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ON THE LIMITING BEHAVIOR OF A HARMONIC OSCILLATOR WITH RANDOM EXTERNAL DISTURBANCE

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ABSTRACT

This paper deals with the limiting behavior of a harmonic oscillator under the external random disturbance that is a process of the white noise type. Influence of noises is investigated in resonance and non-resonance cases.

Key words: Harmonic Oscillator, Instantaneous Energy, Differential Equation of the Second Order, Itô Stochastic Differential Equation.

AMS (MOS) subject classifications: 60H10.

1. Introduction

We investigate the harmonic oscillator as a system of motion described by a linear differential equation of the second order

$$m\ddot{u}(t)+ku(t)=q(t) \text{ while } m>0 \text{ and } k>0,$$

where $q(t)$ is an external disturbance force. In the case, where $q(t)$ is a nonrandom periodic function, the instantaneous energy of the oscillator $\varepsilon(t)=\frac{1}{2}[ku^2(t)+m\dot{u}^2(t)]$ is bounded if the period of the function $q(t)$ is not equal to $2\pi\sqrt{m/k}$ and $\varepsilon(t)\sim t^2$ as $t\to\infty$ if period of function $q(t)$ is equal to $2\pi\sqrt{m/k}$ (resonance).

A model of the random harmonic oscillator with $\varepsilon(t)\sim t$ as $t\to\infty$ was considered by Papanicolau [8] for the case when $q(t)$ is a stationary random process; a model in which $\ln\varepsilon(t)\sim t\to\infty$ was considered by Bendersky and Pastur [1] for the case when $q(t)=0$ and $k=k(t)$ is a stationary random process; a model in which $\varepsilon(t)\sim\sqrt{t}$ as $t\to\infty$ was considered by Kulinich [7] for the case when $q(t)=g(w(t))\dot{w}(t)$, with $\dot{w}(t)$ as a "white" noise, $g(x)$ a nonrandom function and $g^2(x)$ integrable over $\mathbb{R}$.

In the present paper, we consider the external random disturbance of the type $q(t) = f(t)g(w(t))\dot{w}(t)$, where $f(t)$ and $g(x)$ are nonrandom functions and $f^2(t)$ is a periodic function with the period $2L$.

The limiting behavior (for $t\to\infty$) of the joint distribution of the random variables $(u(t),\dot{u}(t))$ the distribution of the random variable $\varepsilon(t)$ is investigated in the following cases:
1) $2L\neq 2\pi\sqrt{m/k}$; 2) $2L=2\pi\sqrt{m/k}$.

It is shown in particular that $\varepsilon(t)\sim t$ if $g^2(x)\sim b\neq 0$ as $|x|\to\infty$ (Theorem 1) and

$E\varepsilon(t) \sim t^{\frac{\alpha+1}{2}}$ if $g^2(x) \sim b(x) \mid x \mid^{\alpha - 1}$, while $\alpha > 0$ and $b(x) = b_1$ for $x > 0$, and $b(x) = b_2$ for $x < 0$ (corollary of Theorem 2).

Let $u(t)$ be the distance of a particle from its equilibrium position. We assume that the particle has mass m and that it is fastened to an immobile support by a spring with the coefficient of stiffness k. Then $u(t)$ satisfies the following equation:

$$m\ddot{u}(t) + ku(t) = q(t) \text{ while } u(0) = u_0 \text{ and } \dot{u}(0) = \dot{u}_0 \ (\dot{u} \equiv \frac{d}{dt}u). \tag{1}$$

Here $q(t)$ is an external force, u_0 is an initial position and $\dot{u}_0$ is an initial velocity of the particle. We assume, then, that $u_0 = 0$, $\dot{u}_0 = 0$ and $q(t) = f(t)g(w(t))\dot{w}(t)$, where $w(t)$ is a Wiener process. In this case, equation (1) can be considered as a system of stochastic Itô equations:

$$\begin{cases} md\dot{u}(t) = -ku(t)dt + f(t)g(w(t))dw(t) \\ du(t) = \dot{u}(t)dt \end{cases} \tag{2}$$

Lemma: *Let function $f(t)$ satisfy the condition, $\mid \int_0^t f(s)ds \mid \leq C$, for every finite t, and let $g(x)$ have the second derivative $g''(x)$ almost everywhere while $\int_0^x \mid g''(v) \mid dv = o(\mid x \mid^{\alpha})$ as $\mid x \mid \to \infty$ with $\alpha > 0$. Then,*

$$\lim_{t\to\infty} t^{-\frac{\alpha+1}{2}} E \mid \int_0^t f(s)g(w(s))ds \mid = 0,$$

where $w(t)$ is a Wiener process.

Proof: Since the functions $f(t)$ and $g''(x)$ are integrable over every bounded domain, because of Krylov [5], we can apply Itô's formula to the process $\Phi(t, w(t))$, where $\Phi(t,x) = \int_0^t f(s)dsg(x)$, and obtain

$$\int_0^t f(s)g(w(s))ds = \int_0^t f(s)dsg(w(t)) - \int_0^t [\int_0^s f(s_1)ds_1]g'(w(s))dw(s)$$

$$-\frac{1}{2}\int_0^t [\int_0^s f(s_1)ds_1]g''(w(s))ds = I_1(t) + I_2(t) + I_3(t).$$

It is easy to see that the following inequalities hold true:

$$t^{-\frac{\alpha+1}{2}} E \mid I_1(t) \mid \leq Ct^{-\frac{\alpha+1}{2}} E \mid g(w(t)) \mid$$

$$E(t^{-\frac{\alpha+1}{2}} I_2(t))^2 = t^{-(\alpha+1)} E \int_0^t [\int_0^s f(s_1)ds_1 g'(w(s))]^2 ds$$

$$\leq C^2 t^{-(\alpha+1)} E \int_0^t [g'(w(s))]^2 ds \tag{3}$$

$$t^{-\frac{\alpha+1}{2}} E \mid I_3(t) \mid \leq \frac{1}{2} Ct^{-\frac{\alpha+1}{2}} E \int_0^t \mid g''(w(s)) \mid ds.$$

Next, applying the Itô formula to the processes $\Phi(w(t))$ and $\Phi_1(w(t))$, where

$$\Phi(x) = 2\int_0^x [\int_0^z (g'(v))^2 dv]dz \text{ and } \Phi_1(x) = 2\int_0^x [\int_0^z \mid g''(v) \mid dv]dz,$$

we obtain the equations

$$t^{-(\alpha+1)}E\int_0^t [g'(w(s))]^2 ds = t^{-(\alpha+1)}E\Phi(w(t))$$

and (4)

$$t^{-\frac{\alpha+1}{2}}E\int_0^t \mid g''(w(s)) \mid ds = t^{-\frac{\alpha+1}{2}}E\Phi_1(w(t)).$$

The conditions of the Lemma require that $g(x) = o(\mid x \mid^{\alpha+1})$, $\Phi(x) = o(\mid x \mid^{2\alpha+1})$ and $\Phi_1(x) = o(\mid x \mid^{\alpha+1})$. When we take into account that $w(t)t^{-\frac{1}{2}}$ for every $t > 0$ is standard normal it is easy to ensure that $E\frac{\mid g(w(t)) \mid}{t^{\frac{\alpha+1}{2}}} \to 0$, $E\frac{\Phi(w(t)}{t^{(\alpha+1)}} \to 0$ and $E\frac{\Phi_1(w(t))}{t^{\frac{\alpha+1}{2}}} \to 0$ as $t \to \infty$. These convergences along with (3) and (4) yield the Lemma. □

In what follows, we assume that $f(t)$ in the equations is a continuously differentiable function and that $f^2(t)$ has period $2L$. Let us denote

$$a_0 = \frac{1}{4L}\int_0^{2L} f^2(t)dt, \quad c_0 = \frac{1}{4L}\int_0^{2L} f^2(t)\cos(2\sqrt{k/m}\ t)dt,$$

$$a_1 = a_0 + c_0, \quad a_2 = a_0 - c_0 \quad \text{and} \quad a_3 = \frac{1}{4L}\int_0^{2L} f^2(t)\sin(2\sqrt{k/m}\,t)dt.$$

Theorem 1: *Let the function $g(x)$ in equation (2) have a second derivative with*

$$\lim_{\mid x \mid \to \infty} \frac{1}{x}\int_0^x g^2(v)dv = b \quad \text{and} \quad \lim_{\mid x \mid \to \infty} \frac{1}{x}\int_0^x \mid g'(v) + g(v)g''(v) \mid dv = 0.$$

1. *Suppose $2L \neq n\pi\sqrt{m/k}$ for any $n = 1,2,\ldots$ or $2L = n_0\pi\sqrt{m/k}$ and at the same time, $c_0 = 0$ and $a_3 = 0$. Then the following hold:*

   a) The joint distribution of the random variables $(u(t)/\sqrt{t}, \dot{u}(t)/\sqrt{t})$, as $t \to \infty$, converges to the distribution of $(\sqrt{\frac{a_0 b}{km}}\zeta_1, \frac{\sqrt{a_0 b}}{m}\zeta_2)$, where ζ_1 and ζ_2 are independent standard normal random variables.

   b) The distribution of the random variable $t^{-1}\varepsilon(t)$, as $t \to \infty$, converges to the exponential distribution with the parameter $m(a_0 b)^{-1}$.

2. *Suppose $2L = n_0\pi\sqrt{m/k}$ and that $c_0 \neq 0$ or $a_3 \neq 0$. Then the following hold:*

   a) $P\{\frac{u(t)}{\sqrt{t}} < x_1, \frac{\dot{u}(t)}{\sqrt{t}} < x_2\} - F_t(x_1, x_2) \to 0$, where for each $t > 0$, $F_t(x_1, x_2)$ is bivariate normal with the density:

$$\rho_t(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\, exp\{-\frac{1}{2(1-r^2)}[Ax_1^2 - 2Bx_1x_2 + Cx_2^2]\}, \tag{5}$$

where

$$A = \frac{sin^2\alpha}{\sigma_1^2} + 2r\frac{sin\,\alpha\,cos\,\alpha}{\sigma_1\sigma_2} + \frac{cos^2\alpha}{\sigma_2^2}, \quad B = -\frac{sin\,\alpha\,cos\,\alpha}{\sigma_1^2} + r\frac{sin^2\alpha - cos^2\alpha}{\sigma_1\sigma_2} + \frac{sin\,\alpha\,cos\,\alpha}{\sigma_2^2},$$

$$C = \frac{cos^2\alpha}{\sigma_1^2} - 2r\frac{sin\,\alpha\,cos\,\alpha}{\sigma_1\sigma_2} + \frac{sin^2\alpha}{\sigma_2^2}, \quad r = \frac{a_3}{\sqrt{a_1 a_2}}, \ \sigma_1^2 = a_1 b, \ \sigma_2^2 = a_2 b \ \text{and}$$

$$\alpha = \sqrt{k/m}\; t.$$

b) The distribution of the random variable $t^{-1}\varepsilon(t)$ converges to the distribution with the density:

$$\rho(x) = \frac{m}{b\sqrt{a_1 a_2 - a_3^2}}\, exp\{-\frac{xm(a_1+a_2)}{2b(a_1a_2 - a_3^2)}\} \otimes$$

$$I_0\left(\frac{xm}{b(a_1a_2 - a_3^2)}\sqrt{\frac{1}{4}(a_1 - a_2)^2 + a_3^2}\right), \quad x > 0, \tag{6}$$

where $I_0(x)$ is the modified Bessel function of the first kind with zero index and $\rho(x) = 0$, when $x < 0$.

Proof: We can write the solution of equation (2) in explicit form [2]:

$$u(t) = \frac{1}{\sqrt{km}}\int_0^t f(s)g(w(s))\sin(\sqrt{k/m}(t-s))dw(s)$$

$$\dot{u}(t) = \frac{1}{m}\int_0^t f(s)g(w(s))\cos(\sqrt{k/m}(t-s))dw(s).$$

Let us introduce the parameter $T \geq T_0 > 0$ and denote

$$u_T(t) = u(tT)/\sqrt{T}, \dot{u}_T(t) = \dot{u}(tT)/\sqrt{T} \text{ and } w_T(t) = w(tT)/\sqrt{T}.$$

Then,

$$u_T(t) = \frac{1}{\sqrt{km}}\left[\gamma_T^{(1)}(t)\sin(\sqrt{k/m}tT) - \gamma_T^{(2)}(t)\cos(\sqrt{k/m}tT)\right]$$

and

$$\dot{u}_T(t) = \frac{1}{m}\left[\gamma_T^{(1)}(t)\cos(\sqrt{k/m}tT) + \gamma_T^{(2)}(t)\sin(\sqrt{k/m}tT)\right], \tag{7}$$

where

$$\gamma_T^{(1)}(t) = \int_0^t g(w_T(s)\sqrt{T})f(sT)\cos(\sqrt{k/m}sT)dw_T(s)$$

and

$$\gamma_T^{(2)}(t) = \int_0^t g(w_T(s)\sqrt{T})f(sT)\sin(\sqrt{k/m}sT)dw_T(s).$$

Since each process $\gamma_T^{(i)}(t)$ for $i = 1,2$ is a martingale with respect to the σ-algebra, $\sigma(w_T(s),$

$s \le t$), and since each satisfies the Skorohod condition of compactness of random processes [9], we can assume, without loss of generality, that $\gamma_T^{(i)}(t) \to \gamma^{(i)}(t)$ for $i = 1,2$ and $w_T(t) \to w(t)$ in probability as $T \to \infty$ at every point $t > 0$, where $w(t)$ is a Wiener process and each $\gamma^{(i)}(t)$ is a martingale with respect to the σ-algebra $\sigma(w(s), s \le t)$.

Thus, (7) implies the convergencies

$$u_T(t) - \frac{1}{\sqrt{km}}\left[\gamma^{(1)}(t)\sin(\sqrt{k/m}tT) - \gamma^{(2)}(t)\cos(\sqrt{k/m}tT)\right] \to 0$$

and (8)

$$\dot{u}_T(t) - \frac{1}{m}\left[\gamma^{(1)}(t)\cos(\sqrt{k/m}tT) + \gamma^{(2)}(t)\sin(\sqrt{k/m}tT)\right] \to 0,$$

in probability, as $T \to \infty$.

Consider now characteristics of martingales:

$$\langle \gamma_T^{(1)}(t) \rangle = \int_0^t g^2(w_T(s)\sqrt{T}) f^2(sT)\cos^2(\sqrt{k/m}sT)ds$$

$$\langle \gamma_T^{(2)}(t) \rangle = \int_0^t g^2(w_T(s)\sqrt{T}) f^2(sT)\sin^2(\sqrt{k/m}sT)ds$$

$$\langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t) \rangle = \frac{1}{2}\int_0^t g^2(w_T(s)\sqrt{T}) f^2(sT)\sin(2\sqrt{k/m}sT)ds.$$

Suppose that for the function $f^2(t)$ the first assumption of Theorem 1 is satisfied. It is easy to verify that, in this case,

$$f^2(t)\cos^2(\sqrt{k/m}t) = a_0 + \alpha_1(t), f^2(t)\sin^2(\sqrt{k/m}t) = a_0 + \alpha_2(t),$$

and (9)

$$\frac{1}{2}f^2(t)\sin(2\sqrt{k/m}t) = \alpha_3(t),$$

where $a_0 = \frac{1}{4L}\int_0^{2L} f^2(s)ds$, and there is a constant $C > 0$ that for all $t \ge 0$ satisfies the inequality,

$$\left| \int_0^t \alpha_i(s) \right| \le C, \quad i = 1,2,3.$$

Then $\langle \gamma_T^{(i)}(t) \rangle = a_0 \int_0^t g^2(w_T(s)\sqrt{T})ds + \int_0^t g^2(w_T(s)\sqrt{t})\alpha_i(sT)ds = I_T(t) + J_T(t)$.

Kulinich [6] implies $I_T(t) \to \beta(t)$ in probability as $T \to \infty$, where $\beta(t) = a_0 bt$, and due to the Lemma, $E \mid J_T(t) \mid \to 0$. Therefore, $\langle \gamma_T^{(i)}(t) \rangle \to a_0 bt$ in probability as $T \to \infty$ for $i = 1,2$. And for the joint characteristic of martingales $\gamma_T^{(1)}$ and $\gamma_T^{(2)}(t)$, we have the equality,

$$\langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t) \rangle = \int_0^t g^2(w_T(s)\sqrt{T})\alpha_3(sT)ds,$$

which, due to the Lemma, implies the convergence,

$$E \mid \langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t) \rangle \mid \to 0 \text{ as } t \to \infty.$$

Hence, for characteristics of the limit martingales we have

$$\langle\gamma^{(i)}(t)\rangle = a_0bt, \quad i=1,2 \text{ and } \langle\gamma^{(1)}(t),\gamma^{(2)}(t)\rangle = 0. \tag{10}$$

It is easy to see that martingales $\gamma^{(1)}(t)$ and $\gamma^{(2)}(t)$ are continuous with probability 1. Therefore, due to [3], there are independent Wiener processes $w^{(1)}(t)$ and $w^{(2)}(t)$ such that

$$\gamma^{(1)}(t) = \sqrt{a_0b}\,w^{(1)}(t) \text{ and } \gamma^{(2)}(t) = \sqrt{a_0b}\,w^{(2)}(t).$$

Thus, taking into consideration convergencies (8), we have

$$P\{u_T(1) < x_1, \dot{u}_T(1) < x_2\}$$
$$- P\left\{\sqrt{\frac{a_0b}{km}}\zeta_T^{(1)} < x_1, \frac{\sqrt{a_0b}}{km}\,\zeta_T^{(2)} < x_2\right\}\to 0 \text{ as } T\to\infty, \tag{11}$$

where

$$\zeta_T^{(1)} = w^{(1)}(1)\sin(\sqrt{k/m}T) - w^{(2)}(1)\cos(\sqrt{k/m}T)$$

and

$$\zeta_T^{(2)} = w^{(1)}(1)\cos(\sqrt{k/m}T) + w^{(2)}(1)\sin\sqrt{k/m}T).$$

Independence of the normally distributed random variables $w^{(1)}(1)$ and $w^{(2)}(1)$ implies that they have a bivariate normal distribution. Hence, due to [4], $\zeta_T^{(1)}$ and $\zeta_T^{(2)}$ are also bivariate normal for every T.

It is easy to verify, that for every T,

$$E\zeta_T^{(i)} = 0, \quad D\zeta_T^{(i)} = 1 \text{ and } E\zeta_T^{(1)}\zeta_T^{(2)} = 0.$$

Therefore, the random variables, $\zeta_T^{(1)}$ and $\zeta_T^{(2)}$, are independent standard normal. Convergence (11) yields the proof of statement 1a) of Theorem 1.

Since for instantaneous energy $\varepsilon(t)$ in system (2) we have the equality,

$$T^{-1}\varepsilon(T) = \frac{1}{2m}([\gamma_T^{(1)}(1)]^2 + [\gamma_T^{(2)}(1)]^2), \tag{12}$$

then, for all $x > 0$,

$$\lim_{T\to\infty} P\{T^{-1}\varepsilon(T) < x\} = P\{\frac{a_0b}{2m}([w^{(1)}(1)]^2 + [w^{(2)}(1)]^2) < x\}.$$

According to Gnedenko [4], the random variable $[w^{(1)}(1)]^2 + [w^{(2)}(1)]^2$ has a χ^2 distribution with two degrees of freedom and it coincides with the exponential distribution with parameter 1/2. Hence, the distribution of the random variable $T^{-1}\varepsilon(T)$, as $T\to\infty$ converges to the exponential distribution with parameter $m[a_0b]^{-1}$. This proves statement 1b) of Theorem 1.

Next, suppose that $2L = n_0\pi\sqrt{m/k}$ and that at least one of the constants, c_0 or a_3, is not equal to zero. Then (9) can be represented in the form:

$$f^2(t)\cos^2(\sqrt{k/m}t) = a_1 + \alpha_1(t), f^2(t)\sin^2(\sqrt{k/m}t) = a_2 + \alpha_2(t)$$

and

$$\frac{1}{2}f^2(t)\sin(2\sqrt{k/m}t) = a_3 + \alpha_3(t).$$

Therefore, in this case we have

$$\langle \gamma_T^{(i)}(t)\rangle = a_i \int\limits_0^t g^2(w_T(s)\sqrt{T})ds + \int\limits_0^t g^2(w_T(s)\sqrt{T})\alpha_i(sT)ds, \quad i = 1,2,$$

and

$$\langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t)\rangle = a_3 \int\limits_0^t g^2(w_T(s)\sqrt{T})ds + \int\limits_0^t g^2(w_T(s)\sqrt{T})\alpha_3(sT)ds.$$

As in the proof of statement 1 of Theorem 1, we obtain characteristics of the limit martingales:

$$\langle \gamma^{(1)}(t)\rangle = a_1 bt,\ \langle \gamma^{(2)}(t)\rangle = a_2 bt \text{ and } \langle \gamma^{(1)}(t), \gamma^{(2)}(t)\rangle = a_3 bt.$$

Also,

$$\gamma^{(i)}(t) = \sqrt{b}[b_{i1}w^{(1)}(t) + b_{i2}w^{(2)}(t)], \quad i = 1,2,$$

where $w^{(1)}(t)$ and $w^{(2)}(t)$ are independent Wiener processes and (b_{i1}, b_{i2}) is the i-th row of the matrix $B^{1/2}$, where $B = \begin{pmatrix} a_1 & a_3 \\ a_3 & a_2 \end{pmatrix}$. The independence of the Wiener processes, $w^{(1)}(t)$ and $w^{(2)}(t)$, implies that random variables $\gamma^{(1)}(1)$ and $\gamma^{(2)}(1)$ have normal distributions with parameters $(0, \sigma_i^2)$, where $\sigma_1^2 = a_1 b$ and $\sigma_2^2 = a_2 b$ are bivariate normal with the coefficient of correlation $r = a_3(a_1 a_2)^{-1}$. Hence, according to Gnedenko [4], the joint density of the random variables,

$$\gamma^{(1)}(1)\sin(\sqrt{k/m}T) - \gamma^{(2)}(1)\cos(\sqrt{k/m}T)$$

and

$$\gamma^{(1)}(1)\cos(\sqrt{k/m}T) + \gamma^{(2)}(1)\sin\sqrt{k/m}T),$$

is of the form (5) with $t = T$. To complete the proof of statement 2*a*) of Theorem 1, we use convergencies (8). Equality (12) implies that the limit distribution of the random variable $T^{-1}\varepsilon(T)$ coincides with the distribution of the absolute value of a bivariate normal random vector. □

Corollary: *Under the conditions of Theorem* 1,

$$\lim_{t\to\infty}(Et^{-1}\varepsilon(t)) = \frac{b}{2m}(a_1 + a_2);$$

$$\lim_{t\to\infty} Dt^{-1}\varepsilon(t) = \frac{b^2}{2m^2}(a_1^2 + a_2^2), \text{ while } a_3 = 0;$$

$$\lim_{t\to\infty} Dt^{-1}\varepsilon(t) = \frac{b^2}{2m^2}(a_0^2 + \tfrac{3}{2}a_3^2), \text{ while } a_3 \neq 0 \text{ and } c_0 = 0;$$

$$\lim_{t\to\infty} Dt^{-1}\varepsilon(t) = \frac{b^2}{2m^2}(a_1^2 + a_2^2 + 3a_3^2 + \beta), \text{ while } a_3 \neq 0 \quad c_0 \neq 0 \text{ and}$$

$$\beta = \frac{2a_3^2}{(a_1 - a_2)^4}\{4a_3^2[a_1^4 - a_1^2\sqrt{a_1^4 - (a_1 - a_2)^2} - \tfrac{1}{2}(a_1 - a_2)^2] - [a_1^4 - (a_1^2 - a_2^2)\sqrt{a_1^4 - (a_1 - a_2)^2}]\} + 2(a_1 a_2 a_3)^2 + a_3^2.$$

In this case we can change the order of limit and expectation (variance). We use the latter, the explicit form of the limit value $\gamma^{(i)}(1)$ for every i and equality (12) to prove the statement. □

Theorem 2: *Let the function $g(x)$ in equation (2) have a second derivative almost everywhere and for some $\alpha > 0$ satisfy the conditions:*

$$\lim_{|x| \to \infty} \left(\frac{1}{|x|^{\alpha}} \int_0^x g^2(v)dv - b(x) \right) = 0, \text{ with } b(x) = \begin{cases} b_1, & x > 0 \\ b_2, & x < 0 \end{cases}$$

and

$$\lim_{|x| \to \infty} \frac{1}{|x|^{\alpha}} \int_0^x |g'(v) + g(v)g''(v)| dv = 0.$$

Then

$$P\left\{ \frac{u(t)}{t^{(\alpha+1)/4}} < x_1, \frac{\dot{u}(t)}{t^{(\alpha+1)/4}} < x_2 \right\} - P\{v(t) < x_1, \dot{v}(t) < x_2\} \to 0 \text{ as } t \to \infty,$$

where $v(t)$ is the position and $\dot{v}(t)$ is the velocity of the homogeneous harmonic oscillator

$$m\ddot{v}(t) + kv(t) = 0, \quad t > 0 \tag{13}$$

with the initial condition

$$v(0) = -\frac{1}{\sqrt{km}}\gamma^{(2)}(1) \text{ and } \dot{v}(0) = \frac{1}{m}\gamma^{(1)}(1).$$

Here each $\gamma^{(i)}(t)$ is a martingale with respect to the σ-algebra $\sigma(w(s), s \le t)$ with characteristics:

$$\langle \gamma^{(i)}(t) \rangle = a_i \beta(t), \quad i = 1,2 \text{ and } \langle \gamma^{(1)}(t), \gamma^{(2)}(t) \rangle = a_3 \beta(t),$$

$$\text{while } \beta(t) = \alpha \int_0^t |w(s)|^{\alpha - 1} b(w(s)) \operatorname{sign} w(s) ds.$$

Proof: The proof is similar to that of (8) in Theorem 1, with the difference that, in this case,

$$u_T(t) = T^{-(\alpha+1)/4} u(tT), \quad \dot{u}_T(t) = T^{-(\alpha+1)/4} \dot{u}(tT),$$

$$\gamma_T^{(1)}(t) = T^{(1-\alpha)/4} \int_0^t g(w_T(s)\sqrt{T}) f(sT) \cos(\sqrt{k/m} sT) dw_T(s),$$

and

$$\gamma_T^{(2)}(t) = T^{(1-\alpha)/4} \int_0^t g(w_T(s)\sqrt{T}) f(sT) \sin(\sqrt{k/m} sT) dw_T(s)$$

with characteristics

$$\langle \gamma_T^{(i)}(t) \rangle = a_i T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T}) ds + T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T}) \alpha_i(sT) ds, \quad i = 1,2,$$

and

$$\langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t) \rangle = a_3 T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T}) ds + T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T}) \alpha_3(sT) ds.$$

Due to the Lemma,

$$\langle \gamma_T^{(i)}(t) \rangle = a_i T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T})ds + o(1)$$

and

$$\langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t) \rangle = a_3 T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T})ds + o(1),$$

where $o(1)$ is such that $E \mid o(1) \mid \to 0$ as $T \to \infty$ for all $t > 0$. Next, Kulinich [6] established that

$$T^{(1-\alpha)/2} \int_0^t g^2(w_T(s)\sqrt{T})ds \to \beta(t)$$

in probability as $T \to \infty$, where

$$\beta(t) = 2[\int_0^{w(t)} \mid v \mid^{\alpha} b(v)dv - \int_0^t \mid w(s) \mid^{\alpha} b(w(s))dw(s)].$$

Since $\alpha > 0$, using Itô's formula, we have

$$\beta(t) = \alpha \int_0^t \mid w(s) \mid^{\alpha - 1} b(w(s)) \operatorname{sign} w(s) ds. \tag{14}$$

Hence,

$$\langle \gamma_T^{(i)}(t) \rangle \to a_i \beta(t), \quad i = 1,2 \text{ and } \langle \gamma_T^{(1)}(t), \gamma_T^{(2)}(t) \rangle \to a_3 \beta(t)$$

in probability as $T \to \infty$. Thus, we obtain convergence (8), where each $\gamma^{(i)}(t)$ is a continuous, with probability 1, martingale with respect to $\sigma(w(s), s \leq t)$, with characteristics:

$$\langle \gamma^{(i)}(t) \rangle = a_i \beta(t), i = 1,2 \text{ and } \langle \gamma^{(1)}(t)(t), \gamma^{(2)}(t) \rangle = a_3 \beta(t),$$

where $\beta(t)$ has the form (14). Using convergence (8) for $t = 1$ and an explicit form of the solution of problem (13), we complete the proof of Theorem 2. □

Corollary: *Under the conditions of Theorem 2,*

$$\lim_{t \to \infty} E t^{-(\alpha+1)/2} \varepsilon(t) = \frac{\alpha(a_1 + a_2)}{2m} \int_0^1 E \mid w(s) \mid^{\alpha - 1} b(w(s)) \operatorname{sign} w(s) ds.$$

This equality is a consequence of the following statements:

1) the equality (12);
2) the equality $E[\gamma_T^{(i)}(t)]^2 = E\langle \gamma_T^{(i)}(t) \rangle$
3) the possibility to change the order of limit and expectation.

Remark: Let $q(x_1, x_2)$ be a joint density of the distribution of $\gamma^{(1)}(1)$ and $\gamma^{(2)}(1)$ and $\rho_t(x_1, x_2)$ be a joint density of the distribution of the position $v(t)$ and the velocity $\dot{v}(t)$ at the moment t, described by (13). Then,

$$\rho_t(x_1, x_3) = q[x_1 \sqrt{km} \sin(\sqrt{k/m}t) + x_2 m \cos(\sqrt{k/m}t),$$
$$- x_1 \sqrt{km} \cos(\sqrt{k/m}t) + x_2 m \sin(\sqrt{k/m}t)] m \sqrt{km}. \tag{15}$$

Using the explicit form of the solution to equation (13) we get

$$\gamma^{(1)}(1) = v(t)\sqrt{km}\sin\left(\sqrt{k/m}t\right) + \dot{v}(t)m\cos\left(\sqrt{k/m}t\right)$$

and

$$\gamma^{(2)}(1) = -v(t)\sqrt{km}\cos\left(\sqrt{k/m}t\right) + \dot{v}(t)m\sin\left(\sqrt{k/m}t\right)$$

which yields (15). □

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