ON AN INTEGRAL OPERATOR ON THE UNIT BALL IN $\mathbb{C}^n$

STEVO STEVIĆ

Let $H(B)$ denote the space of all holomorphic functions on the unit ball $B \subset \mathbb{C}^n$. In this paper, we investigate the integral operator $T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)(dt/t)$, $f \in H(B)$, $z \in B$, where $g \in H(B)$ and $\Re g(z) = \sum_{j=1}^n z_j(\partial g/\partial z_j)(z)$ is the radial derivative of g. The operator can be considered as an extension of the Cesàro operator on the unit disk. The boundedness of the operator on a-Bloch spaces is considered.

1. Introduction

Let U be the unit disc in the complex plane $\mathbb{C}$ and $H(U)$ the space of all analytic functions in U.

For each complex γ with $\mathrm{Re}\,\gamma > -1$ and k nonnegative integer, let A_k^γ be defined as the kth coefficient in the expression

$$\frac{1}{(1-x)^{\gamma+1}} = \sum_{k=0}^\infty A_k^\gamma x^k, \tag{1.1}$$

so that $A_k^\gamma = (\gamma+1)\cdots(\gamma+k)/k!$.

For an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$ on U, the generalized Cesàro operator is defined by

$$\mathscr{C}^\gamma(f)(z) = \sum_{n=0}^\infty \left(\frac{1}{A_n^{\gamma+1}} \sum_{k=0}^n A_{n-k}^\gamma a_k\right) z^n. \tag{1.2}$$

For $\gamma = 0$ we obtain the Cesàro operator on U. The boundedness of the operator on some spaces of analytic functions was considered by a number of authors, see, for example, [8, 10, 13], and the references therein.

The integral form of $\mathscr{C}^0 = \mathscr{C}$ is

$$\mathscr{C}(f)(z) = \frac{1}{z}\int_0^z f(\zeta)\frac{1}{(1-\zeta)}d\zeta = \frac{1}{z}\int_0^z f(\zeta)\left(\ln\frac{1}{(1-\zeta)}\right)' d\zeta, \tag{1.3}$$

or, taking simply as a path the segment joining 0 and z,

$$\mathscr{C}(f)(z) = \int_0^1 f(tz)\left(\ln\frac{1}{(1-\zeta)}\right)'\Bigg|_{\zeta=tz} dt. \tag{1.4}$$

On most holomorphic function spaces the boundedness of the previous operator is equivalent to the boundedness of the operator

$$z\mathscr{C}(f)(z) = \int_0^z \frac{f(\zeta)}{1-\zeta}\,d\zeta. \tag{1.5}$$

Hence, Aleman and Siskakis [2] have introduced and investigated the following natural generalization of operator (1.5):

$$T_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta. \tag{1.6}$$

In [1, 2, 3] were investigated the boundedness and the compactness of the operator on Hardy and Bergman spaces. A natural question is to define a similar integral operator which acts on $H(B)$ (the space of all holomorphic functions in the unit ball B).

Let $z = (z_1,\ldots,z_n)$ and $w = (w_1,\ldots,w_n)$ be points in complex vector space $\mathbb{C}^n$ and

$$\langle z, w \rangle = z_1\bar{w}_1 + \cdots + z_n\bar{w}_n. \tag{1.7}$$

Let dV_N stand for the normalized Lebesgue measure on $\mathbb{C}^n$. For a holomorphic function f we denote

$$\nabla f = \left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right). \tag{1.8}$$

Let $\mathscr{R}f(z) = \sum_{j=1}^n z_j(\partial f/\partial z_j)(z)$ stand for the radial derivative of $f \in H(B)$ (see [7]). It is easy to see that if $f \in H(B)$, $f(z) = \sum_\alpha a_\alpha z^\alpha$, where α is a multi-index, then

$$\mathscr{R}f(z) = \sum_\alpha |\alpha|a_\alpha z^\alpha. \tag{1.9}$$

Let $a > 0$. The a-Bloch space $\mathscr{B}^a = \mathscr{B}^a(B)$ is the space of all $f \in H(B)$ such that

$$b_a(f) = \sup_{z\in B} (1 - |z|^2)^a |\mathscr{R}f(z)| < \infty. \tag{1.10}$$

The little a-Bloch space $\mathscr{B}_0^a = \mathscr{B}_0^a(B)$ consists of all $f \in H(B)$ such that

$$\lim_{|z|\to 1} (1 - |z|^2)^a |\mathscr{R}f(z)| = 0. \tag{1.11}$$

On $\mathscr{B}^a$ the norm is introduced by

$$\|f\|_{\mathscr{B}^a} = |f(0)| + b_a(f). \tag{1.12}$$

With this norm $\mathscr{B}^a$ is a Banach space and $\mathscr{B}_0^a$ is a closed subspace of $\mathscr{B}^a$. If $a = 1$, we denote $\mathscr{B}^a$ and $\mathscr{B}_0^a$ simply by $\mathscr{B}$ and $\mathscr{B}_0$.

The aim of this paper is to investigate the boundedness of the following operator:

$$T_g(f)(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t}, \quad f \in H(B),\ z \in B, \tag{1.13}$$

where $g \in H(B)$, on the a-Bloch spaces. This operator can be considered as a natural extension of operator (1.6) on $H(B)$ (when $n = 1$ we indeed obtain (1.6)). Operator (1.13) has appeared, for the first time, in [6] where its boundedness and compactness are investigated.

Closely related operators to the above mentioned on the unit polydisc were investigated in [4, 5, 9, 11, 12].

In this paper, we prove the following results.

THEOREM 1.1. *Let $g \in H(B)$ and $a \in (0,1)$. Then the following statements are equivalent:*

(a) T_g *is bounded on $\mathscr{B}^a$;*
(b) $\sup_{z \in B}|\Re g(z)|(1 - |z|^2)^a < \infty$.

Moreover $\|T_g\| \asymp \sup_{z \in B}|\Re g(z)|(1 - |z|^2)^a$.

THEOREM 1.2. *Let $g \in H(B)$. Then the following statements are equivalent:*

(a) T_g *is bounded on $\mathscr{B}$;*
(b) T_g *is bounded on $\mathscr{B}_0$;*
(c) $\sup_{z \in B}|\Re g(z)|(1 - |z|^2)\ln 1/(1 - |z|^2) < \infty$;

and the relationship $\|T_g\| \asymp \sup_{z \in B}|\Re g(z)|(1 - |z|^2)\ln 1/(1 - |z|^2)$ *holds.*

THEOREM 1.3. *Let $g \in H(B)$ and $a > 1$. Then the following statements are equivalent:*

(a) T_g *is bounded on $\mathscr{B}^a$;*
(b) $\sup_{z \in B}|\Re g(z)|(1 - |z|^2) < \infty$.

Moreover $\|T_g\| \asymp \sup_{z \in B}|\Re g(z)|(1 - |z|^2)$.

2. Auxiliary results

In order to prove our results, we need some auxiliary results which are incorporated in the following lemmas.

LEMMA 2.1. *For every $f, g \in H(B)$, it holds that*

$$\Re[T_g(f)](z) = f(z)\Re g(z). \tag{2.1}$$

Proof. Assume that the holomorphic function $f\Re g$ has the expansion $\sum_\alpha a_\alpha z^\alpha$. Then

$$\Re[T_g(f)](z) = \Re\int_0^1\sum_\alpha a_\alpha(tz)^\alpha\frac{dt}{t} = \Re\left(\sum_\alpha\frac{a_\alpha}{|\alpha|}z^\alpha\right) = \sum_\alpha a_\alpha z^\alpha, \tag{2.2}$$

which is what we wanted to prove.   □

Lemma 2.2. *Let $f \in \mathcal{B}^a(B)$, $0 < a < \infty$. Then*

$$|f(z)| \le C \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^a}, & a \in (0,1), \\ |f(0)| + \|f\|_{\mathcal{B}^a} \ln \dfrac{e}{1-|z|^2}, & a = 1, \\ |f(0)| + \dfrac{\|f\|_{\mathcal{B}^a}}{\left(1-|z|^2\right)^{a-1}}, & a > 1, \end{cases} \tag{2.3}$$

for some $C > 0$ independent of f.

Proof. Let $|z| > 1/2$, $z = r\zeta$, and $\zeta \in \partial B$. We have

$$\left| f(z) - f\left(\frac{r\zeta}{2}\right) \right| = \left| \int_{1/2}^1 \langle \nabla f(tz), z \rangle dt \right| \le \int_{1/2}^1 \left| \frac{\mathcal{R}f(tz)}{t} \right| dt \tag{2.4}$$

$$\le 4\|f\|_{\mathcal{B}^a} \int_0^1 \frac{|z|dt}{\left(1 - t^2|z|^2\right)^a}.$$

Let $I_a = \int_0^1 (|z|dt/(1 - t^2|z|^2)^a)$. If $a \in (0,1)$, then

$$I_a \le \int_0^1 \frac{|z|dt}{(1-t|z|)^a} = \frac{1 - (1-|z|)^{1-a}}{1-a} \le \frac{1}{1-a}. \tag{2.5}$$

If $a = 1$, then

$$\int_0^1 \frac{|z|dt}{\left(1 - t^2|z|^2\right)^a} = \frac{1}{2}\ln\frac{1+|z|}{1-|z|} \le \frac{1}{2}\ln\frac{4}{1-|z|^2}. \tag{2.6}$$

Finally, if $a > 1$, then

$$I_a \le \int_0^1 \frac{|z|dt}{(1-t|z|)^a} = \frac{1}{a-1}\left(\frac{1}{(1-|z|)^{a-1}} - 1\right) \le \frac{2^{a-1}}{(a-1)(1-|z|^2)^{a-1}}. \tag{2.7}$$

From all of the above we have

$$|f(z)| \le \begin{cases} M\left(\dfrac{1}{2}\right) + \dfrac{4\|f\|_{\mathcal{B}^a}}{1-a}, & a \in (0,1), \\ M\left(\dfrac{1}{2}\right) + 2\|f\|_{\mathcal{B}^a}\ln\dfrac{4}{1-|z|^2}, & a = 1, \\ M\left(\dfrac{1}{2}\right) + \dfrac{2^{a+1}\|f\|_{\mathcal{B}^a}}{(a-1)(1-|z|^2)^{a-1}}, & a > 1, \end{cases} \tag{2.8}$$

where $M(1/2) = \max_{|z|\le 1/2} |f(z)|$.

Let $|z| \leq 1/2$, then, by the mean value property of the function $f(z) - f(0)$ (see [7]), and Jensen's inequality, we obtain

$$
\begin{aligned}
\max_{|z| \leq 1/2} |f(z) - f(0)|^2 &\leq 4^n \int_{|z| \leq 3/4} |f(w) - f(0)|^2 dV_N(w) \\
&\leq 4^n \int_{|z| \leq 3/4} |\Re f(w)|^2 dV_N(w) \\
&\leq 3^n \max_{|z| \leq 3/4} |\Re f(z)|^2.
\end{aligned} \tag{2.9}
$$

The second inequality can be easily proved by using the homogeneous expansion of f. Hence,

$$
M\left(\frac{1}{2}\right) \leq |f(0)| + (\sqrt{3})^n \max_{|z| \leq 3/4} |\Re f(z)| \leq |f(0)| + \frac{2^{4a}(\sqrt{3})^n}{7^a} \|f\|_{\mathscr{B}^a}. \tag{2.10}
$$

From (2.8) and (2.10), the result follows easily when $a \neq 1$. If $a = 1$, then we have

$$
\begin{aligned}
|f(z)| &\leq |f(0)| + \frac{16(\sqrt{3})^n}{7} \|f\|_{\mathscr{B}} + 2\|f\|_{\mathscr{B}} \ln \frac{4}{1 - |z|^2} \\
&\leq \left(\frac{16(\sqrt{3})^n}{7} + \ln 16\right)\left(|f(0)| + \|f\|_{\mathscr{B}} \ln \frac{e}{1 - |z|^2}\right),
\end{aligned} \tag{2.11}
$$

thus finishing the proof. □

3. Proofs of the main results

Proof of Theorem 1.1. Assume that T_g is bounded on $\mathscr{B}^a$. Choose $f_0(z) \equiv 1$. It is clear that $f_0 \in \mathscr{B}_0^a$ and that $\|f_0\|_{\mathscr{B}^a} = 1$. The boundedness of T_g implies

$$
(1 - |z|^2)^a |\Re[T_g(f_0)](z)| = (1 - |z|^2)^a |\Re g(z)| \leq \|T_g\| \|f_0\|_{\mathscr{B}^a} = \|T_g\| < \infty. \tag{3.1}
$$

Hence $g \in \mathscr{B}^a$, as desired.

Assume now that $g \in \mathscr{B}^a$. Then, by Lemma 2.2 we have

$$
\begin{aligned}
(1 - |z|^2)^a |\Re[T_g(f)](z)| &= (1 - |z|^2)^a |f(z)| |\Re g(z)| \\
&\leq \|g\|_{\mathscr{B}^a} C(|f(0)| + \|f\|_{\mathscr{B}^a}) \\
&\leq 2C\|g\|_{\mathscr{B}^a} \|f\|_{\mathscr{B}^a}.
\end{aligned} \tag{3.2}
$$

Taking supremum $z \in B$ in (3.2), we obtain

$$
\|T_g(f)\|_{\mathscr{B}^a} \leq 2C\|g\|_{\mathscr{B}^a} \|f\|_{\mathscr{B}^a}. \tag{3.3}
$$

Hence

$$
\|T_g\| \leq 2C\|g\|_{\mathscr{B}^a}, \tag{3.4}
$$

as desired. □

Proof of Theorem 1.2. First, assume that T_g is bounded on $\mathcal{B}$. From the proof which follows we will see that we also consider the case when T_g is bounded on $\mathcal{B}_0$. For $w \in B$, put $f_w(z) = \ln 1/(1 - \langle z, w \rangle)$. Since

$$(1 - |z|^2) |\mathcal{R} f_w(z)| \leq (1 - |z|^2) |\nabla f_w(z)| = (1 - |z|^2) \left| \frac{w}{1 - \langle z, w \rangle} \right|$$
$$\leq \frac{(1 - |z|^2)}{|1 - \langle z, w \rangle|} \leq 2, \tag{3.5}$$

we have $\| f_w \|_{\mathcal{B}} \leq 2$, for each $w \in B$. On the other hand, we have

$$(1 - |z|^2) |\mathcal{R} f_w(z)| \leq \frac{(1 - |z|^2)}{|1 - \langle z, w \rangle|} \leq \frac{(1 - |z|^2)}{1 - |w|} \longrightarrow 0, \tag{3.6}$$

as $|z| \to 1$. Hence $f_w \in \mathcal{B}_0$, for each $w \in B$.

By Lemma 2.1 we have

$$(1 - |w|^2) |\mathcal{R} g(w)| \ln \frac{1}{1 - |w|^2} = |f_w(w) \mathcal{R} g(w)| (1 - |w|^2)$$
$$= |\mathcal{R} (T_g f_w)(w)| (1 - |w|^2) \tag{3.7}$$
$$\leq \| T_g f_w \|_{\mathcal{B}} \leq 2 \| T_g \|.$$

Taking supremum in (3.7) over $w \in B$, we obtain that conditions (a) and (b) imply (c).

Assume that (c) holds. Since $|f(0)| \leq \| f \|_{\mathcal{B}}$, and by Lemma 2.2, we have

$$|f(z)| \leq C \| f \|_{\mathcal{B}} \left(1 + \ln \frac{1}{1 - |z|^2} \right), \tag{3.8}$$

for some $C > 0$.

Hence

$$|\mathcal{R} [T_g(f)](z)| (1 - |z|^2) = |f(z)| |\mathcal{R} g(z)| (1 - |z|^2)$$
$$\leq C \| f \|_{\mathcal{B}} \left(1 + \ln \frac{1}{1 - |z|^2} \right) |\mathcal{R} g(z)| (1 - |z|^2)$$
$$\leq C \| f \|_{\mathcal{B}} \sup_{|z| \leq 1/2} \left(1 + \ln \frac{1}{1 - |z|^2} \right) |\mathcal{R} g(z)| (1 - |z|^2) \tag{3.9}$$
$$+ C \| f \|_{\mathcal{B}} \sup_{1/2 < |z| < 1} \left(1 + \ln \frac{1}{1 - |z|^2} \right) |\mathcal{R} g(z)| (1 - |z|^2)$$
$$\leq C_1 \| f \|_{\mathcal{B}} \sup_{z \in B} \ln \frac{1}{1 - |z|^2} |\mathcal{R} g(z)| (1 - |z|^2),$$

since (c) implies

$$\sup_{z \in B} |\mathcal{R} g(z)| (1 - |z|^2) < \infty. \tag{3.10}$$

From (3.9) and since $T_g(f)(0) = 0$, (a) follows.

We now prove that (c) implies (b). Since $\ln 1/(1 - |z|) \to \infty$ as $|z| \to 1$, we have that $g \in \mathcal{B}_0$. Hence, by Lemma 2.1, we have that for each polynomial $p(z)$,

$$(1 - |z|^2)\,|\mathfrak{R}[T_g(p)](z)| = (1 - |z|^2)\,|p(z)|\,|\mathfrak{R}g(z)| \leq M_p(1 - |z|^2)\,|\mathfrak{R}g(z)|, \quad (3.11)$$

where $M_p = \sup_{z \in B}|p(z)|$. Since $M_p < \infty$ and $g \in \mathcal{B}_0$, we obtain that for each polynomial $p, T_g(p) \in \mathcal{B}_0$. The set of polynomials is dense in $\mathcal{B}_0$, thus for every $f \in \mathcal{B}_0$ there is a sequence of polynomials (p_n) such that $\|p_n - f\|_{\mathcal{B}} \to 0$. Hence

$$\|T_g p_n - T_g f\|_{\mathcal{B}} \leq \|T_g\|\,\|p_n - f\|_{\mathcal{B}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \quad (3.12)$$

since the operator T_g is bounded. Hence $T_g(\mathcal{B}_0) \subset \mathcal{B}_0$, since $\mathcal{B}_0$ is closed subset of $\mathcal{B}$.

Finally, from (3.7) and (3.9) it follows that

$$\|T_g\| \asymp \sup_{z \in B}\,|\mathfrak{R}g(z)|\,(1 - |z|^2)\ln\frac{1}{1 - |z|^2}. \quad (3.13)$$

$\square$

Proof of Theorem 1.3. Let T_g be bounded on $\mathcal{B}^a$. Let $w \in B$, and $f_w(z) = 1/(1 - \langle z, w \rangle)^{a-1}$. It is clear that $f_w \in \mathcal{B}^a$ and that $\|f_w\|_{\mathcal{B}^a} \leq (a - 1)2^a$. The boundedness of T_g implies

$$\begin{aligned}
(1 - |w|^2)^a\,|\mathfrak{R}[T_g(f_w)](w)| &= (1 - |w|^2)^a\,|\mathfrak{R}g(w)|\,|f_w(w)| \\
&= (1 - |w|^2)\,|\mathfrak{R}g(w)| \\
&\leq \|T_g\|\,\|f_w\|_{\mathcal{B}^a} \\
&= (a - 1)2^a\|T_g\| < \infty.
\end{aligned} \quad (3.14)$$

Hence $\sup_{w \in B}(1 - |w|^2)|\mathfrak{R}g(w)| < \infty$, as desired.

Assume now that $g \in \mathcal{B}$. Then, by Lemma 2.2 we have

$$\begin{aligned}
(1 - |z|^2)^a\,|\mathfrak{R}[T_g(f)](z)| &= (1 - |z|^2)^a\,|f(z)|\,|\mathfrak{R}g(z)| \\
&\leq b_1(g)(1 - |z|^2)^{a-1}C\left(|f(0)| + \frac{\|f\|_{\mathcal{B}^a}}{(1 - |z|^2)^{a-1}}\right) \\
&\leq 2Cb_1(g)\|f\|_{\mathcal{B}^a}.
\end{aligned} \quad (3.15)$$

Hence

$$\|T_g(f)\|_{\mathcal{B}^a} \leq 2Cb_1(g)\|f\|_{\mathcal{B}^a}, \quad (3.16)$$

and consequently $\|T_g\| \leq 2Cb_1(g)$, as desired.

Form (3.14) and (3.16) it follows that $\|T_g\| \asymp \sup_{z \in B}|\mathfrak{R}g(z)|(1 - |z|^2)$. $\square$

References

[1]   A. Aleman and J. A. Cima, *An integral operator on H^p and Hardy's inequality*, J. Anal. Math. **85** (2001), 157–176.

[2]   A. Aleman and A. G. Siskakis, *An integral operator on H^p*, Complex Variables Theory Appl. **28** (1995), no. 2, 149–158.

[3]   —————, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337–356.

[4]   D.-C. Chang and S. Stević, *Estimates of an integral operator on function spaces*, Taiwanese J. Math. **7** (2003), no. 3, 423–432.

[5]   —————, *The generalized Cesàro operator on the unit polydisk*, Taiwanese J. Math. **7** (2003), no. 2, 293–308.

[6]   Z. Hu, *Extended Cesàro operators on mixed norm spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2171–2179.

[7]   W. Rudin, *Function Theory in the Unit Ball of $\mathbf{C}^n$*, Grundlehren der Mathematischen Wissenschaften, vol. 241, Springer-Verlag, New York, 1980.

[8]   K. Stempak, *Cesàro averaging operators*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 1, 121–126.

[9]   S. Stević, *Cesàro averaging operators*, Math. Nachr. **248/249** (2003), 185–189.

[10] —————, *The generalized Cesàro operator on Dirichlet spaces*, Studia Sci. Math. Hungar. **40** (2003), no. 1-2, 83–94.

[11] —————, *The generalized Libera transform on Hardy, Bergman and Bloch spaces on the unit polydisc*, Z. Anal. Anwendungen **22** (2003), no. 1, 179–186.

[12] —————, *Hilbert operator on the polydisk*, Bull. Inst. Math. Acad. Sinica **31** (2003), no. 2, 135–142.

[13] J. Xiao, *Cesàro-type operators on Hardy, BMOA and Bloch spaces*, Arch. Math. (Basel) **68** (1997), no. 5, 398–406.

Stevo Stević: Mathematical Institute of Serbian Academy of Sciences and Arts, Knez Mihailova 35/I, 11000 Beograd, Serbia

E-mail addresses: sstevo@matf.bg.ac.yu ;   sstevic@ptt.yu