Journal of Applied Mathematics and Stochastic Analysis
8, Number 3, 1995, 319-329

EXTREME SOLUTIONS OF NONLINEAR, SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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(Received July, 1994; Revised February, 1995)

ABSTRACT

After establishing a comparison result by means of a new method, we obtain the existence of maximal and minimal solutions for nonlinear, second order integro-differential equations of mixed type in Banach spaces.

Key words: Integro-differential Equations in Banach Space, Kuratowski Measure of Noncompactness, Upper and Lower Solutions, Monotone Iterative Technique.

AMS (MOS) subject classifications: 45J05, 34G20.

1. Introduction

In paper [1], we discussed establishing the existence of the extreme solutions of initial value problems for first order, integro-differential equations of Volterra type in Banach spaces by means of a comparison result. Now, in this paper, we consider the two-point boundary value problem (BVP) for nonlinear, second order integro-differential equation of mixed type in real Banach space E:

$$-u'' = f(t, u, Tu, Su), t \in J; \;\; au(0) - bu'(0) = u_0, cu(1) + du'(1) = u_1, \tag{1}$$

where $J = [0,1]$, $f \in C(J \times E \times E \times E, E)$,

$$(Tu)(t) = \int_0^t k(t,s)u(s)ds, \;\; (Su)(t) = \int_0^1 k_1(t,s)u(s)ds \tag{2}$$

$k \in C(D, R_+)$, $k_1 \in C(J \times J, R_+)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $\mathbb{R}_+$ denotes the set of all nonnegative real numbers, and $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ with $p = ac + ad + bc > 0$, $u_0, u_1 \in E$. Since f contains Su, the method for obtaining a comparison result in paper [1] cannot be applied in this case. In this paper, we use a completely new method to establish a comparison result, and then we obtain the existence of minimal and maximal solutions for BVP (1) by using lower and upper solutions and a measure of noncompactness. As an application, an example of an infinite system for scalar integro-differential equations of mixed type is given.

2. Comparison Result

Let E be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. P is said to be *normal* if there exists a positive constant c such that $\theta \leq x \leq y$ implies $\| x \| \leq c \| y \|$, where θ denotes the zero element of E, and P is said to be *regular* if every nondecreasing and bounded in order sequence in E has a limit, i.e., $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$ implies $\| x_n - x \| \to 0$ as $n \to \infty$ for some $x \in E$. The regularity of P implies the normality of P. For details on cone theory, see [2]. In connection with (1), we consider the linear BVP:

$$-u'' = -Mu - NTu - N_1 Su + g(t),\ t \in J;\ au(0) - bu'(0) = u_0, cu(1) + du'(1) = u_1, \tag{3}$$

where M, N, N_1 are nonnegative constants and $g \in C(J, E)$. Let

$$k^* = \max\{k(t,s): (t,s) \in D\},\ k_1^* = \max\{k_1(t,s): (t,s) \in J \times J\}, \tag{4}$$

and

$$q = \begin{cases} p(4ac)^{-1}, & \text{if } ac \neq 0; \\ p^{-1}(bc + bd), & \text{if } a = 0; \\ p^{-1}(ad + bd), & \text{if } c = 0. \end{cases} \tag{5}$$

Lemma 1: *If*

$$M + Nk^* + N_1 k_1^* < q^{-1}, \tag{6}$$

then the linear BVP (3) *has exactly one solution* $u \in C^2(J, E)$ *given by*

$$u(t) = v(t) + \int_0^1 Q(t,s)v(s)ds + \int_0^1 H(t,s)g(s)ds, \quad t \in J, \tag{7}$$

where

$$v(t) = p^{-1}[(c(1-t) + d)u_0 + (at + b)u_1], \tag{8}$$

$$H(t,s) = G(t,s) + F(t,s), \tag{9}$$

$$G(t,s) = \begin{cases} p^{-1}(at + b)(c(1-s) + d), & t \leq s; \\ p^{-1}(as + b)(c(1-t) + d), & t > s, \end{cases} \tag{10}$$

$$F(t,s) = \int_0^1 Q(t,r)G(r,s)dr, \tag{11}$$

$$Q(t,s) = \sum_{n=1}^{\infty} k_2^{(n)}(t,s), \tag{12}$$

$$k_2^{(n)}(t,s) = \int_0^1 \cdots \int_0^1 k_2(t, r_1)k_2(r_1, r_2) \cdots k_2(r_{n-1}, s)dr_1 \cdots dr_{n-1} \tag{13}$$

and

$$k_2(t,s) = -MG(t,s) - N\int_s^1 G(t,r)k(r,s)dr - N_1\int_0^1 G(t,r)k_1(r,s)dr. \tag{14}$$

All functions $G(t,s)$, $k_2(t,s)$, $k_2^{(n)}(t,s)$, $Q(t,s)$, $F(t,s)$, $H(t,s)$ are continuous on $J\times J$ and the series on the right-hand side of (12) converges uniformly on $J\times J$.

Proof: It is well known that $u\in C^2(J,E)$ is a solution of the linear BVP (3) if and only if $u\in C(J,E)$ is a solution of the following integral equation

$$u(t) = v(t) + \int_0^1 G(t,s)[g(s) - Mu(s) - N(Tu)(s) - N_1(Su)(s)]ds, \tag{15}$$

where $G(t,s)$ is given by (10), i.e.,

$$u(t) = w(t) + \int_0^1 k_2(t,s)u(s)ds, \tag{16}$$

where $k_2(t,s)$ is given by (14) and

$$w(t) = v(t) + \int_0^1 G(t,s)g(s)ds. \tag{17}$$

It is easy to see that

$$0 \le p^{-1}bd \le G(t,s) \le p^{-1}(at+b)(c(1-t)+d) \le q, \quad t,s\in J, \tag{18}$$

where q is defined by (5), and so, by virtue of (14) and (6), we have

$$|\,k_2(t,s)\,| \le q(M + Nk^* + N_1k_1^*) = k_2^* < 1, \quad t,s\in J. \tag{19}$$

It follows from (19) and (13) that

$$|\,k_2^{(n)}(t,s)\,| \le (k_2^*)^n, \quad t,s\in J \quad (n = 1,2,3,\ldots), \tag{20}$$

and consequently, the series in the right-hand side of (12) converges uniformly on $J\times J$ to $Q(t,s)$ and $Q(t,s)$ is continuous on $J\times J$. Let

$$(Au)(t) = w(t) + \int_0^1 k_2(t,s)u(s)ds.$$

Then A is an operator from $C(J,E)$ into $C(J,E)$. By (19), we have

$$\|\,Au - A\bar{u}\,\|_c \le k_2^*\,\|\,u-\bar{u}\,\|_c, \quad u,\bar{u}\in C(J,E).$$

Since $k_2^* < 1$, A is a contractive mapping, A has a unique fixed point u in $C(J,E)$ given by

$$\|\,u_n - u\,\|_c \to 0 \ (n\to\infty), \tag{21}$$

where

$$u_0(t) = w(t), u_n(t) = (Au_{n-1})(t), \quad t\in J \quad (n = 1,2,3,\ldots). \tag{22}$$

It is easy to see that (21) and (22) give

$$u(t) = w(t) + \sum_{n=1}^{\infty}\int_0^1 k_2^{(n)}(t,s)w(s)ds, \quad t\in J,$$

i.e.,

$$u(t) = w(t) + \int_0^1 Q(t,s)w(s)ds, \quad t \in J. \tag{23}$$

Substituting (17) into (23), we get (7) and the proof is complete. □

Lemma 2: (Comparison result) *Let inequality* (6) *be satisfied and*

$$q(M + Nk^* + N_1 k_1^*)(1 - q^2(M + Nk^* + N_1 k_1^*)^2)^{-1}$$
$$\leq \min\left\{p^{-1}q^{-1}bd, b\left(\frac{a}{2}+b\right)^{-1}, d\left(\frac{c}{2}+d\right)^{-1}\right\}. \tag{24}$$

Suppose that $u \in C^2(J,E)$ *satisfies*

$$-u'' \geq -Mu - NTu - N_1 Su, \quad t \in J; \ au(0) - bu'(0) \geq \theta, \ cu(1) + du'(1) \geq \theta. \tag{25}$$

Then $u(t) \geq \theta$ *for* $t \in J$.

Proof: Let $g(t) = -u'' + Mu + NTu + N_1 Su$ and $u_0 = au(0) - bu'(0)$, $u_1 = cu(1) + du'(1)$. Then $g \in C(J,E)$,

$$g(t) \geq \theta, \ t \in J, \tag{26}$$

and

$$u_0 \geq \theta, \quad u_1 \geq \theta. \tag{27}$$

By Lemma 1, (7) holds. From (14) we see that $k_2(t,s) \leq 0$ for $t,s \in J$, and so, (13) implies that $k_2^{(n)}(t,s) \leq 0$ and n is odd and $k_2^{(n)}(t,s) \geq 0$ when n is even. Consequently, by (12) and (20),

$$Q(t,s) \geq \sum_{m=1}^{\infty} k_2^{(2m-1)}(t,s) \geq -k_2^*(1-(k_2^*)^2)^{-1}, \quad t,s \in J. \tag{28}$$

It follows from (9), (11), (18), (28) and (24) that

$$H(t,s) \geq p^{-1}bd - qk_2^*(1-(k_2^*)^2)^{-1} \geq 0, \quad t,s \in J. \tag{29}$$

On the other hand, by virtue of (8), (28), (27) and (24), we have

$$v(t) + \int_0^1 Q(t,s)v(s)ds$$
$$\geq p^{-1}(du_0 + bu_1) - k_2^*(1-(k_2^*)^2)^{-1}p^{-1}\int_0^1 ((c(1-s)+d)u_0 + (as+b)u_1)ds$$
$$= p^{-1}(du_0 + bu_1) - p^{-1}k_2^*(1-(k_2^*)^2)^{-1}\left(\left(\frac{c}{2}+d\right)u_0 + \left(\frac{a}{2}+b\right)u_1\right) \geq \theta, \quad t \in J. \tag{30}$$

Hence, from (7), (29), (26), and (30), we see that $u(t) \geq \theta$ for $t \in J$, and the lemma is proved. □

We also need the following known lemma (see [3], Corollary 3.1(b)):

Lemma 3: *Let* H *be a countable set of strongly measurable functions:* $x: J \rightarrow E$ *such that there exists a* $z \in L(J,R_+)$ *satisfying* $\| x(t) \| \leq z(t)$ *for a.e.* $t \in J$ *and all* $x \in H$. *Then* $\alpha(H(t)) \in L(J,R_+)$ *and*

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \leq 2\int_J \alpha(H(t))dt, \tag{31}$$

where $H(t) = \{x(t): x \in H\}(t \in J)$ and α denotes the Kuratowski measure of noncompactness in E.

Corollary 1: *If $H \subset C(J,E)$ is countable and bounded, then $\alpha(H(t)) \in L(J,R_+)$ and (31) holds.*

Remark 1: The following conclusion is well known: if $H \subset C(J,E)$ is equicontinuous, then $\alpha(H(t)) \in C(J,R_+)$ and

$$\alpha\left(\left\{\int_J x(t)dt : x \in H\right\}\right) \leq \int_J \alpha(H(t))dt.$$

3. Main Theorems

Let us list some conditions for convenience.

(H_1) There exist $v_0, w_0 \in C^2(J,E)$ such that $v_0(t) \leq w_0(t)$ for $t \in J$ and

$$-v_0'' \leq f(t, v_0, Tv_0, Sv_0),\ t \in J;\ av_0(0) - bv_0'(0) \leq u_0, cv_0(1) + dv_0'(1) \leq u_1,$$

$$-w_0'' \geq f(t, w_0, Tw_0, Sw_0), t \in J;\ aw_0(0) - bw_0'(0) \geq u_0, cw_0(1) + dw_0'(1) \geq u_1.$$

(H_2) There exist nonnegative constants M, N and N_1 such that

$$f(t,u,v,w) - f(t,\bar{u},\bar{v},\bar{w}) \geq -M(u-\bar{u}) - N(v-\bar{v}) - N_1(w-\bar{w})$$

whenever $t \in J$, $v_0(t) \leq \bar{u} \leq u \leq w_0(t)$, $(Tv_0)(t) \leq \bar{v} \leq v \leq (Tw_0)(t)$ and $(Sv_0)(t) \leq \bar{w} \leq w \leq (Sw_0)(t)$.

(H_3) There exist nonnegative constants c_1, c_2 and c_3 such that

$$\alpha(f(J,U_1,U_2,U_3)) \leq c_1\alpha(U_1) + c_2\alpha(U_2) + c_3\alpha(U_3)$$

for any bounded $U_i \subset E$ $(i = 1,2,3)$.

In the following, we define the conical segment $[v_0, w_0] = \{u \in C(J,E): v_0(t) \leq u(t) \leq w_0(t)$ for $t \in J\}$.

Theorem 1: *Let cone P be normal and let conditions (H_1), (H_2) and (H_3) be satisfied. Assume that inequalities (6) and (24) hold and*

$$2q(c_1 + c_2k^* + c_3k_1^* + M + Nk^* + N_1k_1^*) < 1. \tag{32}$$

Then there exist monotone sequences $\{v_n\}$, $\{w_n\} \subseteq C^2(J,E)$ which converge uniformly on J to the minimal and maximal solutions $\bar{u}, u^* \in C^2(J,E)$ of BVP (1) in $[v_0, w_0]$, respectively. That is, if $u \in C^2(J,E)$ is any solution of BVP (1) satisfying $u \in [v_0, w_0]$, then

$$v_0(t) \leq v_1(t) \leq \ldots \leq v_n(t) \leq \ldots \leq \bar{u}(t) \leq u(t) \leq u^*(t) \leq \ldots$$

$$\leq w_n(t) \leq \ldots \leq w_1(t) \leq w_0(t),\ t \in J. \tag{33}$$

Proof: For any $h \in [v_0, w_0]$, consider the linear BVP (3) with

$$g(t) = f(t, h(t), (Th)(t), (Sh)(t)) + Mh(t) + N(Th)(t) + N_1(Sh)(t). \quad (34)$$

By Lemma 1, BVP (3) has a unique solution $u \in C^2(J, E)$ which is given by (7). Let $u = Ah$. Then operator $A: [v_0, w_0] \to C(J, E)$ and we shall show that (a) $v_0 \le Av_0$, $Aw_0 \le w_0$ and (b) A is nondecreasing on $[v_0, w_0]$. To prove (a), we set $v_1 = Av_0$ and $w = v_1 - v_0$. By (3) and (34), we have

$$-v_1'' = -Mv_1 - NTv_1 - N_1Sv_1 + f(t, v_0, Tv_0, Sv_0) + Mv_0 + NTv_0 + N_1Sv_0$$

$$= -Mw - NTw - N_1Sw + f(t, v_0, Tv_0, Sv_0), \quad t \in J;$$

$$av_1(0) - bv_1'(0) = u_0, \quad cv_1(1) + dv_1'(1) = u_1,$$

and so, from (H_1) we get

$$-w'' \ge -Mw - NTw - N_1Sw, \quad t \in J; \; aw(0) - bw'(0) \ge \theta, \; cw(1) + dw'(1) \ge \theta.$$

Consequently, Lemma 2 implies that $w(t) \ge \theta$ for $t \in J$, i.e., $Av_0 \ge v_0$. Similarly, we can show $Aw_0 \le w_0$. To prove (b), let $\overline{w} = u_2 - u_1$, where $u_1 = Ah_1$, $u_2 = Ah_2$, $h_1, h_2 \in [v_0, w_0]$, $h_1 \le h_2$. In the same way, we have, by (H_2),

$$-\overline{w}'' = -M\overline{w} - NT\overline{w} - N_1S\overline{w} + f(t, h_2, Th_2, Sh_2) - f(t, h_1, Th_1, Sh_1) + M(h_2 - h_1)$$

$$+ N(Th_2 - Th_1) + N_1(Sh_2 - Sh_1) \ge -M\overline{w} - NT\overline{w} - N_1S\overline{w}, \quad t \in J;$$

$$a\overline{w}(0) - b\overline{w}'(0) = \theta, \quad c\overline{w}(1) + d\overline{w}'(1) = \theta,$$

and hence, Lemma 2 implies that $\overline{w}(t) \ge \theta$ for $t \in J$, i.e., $Ah_2 \ge Ah_1$, and (b) is proved.

Let $v_n = Av_{n-1}$ and $w_n = Aw_{n-1}$ $(n = 1, 2, 3, \ldots)$. By (a) and (b) just proved, we have

$$v_0(t) \le v_1(t) \le \ldots \le v_n(t) \le \ldots \le w_n(t) \le \ldots \le w_1(t) \le w_0(t), \; t \in J, \quad (35)$$

and consequently, the normality of P implies that $V = \{v_n : n = 0, 1, 2, \ldots\}$ is a bounded set in $C(J, E)$. Hence, by (H_3), there is a positive constant c_0 such that

$$\| f(t, v_n(t), (Tv_n)(t), (Sv_n)(t)) + Mv_n(t) + N(Tv_n)(t) + N_1(Sv_n)(t) \| \le c_0,$$

$$t \in J \quad (n = 0, 1, 2, \ldots). \quad (36)$$

By the definition of v_n and (7), (34), we have

$$v_n(t) = v(t) + \int_0^1 G(t, s)v(s)ds$$

$$+ \int_0^1 H(t, s)[f(s, v_{n-1}(s), (Tv_{n-1})(s), (Sv_{n-1})(s)) + Mv_{n-1}(s)$$

$$+ N(Tv_{n-1})(s) + N_1(Sv_{n-1})(s)]ds, \quad t \in J \;\; (n = 1, 2, 3, \ldots). \quad (37)$$

It follows from (36) and (37) that V is equicontinuous on J, and so, the function $m(t) = \alpha(V(t))$ is continuous on J, where $V(t) = \{v_n(t): n = 0, 1, 2, \ldots\} \subset E$. Applying Corollary 1 and Remark 1 to (37) and employing (H_3), we get

$$m(t) \leq 2\int_0^1 |H(t,s)|\alpha(f(s,V(s),(TV)(s),(SV)(s)))ds$$

$$+\int_0^1 |H(t,s)|(M\alpha(V(s)) + N\alpha((TV)(s)) + N_1\alpha((SV)(s)))ds$$

$$\leq \int_0^1 |H(t,s)|[(2c_1 + M)\alpha(V(s)) + (2c_2 + N)\alpha((TV)(s))$$

$$+ (2c_3 + N_1)\alpha((SV)(s))]ds, \quad t \in J. \tag{38}$$

On the other hand, by (9), (11), (12), (18) and (20), we have

$$|H(t,s)| \leq q + qk_2^*(1 - k_2^*)^{-1} = q(1 - k_2^*)^{-1}, \quad t, s \in J. \tag{39}$$

Moreover, by Remark 1,

$$\alpha((TV)(t)) = \alpha\left(\left\{\int_0^t k(t,s)v_n(s)ds: \ n = 0, 1, 2, \ldots\right\}\right)$$

$$\leq \int_0^t \alpha\left(\{k(t,s)v_n(s): n = 0, 1, 2, \ldots\}\right)ds \leq k^*\int_0^t \alpha(V(s))ds \leq k^*\int_0^1 m(s)ds, \ t \in J, \tag{40}$$

and similarly,

$$\alpha((SV)(t)) \leq k_1^*\int_0^1 m(s)ds, \quad t \in J. \tag{41}$$

It follows from (38)-(41) that

$$m(t) \leq q(1 - k_2^*)^{-1}((2c_1 + M) + k^*(2c_2 + N) + k_1^*(2c_3 + N_1))\int_0^1 m(s)ds, \quad t \in J,$$

and so,

$$\int_0^1 m(t)dt \leq q(1 - k_2^*)^{-1}(2c_1 + 2c_2k^* + 2c_3k_1^* + M + Nk^* + N_1k_1^*)\int_0^1 m(s)ds,$$

which implies by virtue of (32) that $\int_0^1 m(t)dt = 0$, and consequently, $m(t) = 0$ for $t \in J$. Thus, by the Ascoli-Arzela theorem (see [4], Theorem 1.1.15), V is relatively compact in $C(J,E)$, and so, there exists a subsequence of $\{v_n\}$ which converges uniformly on J to some $\bar{u} \in C(J,E)$. Since, by (35), $\{v_n\}$ is nondecreasing and P is normal, we see that $\{v_n\}$ itself converges uniformly on J to $\bar{u}$. Now, we have

$$f(t, v_{n-1}, (Tv_{n-1})(t), (Sv_{n-1})(t)) + Mv_{n-1}(t) + N(Tv_{n-1})(t) + N_1(Sv_{n-1})(t)$$

$$\to f(t, \bar{u}(t), (T\bar{u})(t), (S\bar{u})(t)) + M\bar{u}(t) + N(T\bar{u})(t) + N_1(S\bar{u})(t), \quad t \in J, \tag{42}$$

and, by (36),

$$\| f(t,v_{n-1}(t),(Tv_{n-1})(t),(Sv_{n-1})(t))+Mv_{n-1}(t)+N(Tv_{n-1})(t)+N_1(Sv_{n-1})(t)$$
$$-f(t,\bar{u}(t),(T\bar{u})(t),(S\bar{u})(t))-M\bar{u}(t)-N(T\bar{u})(t)-N_1(S\bar{u})(t)\| \leq 2c_0,$$
$$t\in J \quad (n=1,2,3,\ldots). \tag{43}$$

Observing (42) and (43) and taking limits as $n\to\infty$ in (37), we get

$$\bar{u}(t)=v(t)+\int_0^1 G(t,s)v(s)ds$$
$$+\int_0^1 H(t,s)[f(s,\bar{u}(s),(T\bar{u})(s),(S\bar{u})(s))+M\bar{u}(s)+N(T\bar{u})(s)+N_1(S\bar{u})(s)]ds, t\in J,$$

which implies by virtue of Lemma 1 that $\bar{u}\in C^2(J,E)$ and $\bar{u}$ satisfies

$$-\bar{u}''=f(t,\bar{u},T\bar{u},S\bar{u}),t\in J;\quad a\bar{u}(0)-b\bar{u}'(0)=u_0,\quad c\bar{u}(1)+d\bar{u}'(1)=u_1,$$

i.e., $\bar{u}$ is a solution of BVP (1). In the same way, we can show that $\{w_n\}$ converges uniformly on J to some u^* and u^* is a solution of BVP (1) in $C^2(J,E)$.

Finally, let $u\in C^2(J,E)$ be any solution of BVP (1) satisfying $v_0(t)\leq u(t)\leq w_0(t)$ for $t\in J$. Assume that $v_{k-1}(t)\leq u(t)\leq w_{k-1}(t)$ for $t\in J$, and set $\bar{v}=u-v_k$. Then, on account of the definition of v_k and (H_2), we have

$$-\bar{v}''=-M\bar{v}-NT\bar{v}-N_1S\bar{v}+M(u-v_{k-1})+NT(u-v_{k-1})+N_1S(u-v_{k-1})$$
$$+f(t,u,Tu,Su)-f(t,v_{k-1},Tv_{k-1},Sv_{k-1})\geq -M\bar{v}-NT\bar{v}-N_1S\bar{v},\quad t\in J;$$
$$a\bar{v}(0)-b\bar{v}'(0)=\theta,\quad c\bar{v}(1)+d\bar{v}'(1)=\theta,$$

which implies by virtue of Lemma 2 that $\bar{v}(t)\geq\theta$ for $t\in J$, i.e., $v_k(t)\leq u(t)$ for $t\in J$. Similarly, we can show $u(t)\leq w_k(t)$ for $t\in J$. Consequently, by induction, $v_n(t)\leq u(t)\leq w_n(t)$ for $t\in J$ $(n=0,1,2,\ldots)$, and by taking limits, we get $\bar{u}(t)\leq u(t)\leq u^*(t)$ for $t\in J$. Hence, (33) holds and the theorem is proved. □

Theorem 2: *Let cone P be regular and conditions (H_1) and (H_2) be satisfied. Assume that inequalities (6) and (24) holds. Then the conclusions of Theorem 1 hold.*

Proof: The proof is almost the same as that of Theorem 1. The only difference is that instead of using condition (H_3) and inequality (32), the conclusion $m(t)=\alpha(V(t))=0$ $(t\in J)$ is obtained directly by (35) and the regularity of P. □

Remark 2: The condition that P is regular will be satisfied if E is weakly complete (reflexive, in particular) and P is normal (see [2], Theorem 1.2.1 and Theorem 1.2.2, and [5], Theorem 2.2).

4. An Example

Consider the BVP of an infinite system for scalar, second order integro-differential equations of mixed type:

$$\begin{cases} -u_n'' = \dfrac{t}{360\pi^3 n}(1-\pi u_n - \sin\pi(t+u_n))^3 + \dfrac{t}{30n(n+3)^2}(u_{n+1} + tu_{2n-1}^2) \\ \quad -\dfrac{1}{60(n+1)}\left(\displaystyle\int_0^t e^{-ts}u_n(s)ds\right)^2 + \dfrac{t^2}{30(2n+3)}\displaystyle\int_0^1 \cos^2\pi(t-s)u_{2n}(s)ds \\ \quad -\dfrac{1}{60(n+1)}\displaystyle\int_0^1 \cos^2\pi(t-s)u_n(s)ds)^5, \quad 0 \le t \le 1; \\ u_n(0) = u_n'(0), \quad u_n'(1) = 0 \quad (n = 1,2,3,\ldots). \end{cases} \tag{44}$$

Evidently, $u_n(t) \equiv 0$ $(n = 1,2,3,\ldots)$ is not a solution of BVP (44).

Conclusion: BVP (44) has minimal and maximal continuous, twice differentiable solutions satisfying $0 \le u_n \le 2$ for $0 \le t \le 1$ $(n = 1,2,3,\ldots)$.

Proof: Let $E = \ell^\infty = \{u = (u_1,\ldots,u_n,\ldots): \sup_n |u_n| < \infty\}$ with norm $\|u\| = \sup_n |u_n|$ and $P = \{u = (u_1,\ldots,u_n,\ldots) \in \ell^\infty: u_n \ge 0, n = 1,2,3,\ldots\}$. Then P is a normal cone in E and BVP (44) can be regarded as a BVP of the form (1) in E. In this situation, $a = b = d = 1$, $c = 0$, $u_0 = u_1 = (0,\ldots,0,\ldots)$, $k(t,s) = e^{-ts}$, $k_1(t,s) = \cos^2\pi(t-s)$, $u = (u_1,\ldots,u_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$, $w = (w_1,\ldots,w_n,\ldots)$ and $f = (f_1,\ldots,f_n,\ldots)$, in which

$$f_n(t,u,v,w) = \frac{t}{360\pi^3 n}(1-\pi u_n - \sin\pi(t+u_n))^3 + \frac{t}{30n(n+3)^2}(u_{n+1} + tu_{2n-1}^2)$$

$$-\frac{1}{60(n+1)}v_n^2 + \frac{t^2}{30(2n+3)}w_{2n} - \frac{1}{60(n+1)}w_n^5. \tag{45}$$

It is clear that $f \in C(J \times E \times E \times E, E)$, where $J = [0,1]$. Let $v_0(t) = (0,\ldots,0,\ldots)$ and $w_0(t) = (2,\ldots,2,\ldots)$. Then $v_0, w_0 \in C^2(J,E)$, $v_0(t) < w_0(t)$ for $t \in J$, and we have

$$v_0''(t) = w_0''(t) = (0,\ldots,0,\ldots),$$

$$v_0(0) = v_0'(0) = v_0'(1) = w_0'(0) = w_0'(1) = (0,\ldots,0,\ldots), \quad w_0(0) = (2,\ldots,2,\ldots),$$

$$f_n(t,v_0,Tv_0,Sv_0) = \frac{t}{360\pi^3 n}(1-\sin\pi t)^3 \ge 0,$$

$$f_n(t,w_0,Tw_0,Sw_0) = \frac{t}{360\pi^3 n}(1-2\pi-\sin\pi(t+2))^3 + \frac{t}{30n(n+3)^2}(2+4t)$$

$$-\frac{1}{15(n+1)}\left(\int_0^t e^{-ts}ds\right)^2 + \frac{t^2}{30(2n+3)} - \frac{1}{60(n+1)}$$

$$\le \frac{t}{n}\left(\frac{(1-2\pi)^3}{360\pi^3} + \frac{6}{480}\right) + \frac{1}{60(n+1)} - \frac{1}{60(n+1)} \le 0.$$

Consequently, v_0 and w_0 satisfy condition (H_1). On the other hand, for $u = (u_1,\ldots,u_n,\ldots)$, $\bar{u} = (\bar{u}_1,\ldots,\bar{u}_n,\ldots)$, $v = (v_1,\ldots,v_n,\ldots)$, $\bar{v} = (\bar{v}_1,\ldots,\bar{v}_n,\ldots)$, $w = (w_1,\ldots,w_n,\ldots)$ and $\bar{w} = (\bar{w}_1,\ldots,\bar{w}_n,\ldots)$ satisfying $t \in J$, $v_0(t) \le \bar{u} \le u \le w_0(t)$, $(Tv_0)(t) \le \bar{v} \le v \le (Tw_0)(t)$ and $(Sv_0)(t) \le \bar{w} \le w \le (Sw_0)(t)$, i.e., $0 \le \bar{u}_n \le u_n \le 2$,

$$0 \leq \bar{v}_n \leq v_n \leq 2\int_0^t e^{-ts}ds \leq 2 \text{ and } 0 \leq \bar{w}_n \leq w_n \leq 2\int_0^1 \cos^2\pi(t-s)ds = 1 \text{ for } t \in J$$

($n = 1, 2, 3, \ldots$), we have, by (45),

$$f_n(t,u,v,w) - f_n(t,\bar{u},\bar{v},\bar{w}) \geq \frac{t}{360\pi^3 n}[(1-\pi u_n - \sin\pi(t+u_n))^3$$

$$-(1-\pi\bar{u}_n - \sin\pi(t+\bar{u}_n))^3] - \frac{1}{60(n+1)}(v_n^2 - \bar{v}_n^2) - \frac{1}{60(n+1)}(w_n^5 - \bar{w}_n^5). \tag{46}$$

Since

$$\frac{\partial}{\partial s}(1-\pi s - \sin\pi(t+s))^3 = -3\pi(1-\pi s-\sin\pi(t+s))^2(1+\cos\pi(t+s))$$

$$\geq -24\pi^3, \text{ for } 0 \leq t \leq 1, \ \ 0 \leq s \leq 2,$$

$$\frac{\partial}{\partial s}(-s^2) = -2s \geq -4, \ \text{ for } 0 \leq s \leq 2$$

and

$$\frac{\partial}{\partial s}(-s^5) = -5s^4 \geq -5, \ \text{ for } 0 \leq s \leq 1,$$

it follows from (46) that

$$f_n(t,u,v,w) - f_n(t,\bar{u},\bar{v},\bar{w}) \geq -\frac{1}{15n}(u_n - \bar{u}_n) - \frac{1}{15(n+1)}(v_n - \bar{v}_n) - \frac{1}{12(n+1)}(w_n - \bar{w}_n)$$
$$\geq -\frac{1}{15}(u_n - \bar{u}_n) - \frac{1}{30}(v_n - \bar{v}_n) - \frac{1}{24}(w_n - \bar{w}_n), \quad (n = 1,2,3,\ldots).$$

This means that condition (H_2) is satisfied for $M = \frac{1}{15}$, $N = \frac{1}{30}$ and $N_1 = \frac{1}{24}$. We now check condition (H_3). Let $t^{(m)} \in J$ and sequences $\{u^{(m)}\}$, $\{v^{(m)}\}$, $\{w^{(m)}\}$ be bounded in $E = \ell^\infty$. Let $u^{(m)} = (u_1^{(m)}, \ldots, u_n^{(m)}, \ldots)$, $v^{(m)} = (v_1^{(m)}, \ldots, v_n^{(m)}, \ldots)$, $w^{(m)} = (w_1^{(m)}, \ldots, w_n^{(m)}, \ldots)$, and $z^{(m)} = (z_1^{(m)}, \ldots, z_n^{(m)}, \cdots) = f(t^{(m)}, u^{(m)}, v^{(m)}, w^{(m)})$, i.e., $z_n^{(m)} = f_n(t^{(m)}, u^{(m)}, v^{(m)}, w^{(m)})(n, m = 1, 2, 3, \ldots)$. Then, there exists a positive constant r such that

$$|u_n^{(m)}| \leq r, \quad |v_n^{(m)}| \leq r, \quad |w_n^{(m)}| \leq r \quad (n, m = 1,2,3,\ldots),$$

and, by (45),

$$|z_n^{(m)}| \leq \frac{1}{360\pi^3 n}(2+\pi r)^3 + \frac{r(1+r)}{30n(n+3)^2} + \frac{r^2(1+r^3)}{60(n+1)} + \frac{r}{30(2n+3)}, \quad (n, m = 1,2,3,\ldots). \tag{47}$$

Consequently, $\{z_n^{(m)}\}$ is bounded, so, by the diagonal method, we can choose a subsequence $\{m_i\}$ of $\{m\}$ such that

$$z_n^{(m_i)} \to z_n \ \text{ as } \ i \to \infty \ \ (n = 1,2,3,\ldots). \tag{48}$$

From (47), we have

$$|z_n| \leq \frac{1}{360\pi^3 n}(2+\pi r)^3 + \frac{r(1+r)}{30n(n+3)^2} + \frac{r^2(1+r^3)}{60(n+1)} + \frac{r}{30(2n+3)}, \tag{49}$$

and therefore, $z = (z_1, \ldots, z_n, \ldots) \in \ell^{\infty} = E$. For any $\epsilon > 0$, by virtue of (47) and (49), we can choose a positive integer n_0 such that

$$| z_n^{(m_i)} | < \epsilon, \; | z_n | < \epsilon, \; n > n_0 \;\; (i = 1, 2, 3, \ldots). \tag{50}$$

On the other hand, (48) implies that there is a positive integer i_0 such that

$$| z_n^{(m_i)} - z_n | < \epsilon, \;\; i > i_0 \;\; (n = 1, 2, \ldots, n_0). \tag{51}$$

It follows from (50) and (51) that

$$\| z^{(m_i)} - z \| = \sup_n | z_n^{(m_i)} - z_n | \leq 2\epsilon, \quad i > i_0.$$

This means that $\| z^{(m_i)} - z \| \rightarrow 0$ as $i \rightarrow \infty$, and hence, condition (H_3) is satisfied for $c_1 = c_2 = c_3 = 0$.

It is clear that $p = 1$, $q = 2$, $k^* = k_1^* = 1$, and so, it is easy to calculate

$$M + Nk^* + N_1 k_1^* = \frac{17}{120} < \frac{1}{2} = q^{-1},$$

$$q(M + Nk^* + N_1 k_1^*)(1 - q^2(M + Nk^* + N_1 k_1^*)^2)^{-1} = \frac{1020}{3311} < \frac{1}{2}$$

$$= \min\left\{ p^{-1} q^{-1} bd, b\left(\frac{a}{2} + b\right)^{-1}, d\left(\frac{c}{2} + d\right)^{-1} \right\}$$

and

$$2q(c_1 + c_2 k^* + c_3 k_1^* + M + Nk^* + N_1 k_1^*) = \frac{17}{30} < 1.$$

Hence, inequalities (6), (24) and (32) are satisfied, and our conclusion follows from Theorem 1.

References

[1] Guo, Dajun, Initial value problems for integro-differential equations of Volterra type in Banach spaces, *J. Appl. Math. Stoch. Anal.* **7** (1994), 13-23.

[2] Guo, Dajun and Lakshmikantham, V., *Nonlinear Problems in Abstract Cones*, Academic Press, Boston and New York 1988.

[3] Heinz, H.P., On the behavior of measures of noncompactness with respect to differentiation and integration of vector-valued functions, *Nonlinear. Anal. TMA* **7** (1983), 1351-1371.

[4] Lakshmikantham, V. and Leela, S., *Nonlinear Differential Equations in Abstract Spaces*, Pergamon, Oxford 1981.

[5] Yihong, Du, Fixed points of increasing operators in ordered Banach spaces and applications, *Appl. Math.* **38** (1990), 1-20.