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# Stability of Lipschitz Type in Determination of Initial Heat Distribution*

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For the solution $u(x,t) = u(f)(x,t)$ of the equations

$$\left\{\begin{array}{ll} u'(x,t) = \Delta u(x,t), & x \in \Omega,\ t > 0 \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{array}\right\}$$

where $\Omega \subset \mathbb{R}^r$, $2 \le r \le 3$ is a bounded domain with $C^2$-boundary and for an appropriate subboundary $\Gamma$ of $\Omega$ we prove a Lipschitz estimate of $\|f\|_{L^2(\Omega)}$ : For $\mu \in (1, \frac{5}{4})$ and for a positive constant $C$

$$C^{-1}\|f\|_{L^2(\Omega)} \le \left\| \frac{\partial u(f)}{\partial \nu} \right\|_{B_\mu(\Gamma\times(0,\infty))}$$

$$\equiv \int_\Gamma \left\{ \sum_{n=0}^\infty \frac{1}{n!\Gamma(n+2\mu+1)} \right.$$

$$\left. \times \int_0^\infty \left| (p\partial_p^{n+1} + n\partial_p^n) p^{-\frac{3}{2}} \frac{\partial u(f)}{\partial \nu}\left(x, \frac{1}{4p}\right) \right|^2 p^{2n+2\mu-1} dp \right\} dS.$$

The norm $\|\cdot\|_{B_\mu(\Gamma\times(0,\infty))}$ is involved and strong, but it is a natural one in our situation relating to a typical and simple norm for analytic functions. Furthermore, it is acceptable in the sense that $\left\| \frac{\partial u(f)}{\partial \nu} \right\|_{B_\mu(\Gamma\times(0,\infty))} \le C\|f\|_{H^2(\Omega)}$ holds.

*Dedicated to Professor Kyuya Masuda on the occasion of his 60th birthday.

*Keywords:* Heat equation; observation problem; theory of control; stability of Lipschitz type; transform by Reznitskaya; real inversion formula for the Laplace transform.

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## 1 INTRODUCTION AND THE MAIN RESULT

We consider an initial value problem for the heat equation:

$$\left\{\begin{array}{ll} u'(x,t) = \Delta u(x,t), & x \in \Omega,\ t > 0 \\ u(x,0) = f(x), & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{array}\right\} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^r$, $2 \le r \le 3$ is a bounded domain with $C^2$-boundary $\partial\Omega$, $u' = \frac{\partial u}{\partial t}$, $\Delta$ the Laplacian, $\frac{\partial}{\partial \nu}$ is a trace operator (e.g. Adams [1]), that is, if $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial \nu}(x) = \sum_{i=1}^{r} \nu_i(x) \frac{\partial u}{\partial x_i}(x), \quad x \in \partial\Omega,$$

$\nu(x) = (\nu_1(x), \ldots, \nu_r(x))$ being the outward unit normal to $\partial\Omega$ at $x$.

For $f \in L^2(\Omega)$, there exists a unique (strong) solution

$$u = u(f) \in C^0([0,\infty); L^2(\Omega)) \cap C^1((0,\infty); L^2(\Omega))$$

such that $\Delta u \in C^0((0,\infty); L^2(\Omega))$ and $u(\cdot,t)_{|\partial\Omega} = 0$, $t > 0$ (e.g. Pazy [15]).

In this situation, we have the following problems when we consider the heat flux $\frac{\partial u(f)}{\partial \nu}(x,t)$ on a subboundary $\Gamma$ of $\Omega$ as measurements for $t > 0$.

(I) (Uniqueness) What kind of a set $\Gamma \subset \partial\Omega$, does

$$\frac{\partial u(f)}{\partial \nu}(x,t), \quad x \in \Gamma,\ t > 0$$

determine $f(x)$, $x \in \Omega$ uniquely?

(II) (Construction) We wish to represent the initial heat distribution $f(x)$ on $\Omega$ in terms of $\frac{\partial u(f)}{\partial \nu}(x,t)$, $x \in \Gamma$, $t > 0$.

(III) (Stability) Can we estimate $\|f\|_{L^2(\Omega)}$ by $\frac{\partial u(f)}{\partial \nu}(x,t)$, $x \in \Gamma$, $t > 0$? Moreover what norm of $\frac{\partial u(f)}{\partial \nu}(x,t)$, $x \in \Gamma$, $t > 0$ should we choose for the estimation of $\|f\|_{L^2(\Omega)}$?

The determination of initial heat distribution is called an *observation problem.* The observation problem is important also in the theory of control, because it is a dual problem to the controllability problem (e.g. Dolecki and Russell [6]). For the observation problem in the heat equation, we can refer to Cannon [2]. For similar types of problems, the reader can consult Dolecki [5], Mizel and Seidman [13], and Sakawa [17]. In this paper, (I) and (III) are treated, while (II) will be discussed in a succeeding paper.

For the uniqueness, the answer is known under a comfortable assumption:

PROPOSITION 1 *Let $\Gamma \subset \partial\Omega$ satisfy $\Gamma = \partial\Omega \cap U \neq \emptyset$ for an open set $U \subset \mathbb{R}^r$. If*

$$\frac{\partial u(f)}{\partial \nu}(x, t) = 0, \quad x \in \Gamma,\ t > 0,$$

*then $f(x) = 0$, $x \in \Omega$.*

This is proved by the eigenfunction expansion of the solution and the unique continuation theorem for the elliptic operator (e.g. Mizohata [14]), and we can further refer to Georg Schmidt and Weck [8] for the proof.

Now we proceed to the stability. It is easily expected that the stability is very delicate, because the map $f \longmapsto \frac{\partial u(f)}{\partial \nu}$ advances the regularity by the "smoothing" property of the parabolic equation (e.g. Friedman [7]). We have to take a strong norm $\|\cdot\|_*$ for $\frac{\partial u(f)}{\partial \nu}(x, t)$, $x \in \Gamma$, $t > 0$ in order to get an upper estimate of $\|f\|_{L^2(\Omega)}$. More precisely, there are two ways.

(a) We search for a norm $\|\cdot\|_*$ of functions on $\Gamma \times (0, \infty)$ so that

$$\|f\|_{L^2(\Omega)} \le C \left\| \frac{\partial u(f)}{\partial \nu} \right\|_* .$$

(b) Taking a usual norm for $\frac{\partial u(f)}{\partial \nu}$ such as

$$\left\| \frac{\partial u(f)}{\partial \nu} \right\|_{L^2(\Gamma \times (0,\infty))},$$

we search for a stability modulus $\omega \in C^0[0, \infty)$ which is monotone increasing and $\omega(0) = 0$ so that

$$\|f\|_{L^2(\Omega)} \le \omega \left( \left\| \frac{\partial u(f)}{\partial \nu} \right\|_{L^2(\Gamma \times (0,\infty))} \right).$$

In (a) we insist on the stability of Lipschitz type, while we have to admit the choice of a strong norm $\|\cdot\|_*$. In (b), we insist on a usual norm for measurements $\frac{\partial u(f)}{\partial \nu}$, at the cost of a worse stability modulus $\omega$. The latter way (b) seems more pursued in existing papers (e.g. Exercise 11.4 (pp. 144–145) in Cannon [2]), and the estimate of the type

$$\|f\|_{L^2(\Omega)} = O\left(\left[\log \|\text{Data}\|^{-1}\right]^{\alpha}\right),$$

for some constant $\alpha > 0$, is typical (e.g. see p. 147 in [2]).

The purpose of this paper is to pursue the way (a). Our choice of the norm $\|\cdot\|_*$ is, of course, stronger than the $L^2(\Gamma \times (0,\infty))$-norm, but is not extreme in the sense that

$$\left\|\frac{\partial u(f)}{\partial \nu}\right\|_* = O(\|f\|_{L^2(\Omega)}).$$

We shall use the following analytic extension formula

PROPOSITION 2([16]) *For the right half plane $R^+ = \{Z; Z = p+iq, p > 0\}$ and $\mu > \frac{1}{2}$, we have the identity, for the Bergman-Selberg space $H_\mu(R^+)$ comprising all analytic functions $f(Z)$ on $R^+$ with finite norm*

$$\|f\|_{H_\mu(R^+)} = \left\{\frac{1}{\Gamma(2\mu-1)\pi}\iint_{R^+} |f(Z)|^2 (2p)^{2\mu-2} dp dq\right\}^{1/2} < \infty,$$

$$\|f\|^2_{H_\mu(R^+)} = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+2\mu+1)} \int_0^\infty |\partial_p^n (pf'(p))|^2 p^{2n+2\mu-1} dp.$$

*Conversely, any $C^\infty$ function $f(p)$ on the real positive line with a convergent sum in the right hand side can be extended analytically onto $R^+$ and the analytic extension $f(Z)$ satisfying $\lim_{p\to\infty} f(p) = 0$ belongs to $H_\mu(R^+)$ and the identity holds.*

We shall define

$$\|g(x,p)\|^2_{B_\mu(\Gamma\times(0,\infty))} = \int_\Gamma \left\|p^{-\frac{3}{2}} g\left(x, \frac{1}{4p}\right)\right\|^2_{H_\mu(R^+)} dS,$$

the right hand side being convergent. Then we obtain

THEOREM *For an arbitrarily fixed $x_0 \in \mathbb{R}^r$, we set*

$$\Gamma = \{x \in \partial\Omega; (x - x_0)\cdot \nu(x) > 0\} \tag{1.2}$$

*and take*

$$\mu \in \left(1, \frac{5}{4}\right). \tag{1.3}$$

*We assume*

$$r \quad (= \text{the spatial dimension}) \ \leq 3 \tag{1.4}$$

*and*

$$f \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.5}$$

*Then, there exists a constant* $C = C(\Omega, \Gamma, \mu) > 0$ *such that*

$$C^{-1}\|f\|_{L^2(\Omega)} \leq \left\|\frac{\partial u(f)}{\partial \nu}\right\|_{B_\mu(\Gamma\times(0,\infty))} \leq C\|f\|_{H^2(\Omega)}. \tag{1.6}$$

In Theorem, if $r = 1$, then $\Gamma$ is taken as one boundary point of the interval $\Omega$. Here for simplicity, we assume (1.4), that is, the spatial dimension is less than or equal to 3. This is not essential and the condition (1.5) should be replaced by $f \in \mathcal{D}(A^\alpha)$ where $\alpha = \left[\frac{r}{4}\right] + 1$, $(Au)(x) = -\Delta u(x)$ with $\mathcal{D}(A) = \{u \in H^2(\Omega); u_{|\partial\Omega} = 0\}$, $[\beta]$ denotes the greatest integer among ones not exceeding $\beta \in \mathbb{R}$.

## 2 PROOF OF THEOREM

The proof will be divided into three steps.

### 2.1 First Step

We shall discuss the regularity of solutions to the wave equation corresponding to (1.1). First by (1.5) we see that $u(f) \in C^1([0,\infty) \times \overline{\Omega})$ and $\Delta u(f) \in C^0([0,\infty) \times \overline{\Omega})$ (e.g. Theorems 4.3.5 and 4.3.6 in Pazy [15] and the Sobolev embedding theorem (e.g. Adams [1])). We shall consider a corresponding wave equation:

$$\left\{\begin{array}{lll} w''(x,t) = \Delta w(x,t), & & x \in \Omega,\ t > 0 \\ w(x,0) = 0, & w'(x,0) = f(x), & x \in \Omega \\ w(x,t) = 0, & & x \in \partial\Omega, t > 0. \end{array}\right\} \tag{2.1}$$

Again applying the regularity assumption (1.5), by the eigenfunction expansion of the solution $w$ and the Sobolev embedding theorem, we see that there exists a unique solution

$$w = w(f) \in C^0([0,\infty); H^3(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^2([0,\infty); H_0^1(\Omega)), \tag{2.2}$$

and

$$\|w(f)(\cdot,t)\|_{H^2(\Omega)}, \quad \|w(f)'(\cdot,t)\|_{H^2(\Omega)} \le \|f\|_{H^2(\Omega)}, \quad t > 0 \tag{2.3}$$

(e.g. Theorem 1.1.1 in Komornik [10]). The inequalities in (2.3) follow from conservation of energy. We set

$$W(t) = \int_\Gamma \left(\frac{\partial w(f)}{\partial \nu}(x,t)\right)^2 dS \equiv \left\|\frac{\partial w(f)}{\partial \nu}(\cdot,t)\right\|^2_{L^2(\Gamma)}, \quad t > 0.$$

By (2.3) and the trace theorem (e.g. Adams [1]), we have

$$W \in C^1[0,\infty), \quad |W'(t)|, W(t) \le C_1\|f\|^2_{H^2(\Omega)}, \qquad t > 0, \tag{2.4}$$

where $C_1 = C_1(\Omega) > 0$ is a constant independent of $t > 0$.

In fact, $W \in C^0[0,\infty)$ and $W(t) \le C_1\|f\|^2_{H^2(\Omega)}$ is straightforward from (2.3) and the trace theorem. Next, by (2.3),

$$\left(\frac{\partial w(f)}{\partial \nu}\right)' \in C^0([0,\infty); L^2(\Gamma))$$

and

$$\int_\Gamma \left(\left(\frac{\partial w(f)}{\partial \nu}\right)'(x,t)\right)^2 dS \le C_1'\|f\|^2_{H^2(\Omega)}, \quad t > 0,$$

so that

$$W'(t) = 2\int_\Gamma \frac{\partial w(f)}{\partial \nu}(x,t)\left(\frac{\partial w(f)}{\partial \nu}\right)'(x,t)dS, \quad t > 0.$$

Hence $W' \in C^0[0,\infty)$ and by Schwarz's inequality

$$\begin{aligned}|W'(t)| &\le 2\left\|\frac{\partial w(f)}{\partial \nu}(\cdot,t)\right\|_{L^2(\Gamma)}\left\|\left(\frac{\partial w(f)}{\partial \nu}\right)'(\cdot,t)\right\|_{L^2(\Gamma)}\\ &\le 2\sqrt{C_1'}\|f\|_{H^2(\Omega)}\sqrt{C_1'}\|f\|_{H^2(\Omega)}, \quad t > 0,\end{aligned}$$

which proves (2.4).

It follows from (2.4) and $w(f)(\cdot, 0) = 0$ that $W(0) = 0$. Furthermore, we have, using (2.4) and the mean value theorem

$$\begin{aligned} W(t) &= W(t) - W(0) \le t \sup_{0\le s\le t} |W'(s)| \\ &\le C_1 t \|f\|^2_{H^2(\Omega)}, \quad t > 0. \end{aligned} \tag{2.5}$$

By (2.5) and (1.3) we have

$$\begin{aligned} \int_0^\infty W(t)t^{3-4\mu}dt &= \int_0^1 W(t)t^{3-4\mu}dt + \int_1^\infty W(t)t^{3-4\mu}dt \\ &\le C_1\|f\|^2_{H^2(\Omega)}\left(\int_0^1 t^{4-4\mu}dt + \int_1^\infty t^{3-4\mu}dt\right) \\ &= C_1\|f\|^2_{H^2(\Omega)}\left(\frac{1}{5-4\mu} + \frac{1}{4\mu-4}\right) \\ &\equiv C_2\|f\|^2_{H^2(\Omega)}. \end{aligned}$$

That is,

$$\int_0^\infty \int_\Gamma \left(\frac{\partial w(f)}{\partial \nu}(x,t)\right)^2 t^{3-4\mu} dSdt \le C_2\|f\|^2_{H^2(\Omega)}. \tag{2.6}$$

## 2.2 Second Step

By the transform formula by Reznitskaya, we get

$$u(f)(x,t) = \frac{1}{2\sqrt{\pi t^3}}\int_0^\infty \eta \exp\left(-\frac{\eta^2}{4t}\right) w(f)(x,\eta)d\eta, \quad x \in \Omega,\ t > 0 \tag{2.7}$$

(e.g. Section 5 in Chapter VII in Lavrentiev, Romanov and Shishat·skiĭ [11]). By (2.2), (2.3) and the Sobolev embedding theorem, we have

$$\left|\frac{\partial w(f)}{\partial x_i}(x,t)\right| \le C_1\|f\|_{H^2(\Omega)}, \quad x \in \Omega,\ t > 0,$$

that is,

$$\left|\frac{\partial w(f)}{\partial \nu}(x,t)\right| \le C_1\|f\|_{H^2(\Omega)}, \quad x \in \Gamma,\ t > 0.$$

Therefore in (2.7), we can exchange $\int_0^\infty ...d\eta$ and $\frac{\partial}{\partial\nu}$, so that

$$2\sqrt{\pi t^3}\frac{\partial u(f)}{\partial\nu}(x,t) = \int_0^\infty \eta \exp\left(-\frac{\eta^2}{4t}\right)\frac{\partial w(f)}{\partial\nu}(x,\eta)d\eta,$$
$$x \in \Gamma,\ t > 0.$$

Setting $t = \frac{1}{4p}$ and changing independent variables by $s = \eta^2$, we have

$$\frac{\sqrt{\pi}}{2}\frac{1}{p^{\frac{3}{2}}}\frac{\partial u(f)}{\partial\nu}\left(x, \frac{1}{4p}\right) = \int_0^\infty \exp(-sp)\frac{\partial w(f)}{\partial\nu}(x,\sqrt{s})ds,$$
$$x \in \Gamma,\ p > 0. \tag{2.8}$$

We define the Laplace transform of $g \in L^1_{loc}(0,\infty)$ by $(\mathcal{L}g)(p)$:

$$(\mathcal{L}g)(p) = \int_0^\infty \mathrm{e}^{-sp}g(s)ds, \qquad p > 0,$$

the integral existing for $p > 0$. Therefore we can rewrite (2.8) as

$$\frac{\sqrt{\pi}}{2}\frac{1}{p^{\frac{3}{2}}}\frac{\partial u(f)}{\partial\nu}\left(x, \frac{1}{4p}\right) \equiv \widetilde{u}(x,p) = (\mathcal{L}\widetilde{w})(x,p), \quad x \in \Gamma,\ p > 0, \tag{2.9}$$

where

$$\widetilde{w}(x,s) = \frac{\partial w(f)}{\partial\nu}(x,\sqrt{s}), \quad x \in \Gamma,\ s > 0.$$

Then, by using an isometrical identity for the Laplace transform in Byun-Saitoh [3], we obtain

$$\int_0^\infty (\widetilde{w}(x,t))^2 t^{1-2\mu}dt = \|\widetilde{u}(x,\cdot)\|^2_{H_\mu(R^+)}, \quad x \in \Gamma, \tag{2.10}$$

provided that either of the both hand sides is convergent.

Since

$$\int_0^\infty (\widetilde{w}(x,t))^2 t^{1-2\mu}dt = 2\int_0^\infty \left(\frac{\partial w(f)}{\partial\nu}(x,s)\right)^2 s^{3-4\mu}ds, \quad x \in \Gamma,$$

it follows from (2.6) and the Fubini theorem that

$$\int_0^\infty (\widetilde{w}(x,t))^2 t^{1-2\mu}dt < \infty$$

for almost all $x \in \Gamma$. Consequently application of (2.10) yields

$$\int_0^\infty \left(\frac{\partial w(f)}{\partial \nu}(x,s)\right)^2 s^{3-4\mu} ds = \frac{1}{2}\|\widetilde{u}(x,\cdot)\|^2_{H_\mu(R^+)}, \quad a.e. \quad x \in \Gamma,$$

and hence

$$\begin{aligned}\int_\Gamma \int_0^\infty \left(\frac{\partial w(f)}{\partial \nu}(x,t)\right)^2 t^{3-4\mu} dt dS &= \frac{\pi}{8}\int_\Gamma \left\| p^{-\frac{3}{2}} \frac{\partial u(f)}{\partial \nu}\left(x, \frac{1}{4p}\right)\right\|^2_{H_\mu(R^+)} dS \\ &= \frac{\pi}{8}\left\|\frac{\partial u(f)}{\partial \nu}\right\|^2_{B_\mu(\Gamma\times(0,\infty))}. \qquad (2.11)\end{aligned}$$

## 2.3 Third Step

In this step, we apply the stability estimate for the wave equation (2.1):

PROPOSITION 3 (Observability inequality) *Let $\Gamma \subset \partial\Omega$ be defined by (1.2) and let*

$$T > 2 \sup_{x\in\Omega} |x - x_0| \qquad (2.12)$$

*where $x_0 \in \mathbb{R}^r$ is a point which is arbitrarily chosen for specifying the observation subboundary $\Gamma$. Then there exists a constant $C_3 = C_3(\Omega, T, \Gamma) > 0$ such that*

$$\|f\|_{L^2(\Omega)} \le C_3 \left\|\frac{\partial u(f)}{\partial \nu}\right\|_{L^2(\Gamma\times(0,T))}. \qquad (2.13)$$

The estimate (2.13) is proved in Ho [9] and Lions [12]. See also Komornik [10].

Now, by combining (2.6) with (2.11), we have the second inequality in (1.6). Next, we fix $T > 0$ satisfying (2.12). Then, by Proposition 3, since

$$\begin{aligned}\int_\Gamma \int_0^T \left(\frac{\partial w(f)}{\partial \nu}(x,t)\right)^2 dt dS &\le T^{4\mu-3}\int_\Gamma \int_0^T \left(\frac{\partial w(f)}{\partial \nu}(x,t)\right)^2 t^{3-4\mu} dt dS \\ &\le T^{4\mu-3}\int_\Gamma \int_0^\infty \left(\frac{\partial w(f)}{\partial \nu}(x,t)\right)^2 t^{3-4\mu} dt dS,\end{aligned}$$

we obtain

$$
\begin{aligned}
\|f\|^2_{L^2(\Omega)} &\le C_3^2 T^{4\mu-3} \int_\Gamma \int_0^\infty \left( \frac{\partial w(f)}{\partial \nu}(x,t) \right)^2 t^{3-4\mu} dt dS \\
&= \frac{\pi}{8} C_3^2 T^{4\mu-3} \left\| \frac{\partial u(f)}{\partial \nu} \right\|^2_{B_\mu(\Gamma\times(0,\infty))}
\end{aligned}
$$

by (2.11), which is the first inequality in (1.6). Thus we complete the proof of Theorem.

## 3 CONCLUDING REMARKS

(1) For the inversion of the Laplace transform $\mathcal{L}g = h$, the complex one is well-known (e.g. Chapter 31 in Doetsch [4]). The complex form is, however, not adequate in our problem, because the observation data $h$ is real-valued and we have to extend the data analytically, which makes the stability unclear. For an analytical real inversion formula for the Laplace transform, see Byun-Saitoh [3].

(2) One of our keys is the transform formula between a heat equation and a wave equation, through which we reduce the stability in the heat problem to the one in the wave problem. A similar technique is used also in Yamamoto [19]. Thus we do not use the eigenfunction expansion of the solution to the heat equation, which is used in Exercise 11.4 in Cannon [2], Dolecki [5], Mizel and Seidman [13] and Sakawa [17].

(3) The norm $\left\| \frac{\partial u(f)}{\partial \nu} \right\|_{B_\mu(\Gamma\times(0,\infty))}$ for observations is taken over the whole time interval $(0, \infty)$. So far, we do not know whether or not we can reduce the observation time interval to a finite one with keeping the estimate of type (1.6).

(4) In Vu Kim Tuan and Yamamoto [18], for a similar observation problem, the transform formula is considered in terms of a Mellin convolution transform and another stability of Lipschitz type is obtained.

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