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INITIAL AND BOUNDARY VALUE PROBLEMS FOR FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, we study initial and boundary value problems for functional integro-differential equations, by using the Leray-Schauder Alternative.

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AMS (MOS) subject classifications: 34K05, 34K10

1. Introduction

The purpose of this paper is to study the existence of solutions for initial and boundary value problem (IVP and BVP, for short) for functional integro-differential equations. The paper is divided into two parts.

In Section 2 we consider the following IVP for nonlinear Volterra type integro-differential equations

$$x'(t) = A(t, x_t) + \int_0^t k(t,s)f(s, x_s)ds, \quad t \in [0,T] \tag{1.1}$$

$$x_0 = \phi, \tag{1.2}$$

where $A, f: [0,T] \times C \to R^n$ are continuous functions, and for $t \in [0,T]$, $A(t, \cdot)$ is a bounded linear operator from C to R^n, and k is a measurable for $t \geq s \geq 0$ real valued function. Here $C = C([-r,0], R^n)$ is the Banach space of all continuous functions $\phi: [-r,0] \to R^n$ endowed with the sup-norm

$$\| \phi \| = sup\{ | \phi(\theta) | : -r \leq \theta \leq 0\}.$$

Also, for $x \in C([-r,T], R^n)$ we have $x_t \in C$ for $t \in [0,T]$, $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$ and $\phi \in C$.

The results of this section generalize recent results of Ntouyas and Tsamatos [5] when the following degenerate case

$$x'(t) = A(t)x(t) + \int_0^t k(t,s)f(s,x_s)ds, \quad t \in [0,T] \tag{1.1)'$$

$$x_0 = \phi \tag{1.2)'$$

is studied. In Section 3 we study the following BVP for nonlinear Volterra integro-differential equations

$$x'(t) = A(t,x_t) + \int_0^t k(t,s)f(s,x_s)ds, \quad t \in [0,T] \tag{1.3}$$

$$Lx = h, \tag{1.4}$$

where A, f and k are as above and L is a bounded linear operator from a Banach space $C([-r,T],R^n)$ into R^n and $h \in ImL$, the image of L. The results of this section extend previous results on BVP for functional differential equations [2], [3], [4], and [7] to functional integro-differential equations.

2. IVP for Volterra Functional Integro-Differential Equations

In this section we consider the following initial value problem

$$x'(t) = A(t,x_t) + \int_0^t k(t,s)f(t,x_t)ds, \quad 0 \leq t \leq T \tag{2.1}$$

$$x_0 = \phi. \tag{2.2}$$

Before stating our basic existence theorems, we need the following lemma which is an immediate consequence of the Topological Transversality Theorem of Granas [1], known as **"Leray-Schauder alternative"**.

Lemma 2.1: *Let S be a convex subset of a normed linear space E and assume $0 \in S$. Let $F: S \rightarrow S$ be a completely continuous operator, i.e., it is continuous and the image of any bounded set is included in a compact set, and let*

$$E(F) = \{x \in S : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then, either $E(F)$ is unbounded or F has a fixed point.

For the IVP (2.1)-(2.2) we have the following existence theorem.

Theorem 2.2: *Let $f: [0,T] \times C \rightarrow R^n$ be a completely continuous function (i.e., it is continuous and takes closed bounded sets of $[0,T] \times C$ into bounded sets of R^n). Suppose that:*

(HA) There exists a nonnegative integrable function p on $[0,T]$ such that $| A(t,\phi) \leq p(t) \| \phi \|, (t,\phi) \in [0,T] \times C$.

(Hk) There exists a constant M such that $| k(t,s) | \leq M$, $t \geq s \geq 0$.

Also we assume that there exists a constant K such that

$$\| x \|_1 \leq K,$$

for each solution x of

$$x'(t) = \lambda A(t, x_t) + \lambda \int_0^t k(t,s) f(t, x_t) ds, \quad 0 \le t \le T \tag{2.1$_\lambda$}$$

$$x_0 = \phi \tag{2.2}$$

for any $\lambda \in (0,1)$.

Then the initial value problem (2.1)-(2.2) *has at least one solution on* $[-r, T]$.

Proof: We will rewrite (2.1) as follows. For $\phi \in C$ define $\tilde{\phi} \in B$, $B = C([-r,T], R^n)$ by

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & -r \le t \le 0 \\ \phi(0), & 0 \le t \le T. \end{cases}$$

If $x(t) = y(t) + \tilde{\phi}(t)$, $t \in [-r, T]$ it is easy to verify that y satisfies

$$y_0 = 0,$$

$$y(t) = \int_0^t A(s, y_s + \tilde{\phi})ds + \int_0^t \int_0^\tau k(t,s) f(s, y_s + \tilde{\phi}_s) ds d\tau, \quad 0 \le t \le T$$

if and only if x satisfies

$$x(t) = \phi(0) + \int_0^t A(s, x_s)ds + \int_0^t \int_0^\tau k(t,s) f(s, x_s) ds d\tau, \quad 0 \le t \le T$$

and $x_0 = \phi$.

Define $N: B_0 \to B_0$, $B_0 = \{y \in B: y_0 = 0\}$ by

$$Ny(t) = \begin{cases} 0, & -h \le t \le 0 \\ \int_0^t A(s, y_s + \tilde{\phi}_s)ds + \int_0^t \int_0^\tau k(t,s) f(s, y_s + \tilde{\phi}_s) ds d\tau, & 0 \le t \le T. \end{cases}$$

N is clearly continuous. We shall prove that N is completely continuous.

Let $\{h_\nu\}$ be a bounded sequence in B_0, i.e.,

$$\| h_\nu \| \le b, \text{ for all } \nu,$$

where b is a positive constant. We obviously have $\| h_{\nu t} \| \le b$, $t \in [0,T]$, for all ν. Hence we obtain

$$\| Nh_\nu \| \le p_0(b + \| \phi \|) + MM_0 m_0,$$

where

$$p_0 = \int_0^T p(t)dt,$$

$$M_0 = sup\{ \mid f(t,u) \mid : t \in [0,T], \ \| u \| \le b + \| \phi \|)\}.$$

and

$$m_0 = \int_0^T \int_0^\tau p(t) dt d\tau.$$

This means that $\{Nh_\nu\}$ is uniformly bounded.

Moreover, the sequence $\{Nh_\nu\}$ is equicontinuous, since for $t_1, t_2 \in [-r, T]$ we have

$$\mid Nh_\nu(t_1) - Nh_\nu(t_2) \le [p_0(b + \| \phi \|) + MM_0 m_0] \mid t_1 - t_2 \mid .$$

Thus, by the Arzela-Ascoli theorem, the operator N is completely continuous.

Finally, the set $E(N) = \{y \in B_0 : y = \lambda N y, \ \lambda \in (0,1)\}$ is bounded by assumption, since $\| x \|_1 \le K$ implies

$$\| y \|_1 \le K + \| \phi \| .$$

Consequently, by Lemma 2.1, the operator N has a fixed point y^* in B_0. Then $x^* = y^* + \widetilde{\phi}$ is a solution of the IVP (2.1)-(2.2). This proves the theorem.

The applicability of Theorem 2.1 depends upon the existence of a priori bounds for the solutions of the initial value problem $(2.1)_\lambda$-(2.2), which are independent of λ. Conditions on f which imply the desired a priori bounds are given in the following:

Theorem 2.3: *Assume that (HA) and (Hk) hold. Also assume that*

(Hf) *There exists a continuous function m such that $\mid f(t,\phi) \mid \le m(t)\Omega(\| \phi \|)$, $0 \le t \le T, \phi \in C$ where Ω is a continuous nondecreasing function defined on $[0,\infty)$ and positive on $(0,\infty)$.*

Then, the initial value problem (2.1)-(2.2) has a solution on $[-r,T]$ provided

$$\int_0^T m_1(s) ds < \int_{\| \phi \|}^{\infty} \frac{ds}{s + \Omega(s)}, \ m_1(t) = sup\{1, p(t), Mm(t)\}.$$

Proof: To prove the existence of a solution of the IVP (2.1)-(2.2), we apply Theorem 2.1. In order to apply this theorem, we must establish the a priori bounds for the solutions of the IVP $(2.1)_\lambda$-(2.2). Let x be a solution of $(2.1)_\lambda$. From

$$x(t) = \phi(0) + \lambda \int_0^t A(s, x_s) ds + \lambda \int_0^t \int_0^\tau k(t,s) f(s, x_s) ds d\tau, \ 0 \le t \le T$$

we have

$$\mid x(t) \mid \ \le \ \mid \phi(0) \mid \ + \lambda \int_0^t \mid A(s, x_x) \mid ds + \int_0^t \int_0^\tau \mid k(t,s) \mid \ \mid f(s, x_s) \mid ds d\tau, \ 0 \le t \le T,$$

from which, by (HA), (Hf), and (Hk), we get

$$\mid x(t) \mid \ \le \ \| \phi \| \ + \int_0^t p(s) \| x_s \| ds + M \int_0^t \int_0^\tau m(s) \Omega(\| x_s \|) ds d\tau.$$

We consider the function μ given by

$$\mu(t) = sup\{\, |x(s)| : \ -r \le s \le t\}, \ \ 0 \le t \le T.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0, t]$, by the previous inequality we have

$$\mu(t) \le \|\phi\| + \int_0^t p(s)\mu(s)ds + M\int_0^t\int_0^\tau m(s)\Omega(\mu(s))dsd\tau, \ 0 \le t \le T \tag{2.3}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and (2.3) obviously holds.

Denoting by $u(t)$ the right-hand side of (2.3) we have

$$\mu(t) \le u(t), \ \ 0 \le t \le T,$$

$$u(0) = \|\phi\|,$$

and

$$\begin{aligned} u'(t) &= p(t)\mu(t) + M\int_0^t m(s)\Omega(\mu(s))ds \\ &\le p(t)u(t) + M\int_0^t m(s)\Omega(u(s))ds \\ &\le m_1(t)[u(t) + \int_0^t \Omega(u(s))ds], \ \ 0 \le t \le T. \end{aligned}$$

Let

$$v(t) = u(t) + \int_0^t \Omega(u(s))ds, \ \ 0 \le t \le T.$$

Then

$$v(0) = u(0), \ \ u(t) \le v(t), \ \ u'(t) \le m_1(t)v(t), \ \ 0 \le t \le T$$

and

$$\begin{aligned} v'(t) &= u'(t) + \Omega(u(t)) \\ &\le m_1(t)v(t) + \Omega(v(t)) \\ &\le m_1(t)[v(t) + \Omega(v(t))], \ \ 0 \le t \le T \end{aligned}$$

or

$$\frac{v'(t)}{v(t) + \Omega(v(t))} \le m_1(t), \quad 0 \le t \le T.$$

This implies

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \Omega(s)} \le \int_0^T m_1(t)dt < \int_{v(0)}^{\infty} \frac{ds}{s + \Omega(s)}, \ \ 0 \le t \le T.$$

This inequality implies that there is a constant K such that $u(t) \leq K$, $t \in [0,T]$, and hence $\mu(t) \leq K$, $t \in [0,T]$. Therefore,

$$\| x \|_1 \leq K, \tag{2.4}$$

and the proof of the theorem is complete.

By applying Theorem 2.3, we have the following result which concerns the global existence of solutions for the IVP (1.1)-(1.2). The proof is omitted since it is similar to that of Theorem 2.3 of [5].

Theorem 2.4: *Assume that* (HA) *and* (Hk) *hold. Also assume that*

$(Hf)'$ *There exists a continuous function* m *such that* $|f(t,\phi)| \leq m(t)\Omega(\|\phi\|)$, $0 \leq t < \infty$, $\phi \in C$, *where* Ω *is a continuous nondecreasing function defined on* $[0,\infty)$ *and positive on* $(0,\infty)$,

and

$$\int^{\infty} \frac{ds}{s+\Omega(s)} = +\infty.$$

Then the initial value problem

$$x'(t) = A(t,x_t) + \int_0^t k(t,s)f(t,x_t)ds, \quad t \geq 0 \tag{2.1$'$}$$

$$x_0 = \phi \tag{2.2$'$}$$

has a solution defined on $[0,\infty)$.

Consider now the following special case of initial value problem (2.1)-(2.2), i.e.,

$$x'(t) = A(t)x(t) + \int_0^t k(t,s)f(t,x_t)ds, \quad 0 \leq t \leq T \tag{2.5}$$

$$x_0 = \phi, \tag{2.6}$$

where $A(t)$ is an $n \times n$ continuous matrix for $t \in [0,T]$ and f is a continuous mapping from $[0,T] \times C$ to R^n.

Any solution of this problem may be represented as follows:

$$x(t) = \Phi(t)\Phi^{-1}(t)\phi(0) + \int_0^t \Phi(t)\Phi^{-1}(t) \int_0^{\tau} k(t,s)f(s,x_s)dsd\tau, \quad 0 \leq t \leq T,$$

where $\Phi(t)$ is the fundamental matrix of solutions of the homogeneous system $x'(t) = A(t)x(t)$, $0 \leq t \leq T$. $\Phi(t)$ is extended to $[-r,0]$ by I, the identity matrix.

Let $M_1 = max\{sup\,|\Phi(t)\Phi^{-1}(t)| : t,s \in [0,T], 1\}$. Using this formula, we obtain the following theorem proved earlier in [5].

Theorem 2.5: *If* (Hf) *and* (Hk) *hold, then the initial value problem* (2.5)-(2.6) *has at least one solution on* $[-r,T]$, *provided that*

$$MM_1 \int_0^T \int_0^t m(s)dsdt < \int_{M_1 \|\phi\|}^{\infty} \frac{ds}{\Omega(s)}.$$

3. BVP For Volterra Functional Integro-Differential Equations

Consider in this section the following BVP for nonlinear Volterra type integro-differential equations

$$x'(t) = A(t, x_t) + \int_0^t k(t,s)f(s, x_s)ds, \quad t \in [0,T] \tag{3.1}$$

$$Lx = h, \tag{3.2}$$

where A, f and k are as in the previous section and L is a bounded linear operator from a Banach space $C([-r,T], R^n)$ into R^n and $h \in ImL$, is the image of L.

We will now introduce some necessary preliminaries. Consider a linear nonhomogeneous system of differential equations

$$x'(t) = A(t, x_t) + g(t) \tag{3.3}$$

$$x_0 = \phi \tag{3.4}$$

for which we assume that (HA) holds.

For any initial function $\phi \in C$ we denote by $x(\phi, g)(t)$, the solution of (3.3) satisfying $x(\phi, g) = \phi$. For each $\phi \in C$ and g as above, the initial value problem (3.3)-(3.4) has a unique solution $x(\phi, g)$ defined on $[-r, T]$ such that

$$x(\phi, g)(t) = x(\phi, 0)(t) + \int_0^t U(t,s)g(s)ds, \quad t \in [0,T], \tag{3.5}$$

where $U(t,s)$ is the fundamental matrix of $x'(t) = A(t, x_t)$. Denote by $|U(t,s)|$, the operator norm of the matrix $U(t,s)$ and set

$$P = sup\{ |U(t,s)| : 0 \le s, t \le T\}.$$

Set $S: C \rightarrow C([-r,T], R^n)$ be the solution mapping defined by

$$S\phi = x(\phi, 0).$$

Then S is a bounded linear operator and hence the composite mapping $L_S = LS$ is a bounded linear operator from C into R^n. We assume that

(HL) *There exists a bounded linear operator* $L_S^*: R^n \rightarrow C$ *such that* $L_S L_S^* L_S = L_s$.

Therefore L_S^* is the generalized inverse of L_S. Then any solution to the BVP (3.1)-(3.2) is a fixed point of the operator F with

$$Fx = F_1x + F_2x,$$

where

$$(F_1x)(t) = SL_S^*(h - LF_2x)(t), \quad -r \leq t \leq T, \tag{3.6}$$

and

$$F_2x)(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t \int_0^\tau U(t,s)k(t,s)f(s,x_s)dsd\tau, & 0 \leq t \leq T. \end{cases} \tag{3.7}$$

For a proof of this fact, the reader is referred to Kaminogo [4].

Now, we present our main result on the existence of solutions of the BVP (3.1)-(3.2).

Theorem 3.1: *Assume that (HA), (Hk), (Hf) and (HL) hold, then, if*

$$MP(|SL_S^*| \, |L| + 1)\int_0^T \int_0^\tau m(s)dsd\tau < \int_c^\infty \frac{ds}{\Omega(s)}, \; c = max\{|SL_S^*| \, |h|, \|\phi\|\}$$

the BVP (3.1)-(3.2) has at least one solution on $[-r,T]$.

Proof: To prove the existence of a solution of the BVP (3.1)-(3.2), we apply Lemma 2.1. In order to apply this lemma, we must establish the a priori bounds for the BVP $(3.1)_\lambda$-$(3.2)_\lambda$. Let x be a solution of the BVP $(3.1)_\lambda$-$(3.2)_\lambda$. Then,

$$x(t) = \lambda\{SL_S^*(h - LF_2x)(t) + (F_2x)(t)\}, \; t \in [0,T]$$

where $F_2(t)$ is given by (3.6). From this, we get

$$|x(t)| \leq |SL_S^*|(|h| + |L| P\int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsd\tau)$$

$$+ P\int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsd\tau$$

$$\leq |SL_S^*| \, |h| + P(|SL_S^*| \, |L| + 1)\int_0^t \int_0^\tau m(s)\Omega(\|x_s\|)dsd\tau, \; 0 \leq t \leq T.$$

As in Theorem 2.3, we consider the function μ given by

$$\mu(t) = sup\{|x(s)|: \; -r \leq s \leq t\}, \; 0 \leq t \leq T.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |x(t^*)|$. If $t^* \in [0,t]$, by the previous inequality we have

$$\mu(t) = |x(t^*)| \leq |SL_S^*| \, |h| + P(|SL_S^*| \, |L| + 1)\int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsd\tau$$

$$\leq c + P(|SL_S^*| \, |L| + 1)\int_0^t \int_0^\tau m(s)\Omega(\mu(s))dsd\tau,$$

where $c = max\{\mid SL_S^* \mid \mid h \mid, \| \phi \| \}$.

If $t^* \in [-r, 0]$, then $\mu(t) = \| \phi \|$ and the previous inequality obviously holds true.

Denoting by $u(t)$ the right-hand side of the above inequality, we have

$$\mu(t) \le u(t), \quad 0 \le t \le T,$$

$$u(0) = c,$$

and

$$u'(t) = P(\mid SL_S^* \mid \mid L \mid + 1) \int_0^t m(s)\Omega(\mu(s))ds$$

$$\le P(\mid SL_S^* \mid \mid L \mid + 1) \int_0^t m(s)\Omega(u(s))ds$$

$$\le P(\mid SL_S^* \mid \mid L \mid + 1)\Omega(u(t)) \int_0^t m(s)ds, \ \ 0 \le t \le T.$$

or

$$\frac{v'(t)}{\Omega(u(t))} \le P(\mid SL_S^* \mid \mid L \mid + 1) \int_0^t m(s)ds, \ \ 0 \le t \le T.$$

Then,

$$\int_{u(0)}^{u(t)} \frac{ds}{\Omega(s)} \le P(\mid SL_S^* \mid \mid L \mid + 1) \int_0^T \int_0^\tau m(s)dsd\tau < \int_{u(0)}^{\infty} \frac{ds}{\Omega(s)}, \ \ 0 \le t \le T.$$

This inequality implies that there is a constant K such that $u(t) \le K$, $t \in [0, T]$, and hence $\mu(t) \le K$, $t \in [0, T]$. Since for every $t \in [0, T]$, $\| x_t \| \le \mu(t)$, we have

$$\| x \|_1 \le K,$$

where K depends only on T and the functions m and Ω.

In the second step, we notice that any solution of the BVP (3.1)-(3.2) is a fixed point of the operator F with

$$Fx = SL_S^*(h - LF_2x) + F_2x$$

which is a completely continuous operator ([4]).

Finally, the set $E(F) = \{x \in B\colon \ x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$ is bounded, since in the first step we have proved that $\| x \|_1 \le K$.

Consequently, by Lemma 2.1, the BVP (3.1)-(3.2) has at least one solution, completing the proof of the theorem.

We shall now consider equation (3.1) when the linear part $A(t, x_t)$ is not a functional on C. More precisely, we shall consider the functional differential equation of the form

$$x'(t) = A(t)x(t) + \int_0^t k(t,s)f(s,x_s)ds, \quad t \in [0,T], \tag{3.8}$$

where $A(t)$ is a continuous $n \times n$ matrix for $t \in [0,T]$.

Let us assume that $\Phi(t)$ is the fundamental matrix of solutions of the homogeneous system

$$x'(t) = A(t)x(t), \qquad 0 \leq t \leq T \tag{3.9}$$

with $\Phi(0) = I$, the identity matrix. $\Phi(t)$ is extended to $[-r, 0]$ by I. We denote by L_0, the $n \times n$ matrix whose elements are the values of L on the corresponding columns of $\Phi(t)$. Assume that L_0 is nonsingular with inverse L_0^{-1}. Then it is well known (Opial [6]) that:

(I) The BVP (3.8)-(3.2) has a solution for any $h \in R^n$, if and only if, the corresponding homogeneous BVP

$$x'(t) = A(t)x(t)$$

$$Lx = 0$$

has only the trivial solution $x(t) = 0$.

(II) The solution of the BVP (3.8)-(3.2) is unique and is given by the explicit formula

$$x(t) = \Phi(t)L_0^{-1}(h - LF_2(t)) + F_2(t),$$

where

$$F_2(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \int_0^t \Phi(t) \int_0^\tau \Phi^{-1}(s)k(t,s)f(s,x_s)ds d\tau, & 0 \leq t \leq T. \end{cases}$$

Let

$$\alpha = sup\{ \mid \Phi(t) \mid : \ 0 \leq t \leq T\},$$

$$\beta = sup\{ \mid \Phi^{-1}(t) \mid : 0 \leq t \leq T\}.$$

Then we have:

Theorem 3.2: *Assume that (Hf) and (Hk) hold. Assume also that the linear operator L is such that the operator L_0 has a bounded inverse L_0^{-1}.*

Then if

$$\alpha\beta M(\alpha \mid L_0^{-1} \mid \mid L \mid + 1) \int_0^T \int_0^t m(s)ds < \int_c^\infty \frac{ds}{\Omega(s)}, \tag{3.10}$$

where $c = max\{\alpha \mid L_0^{-1} \mid \mid h \mid, \| \phi \| \}$, the BVP (3.8)-(3.2) has at least one solution.

Proof: The proof is similar to that of the previous theorem and it is omitted.

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