, resp.) solution to $SI(F,G<h)$ and $x_0 = y$ ($x_\alpha = y$, resp.). An initial-value problem for $SI(F,G,H)$ mentioned above will be denoted by $SI_y(F,G,H)$ ($S_y^{\alpha,\beta}(F,G,H)$, resp.). In what follows, we denote a set of all global (local on $[\alpha,\beta]$ solutions to $SI_y(F,G,H)$ by $\Lambda_y(F,G,H)$ ($\Lambda_y^{\alpha,\beta}(F,G,H)$, resp.).

Suppose F,G and H satisfy the following conditions:

$(\mathcal{A}_1)$ (*i*) $F = \{(F_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$ are such that mappings $\mathbb{R}^+ \times \Omega \times \mathbb{R}^n \ni (t,\omega,x) \to F_t(x)(\omega) \in Comp(\mathbb{R}^n)$, $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n \ni (t,\omega,x) \to G_t(x)(\omega) \in Comp(\mathbb{R}^n)$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \ni (t,\omega,z,x) \to H_{t,z}(x)(\omega) \in Comp(\mathbb{R}^n)$ are $\Sigma \otimes \mathfrak{B}^n$ and $\tilde{\Sigma} \otimes \mathfrak{B}^n$-measurable, respectively, where Σ and $\tilde{\Sigma}$ are σ-algebras on $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$ defined above,

(*ii*) $(F_t(x))_{t \geq 0}$, $(G_t(x))_{t \geq 0}$ and $(H_{x,z}(x))_{t \geq 0, z \in \mathbb{R}^n}$ are square integrable bounded for fixed $x \in \mathbb{R}^n$.

Corollary 1: *For every $(x_t)_{t \geq 0} \in D$ and F,G,H satisfying $(\mathcal{A}_1)$ one has $F \circ x$, $G \circ x \in \mathcal{M}^2_{s-\nu}(\mathfrak{F}_t)$ and $H \circ x \in \mathcal{M}^2_{s-\nu}(\mathfrak{F}_t, q)$.* □

Now, define a linear mapping Φ on $\mathcal{M}^2(\mathfrak{F}_t) \times \mathcal{M}^2(\mathfrak{F}_t \times \mathcal{M}^2(\mathfrak{F}_t, q)$ by taking $\Phi(f,g,h) = \int_0^t f_\tau d\tau + \int_0^t g_\tau dw_\tau + \int_0^t \int_{\mathbb{R}^n} h_{\tau,z} \tilde{\nu}(d\tau, dz))_{t \geq 0}$ to each $(f,g,h) \in \mathcal{M}^2(\mathfrak{F}_t) \times \mathcal{M}^2(\mathfrak{F}_t) \times \mathcal{M}^2(\mathfrak{F}_t, q)$.
It is clear that Φ maps $\mathcal{L}_n^2 \times \mathcal{L}_n^2 \times \mathcal{W}_n^2$ into D.

In what follows, we shall deal with $F = \{(F_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$, $G = \{(G_t(x))_{t \geq 0}: x \in \mathbb{R}^n\}$ and $H = \{(H_{t,z}(x))_{t \geq 0, z \in \mathbb{R}^n}: x \in \mathbb{R}^n\}$ satisfying conditions $(\mathcal{A}_1)$ and any one of the following conditions.

$(\mathcal{A}_2)$ There are $k, \ell \in \mathcal{L}_1^2$ and $m \in \mathcal{W}_1^2$ such that $\| \int_0^\infty h[(F \circ x)_t, (F \circ y)_t] dt \|_{\mathcal{L}_1^2} \leq E\int_0^\infty k_t \mid x_t - y_t \mid dt$, $\| h(G \circ x, G \circ y) \|_{\mathcal{L}_1^2} \leq E\int_0^\infty \ell_t \mid x_t - y_t \mid dt$ and $\| h(H \circ x, H \circ y) \|_{\mathcal{W}_1^2} \leq E\int_0^\infty \int_{\mathbb{R}^n} m_{t,z} \mid x_t - y_t \mid dt q(dz)$ for all $x, y \in D$.

$(\mathcal{A}_3)$ There are $k, \ell \in L^2(\mathfrak{B}_+)$ and $m \in L^2(\mathfrak{B}_+ \times \mathfrak{B}^n)$ such that $h(F_t(x_2)(\omega), F_t(x_1)(\omega)) \leq k(t) \mid x_1 - x_2 \mid$, $h(G_t(x_2)(\omega), G_t(x_1)(\omega)) \leq \ell(t) \mid x_1 - x_2 \mid$ and $h(H_{t,z}(x_2)(\omega), H_{t,z}(x_1)(\omega)) \leq m(t,z) \mid x_1 - x_2 \mid$ *a.e.*, each $t \geq 0$ and x_1, $x_2 \in \mathbb{R}^n$.

Lemma 1: *Let $\varphi \in L^2(\Omega, \mathfrak{F}_0, \mathbb{R}^n)$. Suppose F,G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ or $(\mathcal{A}_3)$. Let $x^n = \varphi + \Phi(f^{n-1}, g^{n-1}, h^{n-1})$, each $n = 1,2,\ldots$, with $(f^0, g^0, h^0) \in \mathcal{I}(F \circ 0) \times \mathcal{I}(G \circ 0) \times \mathcal{I}_q(H \circ 0)$ and $(f^n, g^n, h^n) \in \mathcal{I}(F \circ x^n) \times \mathcal{I}(G \circ x^n) \times \mathcal{I}_q(H \circ x^n)$ satisfying $\mid f_t^{n-1}(\omega) - f_t^n(\omega) \mid = dist(f_t^{n-1}(\omega), (F \circ x^n)_t(\omega))$, $\mid g_t^{n-1}(\omega) - g_t^n(\omega) \mid = dist(g_t^{n-1}(\omega), (G \circ x^n)_t(\omega))$ and $\mid h_{t,z}^{n-1}(\omega) - h_{t,z}^n(\omega) \mid = dist(h_{t,z}^{n-1}(\omega), (H \circ x^n)_{t,z}(\omega))$, on $\mathbb{R}_+ \times \Omega$ and $\mathbb{R}_+ \times \Omega \times \mathbb{R}^n$, respectively. If $L := \| \int_0^\infty k_t dt \|_{L_1^2} + 2 \| \int_0^\infty \ell_t dt \|_{L_1^2} + 2 \| \int_0^\infty \int_{\mathbb{R}^n} m_{\tau,z} d\tau q(dz) \|_{L_1^2} < 1$ or $L' := \mid k \mid_1 + 2 \mid \ell \mid_2 + 2 \| m \|_2 < 1$, respectively then $(x^n)_{n=1}^\infty$ is a Cauchy sequence of $(D, \| \cdot \|_\ell)$.*

Proof: Let $(x^n)_{n=1}^{\infty}$ be such as above. By $(\mathcal{A}_2)$ it follows

$$E\left[\sup_{t\geq 0}\left|\int_0^t (f_\tau^n - f_\tau^{n-1})d\tau\right|\right]^2 \leq E\left[\int_0^\infty |f_\tau^n - f_\tau^{n-1}|\, d\tau\right]^2$$

$$\leq E\left[\int_0^\infty \bar{h}((F\circ x^n)_\tau, (F\circ x^{n-1})_\tau)d\tau\right]^2 \leq \left(E\int_0^\infty k_\tau\, |x_\tau^n - x_\tau^{n-1}|\, d\tau\right)^2$$

$$\leq \left[E\left(\sup_{t\geq 0} |x_t^n - x_t^{n-1}| \cdot \int_0^\infty k_\tau d\tau\right)\right]^2 \leq E\left(\int_0^\infty k_\tau d\tau\right)^2 \cdot \| x^n - x^{n-1}\|_{\ell}^2.$$

Similarly, by Doob's inequality, we obtain

$$E\left[\sup_{t\geq 0}\left|\int_0^t (g_\tau^n - g_\tau^{n-1})dw_\tau\right|\right]^2 \leq 4E\int_0^\infty |g_\tau^n - g_\tau^{n-1}|^2 d\tau$$

$$\leq 4E\int_0^\infty [\bar{h}((G\circ x^n)_\tau, (G\circ x^{n-1})_\tau)]^2 d\tau \leq 4\left(E\int_0^\infty \ell_\tau\, |x_\tau^n - x_\tau^{n-1}|\, d\tau\right)^2$$

$$\leq 4\left[E\left(\sup_{t\geq 0} |x_t^n - x_t^{n-1}| \cdot \int_0^\infty \ell_\tau d\tau\right)\right]^2 \leq 4E\left(\int_0^\infty \ell_\tau d\tau\right)^2 \cdot \| x^n - x^{n-1}\|_{\ell}^2.$$

Quite similarly we also get

$$E\left[\sup_{t\geq 0}\left|\int_0^t \int_{\mathbb{R}^n} (h_\tau^n - h_{\tau,z}^{n-1})\tilde{\nu}(d\tau, dz)\right|\right]^2$$

$$\leq 4E\left(\int_0^\infty \int_{\mathbb{R}^n} m_{\tau,z} d\tau q(dz)\right)^2 \cdot \| x_\tau^n - x_\tau^{n-1}\|_{\ell}^2.$$

Therefore, $\| x^{n+1} - x^n \|_\ell \leq L^n \| x^1 \|_\ell$, where L is such as above. This implies that

$$\| x^m - x^n \|_\ell \leq \frac{L^n \cdot \| x^1 \|_\ell}{1-L},$$

each $m > n \geq 1$. Using conditions $(\mathcal{A}_3)$ instead of $(\mathcal{A}_2)$ we also get

$$\| x^m - x^n \|_\ell \leq \frac{(L')^n \cdot \| x^1 \|_\ell}{1-L'},$$

for $m > n \geq 1$. Therefore, $\| x^m - x^n \|_\ell \to 0$ as $n\to\infty$. □

Lemma 2: *Let $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. Suppose F, G and H satisfy $(\mathcal{A}_1)$ and $(\mathcal{A}_3)$. IF $L: = |k|_1 + 2|\ell|_2 + 2\|m\|_2 < 1$, then $\Lambda_\varphi(F,G,H) \neq \emptyset$.*

Proof: Let $(x^n)_{n=1}^\infty$ be such as in Lemma 1 and let $x = \lim_{n\to\infty} x^n$. The existence of such a sequence follows immediately from the measurable selection theorem given in [3] (see Th. II, 3.13). We shall now show that $(f^n)_{n=1}^\infty$, $(g^n)_{n=1}^\infty$ and $(h^n)_{n=1}^\infty$ are Cauchy sequences of $\mathcal{L}_n^2$ and $\mathcal{W}_n^2$, respectively. Indeed, one obtains

$$\| f^m - f^n \|_{\mathcal{L}_n^2} = \sum_{j=n+1}^{m} [\| f^j - f^{j-1} \|_{\mathcal{L}_n^2}]^{1/2}$$

$$\leq \sum_{j=n+1}^{m} [E \int_0^\infty h^2((F \circ x^j)_\tau, (F \circ x^{j-1})_\tau) d\tau]^{1/2}$$

$$\leq \sum_{j=n+1}^{m} |k|_2 \| x^j - x^{j-1} \|_\ell \leq \sum_{j=n+1}^{m} L^{j-1} |k|_2 \| x^1 \|_\ell \leq \frac{L^n |k|_2 \| x^1 \|_\ell}{1-L}.$$

Therefore, $(f^n)_{n=1}^\infty$ is a Cauchy sequence of $\mathcal{L}_n^2$. Quite similarly, it also follows that $(g^n)_{n=1}^\infty$ and $(h^n)_{n=1}^\infty$ are Cauchy sequences of $\mathcal{L}_n^2$ and $\mathcal{W}_n^2$, respectively. Let $f, g \in \mathcal{L}_n^2$ and $h \in \mathcal{W}_n^2$ be such that $\| f^n - f \|_{\mathcal{L}_n^2} \to 0$, $\| g^n - g \|_{\mathcal{L}_n^2} \to 0$ and $| h^n - h |_{\mathcal{W}_n^2} \to 0$ as $n \to \infty$. One gets $\| x^n - \varphi - \Phi(f,g,h) \|_\ell \to 0$ as $n \to \infty$. Therefore, $x = \varphi + \Phi(f,g,h)$. To prove that

$$x_t - x_s \in \int_s^t (F \circ x)_\tau d\tau + \int_s^t (G \circ x)_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} (H \circ x)_{\tau, z} \tilde{\nu}\,(d\tau, dz)$$

for every $0 \leq s < t < \infty$ it suffices only to verify that $(f,g,h) \in \mathcal{S}(F \circ x) \times \mathcal{S}(G \circ x) \times \mathcal{S}_q(H \circ x)$. For this aim, denote by $Dist(a, B)$ and $\bar{H}$ the distance of $a \in \mathcal{L}_n^2$ to a nonempty set $B \subset \mathcal{L}_n^2$ and the Hausdorff subdistance, respectively induced by the norm of $\mathcal{L}_n^2$. Now let v be a fixed element of $\mathcal{S}(F \circ x^n)$. Select $u \in \mathcal{S}(F \circ x)$ such that $| v_\tau(\omega) - u_\tau(\omega) | = dist(v_\tau(\omega), (F \circ x)_\tau(\omega))$ for $(\tau, \omega) \in \mathbb{R}_+ \times \Omega$. Then

$$Dist(v, \mathcal{S}(F \circ x)) \leq \| v - u \|_{\mathcal{L}_n^2}$$

$$\leq \left(E \int_0^\infty h^2((F \circ x^n)_\tau(\omega)),\ (F \circ x)_\tau(\omega)) d\tau \right)^{\frac{1}{2}} \leq |k|_2 \| x^n - x \|_\ell,$$

which implies $\bar{H}(\mathcal{S}(F \circ x^n),\ \mathcal{S}(F \circ x)) \leq |k|_2 \| x^n - x \|_\ell$, each $n = 1, 2, \ldots$. Thus $\bar{H}(\mathcal{S}(F \circ x^n),$ $\mathcal{S}(F \circ px)) \to 0$ as $n \to \infty$. In a similar way we also get $\bar{H}(\mathcal{S}(G \circ x^n),\ \mathcal{S}(G \circ x) \to 0$ and $\bar{H}(\mathcal{S}_q(H \circ x^n), \mathcal{S}_q(H \circ x)) \to 0$ as $n \to \infty$. Now we get

$$Dist(f, \mathcal{S}(F \circ x)) \leq \| f - f_n \|_{\mathcal{L}_n^2} + Dist(f_n, \mathcal{S}(F \circ x^{n-1}))$$
$$+ \bar{H}(\mathcal{S}(F \circ x^{n-1}), \mathcal{S}(F \circ x))$$

for $n = 1, 2, \ldots$, which implies that $Dist(f, \mathcal{S}(F \circ x)) = 0$. But, $\mathcal{S}(F \circ x)$ is a nonempty closed subset of $\mathcal{L}_n^2$. Therefore, $f \in \mathcal{S}(F \circ x)$. In a similar way we can also verify that $g \in \mathcal{S}(G \circ x)$ and $h \in \mathcal{S}_q(H \circ x)$. □

Lemma 3: *Let* $0 \leq \alpha < \beta < \infty$ *and* $\varphi \in L^2(\Omega, \mathcal{F}_\alpha, \mathbb{R}^n)$. *Suppose* F, G, *and* H *satisfy* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_3)$. *If* $L_{\alpha,\beta} := |\mathbb{I}_{[\alpha,\beta]}k|_1 + 2|\mathbb{I}_{[\alpha,\beta]}\ell|_2 + 2\|\mathbb{I}_{[\alpha,\beta]}m\|_2 < 1$ *then* $\Lambda_\varphi^{\alpha,\beta}(F,G,H) \neq \emptyset$.

Proof: The proof follows immediately from Lemma 2 applied to $F^{\alpha\beta} = \mathbb{I}_{[\alpha,\beta]}F$, $G^{\alpha\beta} = \mathbb{I}_{[\alpha,\beta}G$ and $H^{\alpha\beta} = \mathbb{I}_{[\alpha,\beta]}H$. □

Lemma 4: *Let* $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$ *and let* $(\tau_n)_{n=1}^\infty$ *be a sequence of positive number increasing to* $+\infty$. *Suppose* F, G *and* H *satisfy* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_3)$. *If* $x^1 \in \Lambda_\varphi^{0,\tau_1}(F,G,H)$ *and* $x^{n+1} \in \Lambda_{x^n_{\tau_n}}^{\tau_n,\tau_{n+1}}(F,G,H)$ *for* $n = 1,2,\ldots$, *then* $x = \sum_{n=1}^\infty \mathbb{I}_{[\tau_{n-1},\tau_n)}x^n$ *belongs to* $\Lambda_\varphi(F,G,H)$, *where* $\tau_0 = 0$.

Proof: It is clear that $x_0 = \varphi$ because $x_0 = x_0^1 = \varphi$. Let $0 \leq s < t < \infty$ be fixed and suppose $s \in [\tau_{k-1}, \tau_k)$, and $t \in [\tau_{m-1}, \tau_m)$, for $1 \leq k < m$. One obtains

$$x_t - x_s = (x_t^m - x^m_{\tau_{m-1}}) + (x^{m-1}_{\tau_{m-1}} - x^{m-1}_{\tau_{m-2}}) + \ldots + (x^{k+1}_{\tau_{k=1}} - x^{k+1}_{\tau_k}) + (x^k_{\tau_k} - x^k_s).$$

Let $(f^j, g^j, h^j) \in S(F \circ x^j) \times S(G \circ x^j) \times S_q(H \circ x^j)$, each $j = k, k+1, \ldots, m$ be such that

$$x_t^m - x^m_{\tau_{m-1}} = \int_{\tau_{m-1}}^t f_\tau^m d\tau + \int_{\tau_{m-1}}^t g_\tau^m dw_\tau + \int_{\tau_{m-1}}^t \int_{\mathbb{R}^n} h_\tau^m \widetilde{\nu}\,(d\tau, dz),$$

$$x^j_{\tau_j} - x^j_{\tau_{j-1}} = \int_{\tau_{j-1}}^{\tau_j} f_\tau^j d\tau + \int_{\tau_{j-1}}^{\tau_j} g_\tau^j dw_\tau + \int_{\tau_{j-1}}^{\tau_j} \int_{\mathbb{R}^n} h_\tau^j \widetilde{\nu}\,(dt, dz),$$

each $j = k+1, \ldots, m-1$, and

$$x^k_{\tau_k} - x^k_s = \int_s^{\tau_k} f_\tau^k d\tau + \int_s^{\tau_k} g_\tau^k dw_\tau + \int_s^{\tau_k} \int_{\mathbb{R}^n} h_\tau^k \widetilde{\nu}\,(d\tau, dz).$$

Let $f = \mathbb{I}_{[0,\tau_{k-1})}f^k + \sum_{j=k}^m \mathbb{I}_{[\tau_{j-1},\tau_j)}f^j + \mathbb{I}_{[\tau_m,\infty)}f^m$, $g = \mathbb{I}_{[0,\tau_{k-1})}g^k + \sum_{j=k}^m \mathbb{I}_{[\tau_{j-1},\tau_j)}g^j + \mathbb{I}_{[\tau_m,\infty)}g^m$ and $h = \mathbb{I}_{[0,\tau_{k-1})}h^k + \sum_{j=k}^m \mathbb{I}_{\tau_{j-1},\tau_j)}h^j + \mathbb{I}_{[\tau_m,\infty)}h^m$. It is clear that $(f,g,h) \in S(F \circ x) \times S(Goc) \times S_q(H \circ x)$ and $x_t - x_s = \int_s^t f_\tau d\tau + \int_s^t g_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} h_{\tau,z}\widetilde{\nu}\,(d\tau, dz)$. Therefore

$$x_t - x_s \in \int_s^t (F \circ x)_\tau d\tau + \int_s^t (G \circ x)_\tau dw_\tau + \int_s^t \int_{\mathbb{R}^n} (H \circ x)_{\tau,z} \widetilde{\nu}\,(d\tau, dz).$$ □

We can prove now the main result of this paper.

Theorem 5: *Let* $\varphi \in L^2(\Omega, \mathcal{F}_0, \mathbb{R}^n)$. *Suppose* F, G *and* H *satisfy* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_3)$. *Then* $\Lambda_\varphi(F,G,H) \neq \emptyset$.

Proof: Let $(\tau_n)_{n=1}^\infty$ be a sequence of positive numbers increasing to ∞. Select a positive number σ such that $L_{k\sigma,(k1)\sigma} < 1$ for $k = 0,1,\ldots$, where $L_{k\sigma,(k+1)\sigma}$ is such as in Lemma 3. Suppose a positive integer n_1 is such that $n_1\sigma < \tau_1 \leq (n_1+1)\sigma$. By virtue of Lemma 3, there is $z^1 \in \Lambda_\varphi^{0,\sigma}(F,G,H)$. By the same argument, there is $z^2 \in \Lambda_{z_\sigma^1}^{\sigma,2\sigma}(F,G,H)$. Continuing the above procedure we can finally find a $z^{n_1+1} \in \Lambda_{z_{n_1\sigma}^{n_1}}^{n_1\sigma,\tau_1}(F,G,H)$. Put

$$x^1 = \sum_{k=0}^{n_1-1} \mathbb{I}_{[k\sigma,(k+1)\sigma)} z^{k+1} + \mathbb{I}_{[n_1\sigma,\tau_1]} z^{n_1+1} + \mathbb{I}_{(\tau_1,\infty)} z_{\tau_1}^{n_1+1}.$$

Similarly as in the proof of Lemma 4, we can easily verity that $x^1 \in \Lambda_\varphi^{0,\tau_1}(F,G,H)$. Repeating the above procedure to the interval $[\tau_1,\tau_2]$, we can find $x^2 \in \Lambda_{x_{\tau_1}^1}^{\tau_1,\tau_2}(F,G,H)$. Continuing this process, we can define a sequence (x^n) of D satisfying conditions of Lemma 4. Therefore $\Lambda_\varphi(F,G,H) \neq \emptyset$. □

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