MULTIPLE SOLUTIONS FOR IMPULSIVE SEMILINEAR FUNCTIONAL AND NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS IN HILBERT SPACE

M. BENCHOHRA, J. HENDERSON, S. K. NTOUYAS, AND A. OUAHAB

Received 1 April 2003 and in revised form 30 April 2003

The well-known Krasnoselskii twin fixed point theorem is used to investigate the existence of mild solutions for first- and second-order impulsive semilinear functional and neutral functional differential equations in Hilbert spaces.

1. Introduction

This paper is concerned with the existence of mild solutions of some classes of initial value problem for first- and second-order impulsive semilinear functional and neutral functional differential equations. Initially, we will consider initial value problems for first-order impulsive semilinear functional differential equations

$$\begin{aligned} y'(t) - Ay(t) &= f(t, y_t), \quad \text{a.e. } t \in J := [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} &= I_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ y(t) &= \phi(t), \quad t \in [-r,0], \end{aligned} \tag{1.1}$$

where $f : J \times D \to H$ is a given function, $D = \{\psi : [-r,0] \to H \mid \psi$ is continuous everywhere except for a finite number of points $\bar{t}$ at which $\psi(\bar{t})$ and $\psi(\bar{t}^+)$ exist and $\psi(\bar{t}^-) = \psi(\bar{t})\}$, $\phi \in D$, $0 < r < \infty$, A is a densely defined operator generating a semigroup $\{T(t) : t \geq 0\}$ of bounded linear operators from H into H, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k \in C(H,H)(k = 1,2,\dots,m)$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively, and H is a real Hilbert space with norm $\|\cdot\|$ inherited from the scalar product $\langle \cdot,\cdot \rangle$. For any continuous function y defined on $[-r,b] - \{t_1,\dots,t_m\}$ and any $t \in [0,b]$, we denote by y_t the element of D defined by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r,0]$. Here $y_t(\cdot)$ represents the history of the state from time $t-r$ up to the present time t.

Later, we study the second-order impulsive semilinear functional differential equations of the form

$$\begin{aligned} y''(t) - Ay(t) &= f(t, y_t), \quad \text{a.e. } t \in J = [0,b],\ t \neq t_k,\ k = 1,\dots,m, \\ \Delta y|_{t=t_k} &= I_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ \Delta y'|_{t=t_k} &= \bar{I}_k\big(y(t_k^-)\big), \quad k = 1,\dots,m, \\ y(t) &= \phi(t), \quad t \in [-r,0], \qquad y'(0) = \eta, \end{aligned} \tag{1.2}$$

Journal of Inequalities and Applications 2005:2 (2005) 189–205
DOI: 10.1155/JIA.2005.189

where f, I_k, and ϕ are as in problem (1.1), $\bar{I}_k \in C(H,H)$, $\eta \in H$, and A is an operator generating a family of linear bounded cosine operators $C(t)$, $t \geq 0$.

Sections 5 and 6 are devoted to the existence of solutions for initial value problems for first- and second-order impulsive semilinear neutral functional differential equations. In Section 5, we consider first-order impulsive semilinear neutral functional equations of the form

$$\begin{gathered}
\frac{d}{dt}\left[y(t)-g(t,y_t)\right]-Ay(t)=f(t,y_t), \quad \text{a.e. } t \in J=[0,b]\ t \neq t_k,\ k=1,\ldots,m, \\
\Delta y|_{t=t_k}=I_k\big(y(t_k^-)\big), \quad k=1,\ldots,m, \\
y(t)=\phi(t), \quad t \in [-r,0],
\end{gathered} \tag{1.3}$$

where f, I_k, A, and ϕ are as in problem (1.1). In Section 6, we study the second-order problem

$$\begin{gathered}
\frac{d}{dt}\left[y'(t)-g(t,y_t)\right]-Ay(t)=f(t,y_t), \quad \text{a.e. } t \in [0,b]\ t \neq t_k,\ k=1,\ldots,m, \\
\Delta y|_{t=t_k}=I_k\big(y(t_k^-)\big), \quad k=1,\ldots,m, \\
\Delta y'|_{t=t_k}=\bar{I}_k\big(y(t_k^-)\big), \quad k=1,\ldots,m, \\
y(t)=\phi(t), \quad t \in [-r,0], \qquad y'(0)=\eta,
\end{gathered} \tag{1.4}$$

where f, I_k, $\bar{I}_k$, A, η, and ϕ are as in problems (1.1) and (1.2).

Differential and partial differential equations with impulses are a basic tool to study evolution processes that are subjected to abrupt changes in their state. Such equations arise naturally from a wide variety of applications, such as space-craft control, inspection processes in operations research, drug administration, and threshold theory in biology. There has been a significant development in the last few years; see the monographs by Bainov and Simeonov [2], Lakshmikantham et al. [11], and Samoilenko and Perestyuk [16], the papers by Ahmed [1], Liu and Zhang [13], Liu [12], Erbe et al. [7] and the survey paper by Rogovchenko [15]. A natural generalization of impulsive ordinary and partial differential equations is impulsive functional differential and functional partial differential equations. In spite of the great possibilities for applications, the theory of these equations is developing rather slowly due to a series of difficulties of technical and theoretical character.

The impulsive neutral and semilinear neutral differential equations were studied by Benchohra et al. in [3, 4, 5, 6]. The goal of this paper is to prove the existence of multiple solutions for impulsive functional semilinear and neutral semilinear functional differential equations. Our approach here is based on the Krasnoselskii twin fixed point theorem (see [10]).

This paper will be divided into six sections. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we establish an existence theorem for (1.1). In Section 4, we will establish an existence theorem for (1.2). In Sections 5 and 6, we study the existence of mild solutions for first and second impulsive semilinear neutral functional differential equations, respectively.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

D is the Banach space with the norm

$$\|\phi\|_D := \sup\{||\phi(\theta)|| : -r \leq \theta \leq 0\}. \tag{2.1}$$

$B(H)$ is the Banach space of all linear bounded operators from H into H with norm

$$\|N\|_{B(H)} := \sup\{||N(y)|| : \|y\| = 1\}. \tag{2.2}$$

A measurable function $y : J \to H$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. (For properties of the Bochner integral, see, e.g., Yosida [19].)

$L^1(J,H)$ denotes the Banach space of functions $y : J \to H$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt. \tag{2.3}$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(H)$ is a strongly continuous cosine family if:

(1) $C(0) = I$ (I is the identity operator in H),

(2) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s,t \in \mathbb{R}$,

(3) the map $t \mapsto C(t)y$ is strongly continuous for each $y \in H$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated with the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in H,\ t \in \mathbb{R}. \tag{2.4}$$

The infinitesimal generator $A : H \to H$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2}C(t)y|_{t=0}. \tag{2.5}$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [9], Fattorini [8], and to the papers of Travis and Webb [17, 18]. For properties of semigroup theory, we refer the interested reader to the book of Pazy [14].

Definition 2.1. A map $f : J \times D \to H$ is said to be L^1-Carathéodory if

(i) $t \mapsto f(t,u)$ is measurable for each $u \in D$;

(ii) $u \mapsto f(t,u)$ is continuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $h_q \in L^1(J,\mathbb{R}_+)$ such that

$$||f(t,u)|| \leq h_q(t) \quad \forall \|u\| \leq q \text{ and for almost all } t \in J. \tag{2.6}$$

Our consideration is based on the following twin fixed point theorem given by Krasnoselskii (see Guo and Lakshmikantham [10]).

Theorem 2.2. *Let E be a Banach space, $C \subset E$ a cone of E, and $R > 0$ a constant. Let $C_R = \{y \in C : \|y\| < R\}$ and let $N : C_R \to C$ be a completely continuous operator where $0 < r < R$. If*

(A1) $\|N(y)\| < \|y\|$ *for all* $y \in \partial C_r$,

(A2) $\|N(y)\| > \|y\|$ *for all* $y \in \partial C_R$,

then N has at least two fixed points y_1, y_2 in $\overline{C}_R$. Furthermore,

$$||y_1|| < r, \qquad r < ||y_2|| \leq R. \tag{2.7}$$

In what follows we will assume that f is an L^1-Carathéodory function. In order to define its mild solution we will consider the space

$$\begin{aligned} PC = \{y : [0,b] \longrightarrow H : y_k \in C(J_k, H),\ k = 0,\dots,m \\ \text{and } \exists y(t_k^-),\quad y(t_k^+),\ k = 1,\dots,m \text{ with } y(t_k^-) = y(t_k)\}, \end{aligned} \tag{2.8}$$

which is a Banach space with the norm

$$\|y\|_{PC} = \max\{||y_k||_{J_k},\ k = 0,\dots,m\}, \tag{2.9}$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 0,\dots,m$. Set $\Omega := D \cup PC$. Then Ω is a Banach space with norm

$$\|y\|_\Omega = \sup\{||y(t)|| :\ t \in [-r,b]\}. \tag{2.10}$$

3. First-order impulsive functional differential inclusions

The main result of this section is devoted to the IVP (1.1). Before stating and proving the main result, we give the definition of a mild solution of the IVP (1.1).

Definition 3.1. A function $y \in \Omega$ is said to be a mild solution of (1.1) if $y(t) = \phi(t)$, $t \in [-r,0]$ and y is a solution of impulsive integral equation

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \tag{3.1}$$

The following hypotheses are assumed hereafter.

(H1) $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}$, $t \geq 0$, which is compact for $t > 0$ and there is a constant $M \geq 1$ such that

$$||T(t)||_{B(H)} \leq M \quad \text{for each } t \geq 0. \tag{3.2}$$

(H2) There exist constants c_k such that

$$||I_k(y)|| \leq c_k, \quad k = 1,\dots,m \text{ for each } y \in H. \tag{3.3}$$

(H3) There exist a continuous nondecreasing function

$$\psi : [0,\infty) \longrightarrow (0,\infty), \quad p \in L^1([0,b],\mathbb{R}_+), \tag{3.4}$$

and a nonnegative number $r > 0$ such that

$$M\|\phi\| + M\sum_{k=1}^{m} c_k + M\psi(r)\|p\|_{L^1} < r. \tag{3.5}$$

(H4) There exists $R > r$, such that for any $z \in \Omega$ with $\|z\|_\Omega \leq R$ and $t \in [0,b]$ and for all $(x,u) \in H \times D$ we have

$$\left\langle z(t), T(t)\phi(0) + \int_0^t T(t-s)f(s,u)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x) \right\rangle \geq 0, \tag{3.6}$$

and for each $y \in \Omega$ such that $\|y\|_\Omega = R$,

$$\left| \left\langle y(t), T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)) \right\rangle \right| \geq R^2. \tag{3.7}$$

Theorem 3.2. *Suppose that hypotheses (H1)–(H4) are satisfied. Then the impulsive initial value problem (1.1) has at least two solutions.*

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)ds & \\ + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), & \text{if } t \in [0,b]. \end{cases} \tag{3.8}$$

We will show that N is completely continuous. The proof will be given in several steps.
Step 1. N sends bounded sets into bounded sets in Ω.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $y \in B_q := \{y \in \Omega : \|y\|_\Omega \leq q\}$ one has $\|N(y)\|_\Omega \leq l$. So choose $y \in B_q$. Then, for each $t \in [0,b]$,

$$N(y)(t) = T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \tag{3.9}$$

By (H1)-(H2) we have, for each $t \in [0,b]$,

$$\begin{aligned} ||N(y)(t)|| &\leq M||\phi(0)|| + M\int_0^t ||f(s,y_s)||ds + M\sum_{k=1}^{m} c_k \\ &\leq M\|\phi\| + M||\varphi_q||_{L^1} + M\sum_{k=1}^{m} c_k := l. \end{aligned} \tag{3.10}$$

Step 2. N sends bounded sets in Ω into equicontinuous sets.

Let $u_1, u_2 \in J$ and $\epsilon > 0$ with $0 < \epsilon < u_1 < u_2$, let $B_q := \{y \in \Omega : \|y\|_\Omega \leq q\}$ be a bounded set in Ω and let $y \in B_q$. Then we have

$$\begin{aligned}
||N(y)(u_2) - N(y)(u_1)|| \leq{}& ||T(u_2)\phi(0) - T(u_1)\phi(0)|| \\
&+ M\int_{u_1}^{u_2} \varphi_q(s)ds \\
&+ \int_0^{u_1-\epsilon} ||[T(u_1 - s) - T(u_2 - s)]\varphi_q(s)||ds \\
&+ \int_{u_1-\epsilon}^{u_1} ||[T(u_1 - s) - T(u_2 - s)]\varphi_q(s)||ds \\
&+ M \sum_{0<t_k<\tau_2-\tau_1} c_k + \sum_{0<t_k<\tau_1} c_k ||T(\tau_2 - t_k) - T(\tau_1 - t_k)||.
\end{aligned} \tag{3.11}$$

As $u_2 \to u_1$, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator, and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. As a consequence of the Arzelá-Ascoli theorem, it suffices to show that N maps B_q into a precompact set in H. Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$, we define

$$\begin{aligned}
N_\epsilon(y)(t) = T(t)y_0 + T(\epsilon)\int_0^{t-\epsilon} T(t - s - \epsilon) f(s, y_s)ds \\
+ T(\epsilon) \sum_{0<t_k<t} T(t - t_k - \varepsilon) I_k(y(t_k^-)).
\end{aligned} \tag{3.12}$$

Since $T(t)$ is a compact operator, the set $\{N_\epsilon(y)(t); y \in B_q\}$ is precompact in H for every ϵ, $0 < \epsilon < t$. Moreover, we have

$$|N_\epsilon(y)(t) - N(y)(t)| \leq M\int_{t-\epsilon}^{t} \varphi_q(s)ds + \sum_{t-\epsilon<t_k<t} Mc_k. \tag{3.13}$$

Therefore there are precompact sets arbitrarily close to the set $\{N(y)(t) : y \in B_q\}$. Hence the set $\{N(y)(t) : y \in B_q\}$ is precompact in H. The equicontinuity for the cases $u_1 < u_2 \leq 0$ and $u_1 \leq 0 \leq u_2$ is obvious.

Step 3. N is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned}
||N(y_n)(t) - N(y)(t)|| &\leq M\int_0^t ||f(s, y_{n,s}) - f(s, y_s)||ds \\
&\quad + M \sum_{0<t_k<t} ||I_k(y_n(t_k)) - I_k(y(t_k^-))|| \\
&\leq M\int_0^b ||f(s, y_{n,s}) - f(s, y_s)||ds \\
&\quad + M \sum_{0<t_k<t} ||I_k(y_n(t_k)) - I_k(y(t_k^-))||.
\end{aligned} \tag{3.14}$$

Since the functions I_k, $k = 1,\dots,m$, are continuous and f is L^1-Carathéodory, then

$$\begin{aligned}||N(y_n) - N(y)||_\Omega \le{}& M||f(\cdot,y_n) - f(\cdot,y)||_{L^1}\\ &+ M\sum_{k=1}^{m}||I_k(y_n(t_k^-)) - I_k(y(t_k^-))|| \longrightarrow 0\end{aligned} \tag{3.15}$$

as $n \to \infty$. As a consequence of steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : \Omega \to \Omega$ is completely continuous.

Let

$$C = \{y \in \Omega : \langle y(t), z(t)\rangle \ge 0 \text{ for } t \in [0,b],\ \forall z \in \Omega \text{ with } \|z\|_\Omega \le R\}. \tag{3.16}$$

Then C is a cone in Ω. Let $y, z \in C_R$. Then, by (H4), we have

$$\left\langle z(t), T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k))\right\rangle \ge 0. \tag{3.17}$$

Thus N maps C_R into C and it is a completely continuous map. Now it remains to show that the hypotheses of Theorem 2.2 are satisfied. First notice that condition (A1) of Theorem 2.2 holds since for $y \in \partial C_r$, we have from (H1)–(H3)

$$\begin{aligned}||N(y)(t)|| &\le M||\phi(0)|| + \int_0^t M||f(s,y_s)||ds + M\sum_{0<t_k<t}|I_k(y(t_k))|\\ &\le M\|\phi\| + M\sum_{k=1}^{m} c_k + M\psi(r)\|p\|_{L^1} < r.\end{aligned} \tag{3.18}$$

Finally to see that (A2) of Theorem 2.2 holds, let $y \in \partial C_R$, that is, $\|y\|_\Omega = R$. Then from (H4) we have

$$\left|\left\langle y(t), T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-))\right\rangle\right| \ge R^2. \tag{3.19}$$

By the Schwarz inequality we have

$$\begin{aligned}&\left|\left\langle y, T(t)\phi(0) + \int_0^t T(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-))\right\rangle\right|\\ &\quad\le ||y(t)||||N(y)(t)||\\ &\quad\le \|y\|_\Omega||N(y)||_\Omega\\ &\quad= R||N(y)||_\Omega.\end{aligned} \tag{3.20}$$

Hence

$$||N(y)||_\Omega \ge R := \|y\|_\Omega. \tag{3.21}$$

Thus condition (A2) of Theorem 2.2 holds. The Krasnoselskii twin fixed point theorem implies that N has at least two fixed points y_1, y_2 which are mild solutions to problem (1.1). Furthermore, we have

$$y_1 \in C_r, \qquad y_2 \in C_R - C_r. \tag{3.22}$$

□

4. Second-order impulsive functional differential inclusions

In this section, we give an existence result for the IVP (1.2).

Definition 4.1. A function $y \in \Omega$ is said to be a mild solution of (1.2) if $y(t) = \phi(t)$, $t \in [-r,0]$, $y'(0) = \eta$, and satisfies the following impulsive integral equation:

$$\begin{aligned} y(t) = {} & C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,y_s)\,ds \\ & + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\bar{I}_k(y(t_k^-))\right]. \end{aligned} \tag{4.1}$$

Theorem 4.2. *Assume that (H2) and the following conditions are satisfied.*

(H5) *There exists nonnegative constants d_k such that*

$$\|\bar{I}_k(y)\| \le d_k \text{ for each } y \in H,\ k = 1,\dots,m. \tag{4.2}$$

(H6) *$A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in J\}$ which is compact for $t > 0$, and there exists a constant $M_1 > 0$ such that $\|C(t)\|_{B(H)} < M_1$ for all $t \in \mathbb{R}$.*

(H7) *There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, $p \in L^1([0,b],\mathbb{R}_+)$, and nonnegative number $r^* > 0$ such that*

$$M_1\|\phi\| + bM_1\|\eta\| + bM_1\psi(r^*)\|p\|_{L^1} + \sum_{k=1}^{m} M_1[c_k + bd_k] < r^*. \tag{4.3}$$

(H8) *There exist $R^* > r^*$, such that for each $z \in \Omega$ with $\|z\|_\Omega \le R^*$, $t \in [0,b]$ and for all $(x,u) \in H \times D$,*

$$\begin{aligned} \Big\langle z(t), C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,u)\,ds \\ + \sum_{0<t_k<t} \left[C(t-t_k)I_k(x) + S(t-t_k)\bar{I}_k(x)\right] \Big\rangle \ge 0, \end{aligned} \tag{4.4}$$

and for any $y \in \Omega$ such that $\|y\|_\Omega = R^*$,

$$\begin{aligned} \Big| \Big\langle y(t), C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,y_s)\,ds \\ + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\bar{I}_k(y(t_k))\right] \Big\rangle \Big| \ge R^{*2}. \end{aligned} \tag{4.5}$$

Then the IVP (1.2) has at least two mild solutions.

Proof. Transform the problem (1.2) into a fixed point problem. Consider the operator $N_1 : \Omega \to \Omega$ defined by

$$N_1(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,y_s)\,ds & \\ + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k^-))\right], & \text{if } t \in [0,b]. \end{cases} \tag{4.6}$$

As in Theorem 3.2 we can show that N_1 is completely continuous.

Let

$$C = \left\{ y \in \Omega : \langle y(t), z(t) \rangle \geq 0 \text{ for } t \in [0,b],\ \forall z \in \Omega \text{ with } \|z\|_\Omega \leq R^* \right\} \tag{4.7}$$

be a cone in Ω. Let $y, z \in C_{R^*}$, then by (H8) we have

$$\begin{aligned} \Big\langle z(t), C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,y_s)\,ds & \\ + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k)) + S(t-t_k)\overline{I}_k(y(t_k))\right] \Big\rangle & \geq 0. \end{aligned} \tag{4.8}$$

Thus $N(C_{R^*}) \subset C$ and $N : C_{R^*} \to C$ is a completely continuous map. Now it remains to show that the hypotheses of Theorem 2.2 are satisfied. First notice that condition (A1) of Theorem 2.2 holds since for $y \in \partial C_{r^*}$, we have from (H2), (H5), (H6), and (H7)

$$\begin{aligned} \|N_1(y)(t)\| \leq M_1\|\phi\| + bM_1\|\eta\| + bM_2\psi(r^*)\|p\|_{L^1} & \\ + \sum_{k=1}^{m} M_1\left[c_k + bd_k\right] \leq r^*. & \end{aligned} \tag{4.9}$$

Finally, to see that Theorem 2.2 (A2) holds, let $y \in \partial C_{R^*}$, $\|y\|_\Omega = R^*$. Then from (H8) we have

$$\begin{aligned} \Big| \Big\langle y(t), C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,y_s)\,ds & \\ + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k))\right] \Big\rangle \Big| & \geq R^{*2}. \end{aligned} \tag{4.10}$$

By the Schwarz inequality we have

$$\begin{aligned} & \Big| \Big\langle y(t), C(t)\phi(0) + S(t)\eta + \int_0^t S(t-s)f(s,y_s)\,ds \\ & \quad + \sum_{0<t_k<t} \left[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k))\right] \Big\rangle \Big| \\ & \quad \leq \|y\|_\Omega \|N_1(y)\|_\Omega = R^*\|N_1(y)\|_\Omega. \end{aligned} \tag{4.11}$$

Thus

$$\|N_1(y)\|_\Omega \geq R^*. \tag{4.12}$$

Hence condition (A2) of Theorem 2.2 holds. Then Krasnoselskii's twin fixed point theorem implies that N has at least two fixed points y_1, y_2 which are mild solutions to problem (1.2). Furthermore, we have

$$y_1 \in C_{r^*}, \qquad y_2 \in C_{R^*} - C_{r^*}. \tag{4.13}$$

□

5. First-order impulsive neutral functional differential inclusions

We start by defining what we mean by a solution of IVP (1.3).

Definition 5.1. A function $y \in \Omega$ is said to be a mild solution of (1.3) if $y(t) = \phi(t)$, $t \in [-r,0]$, the restriction of $y(\cdot)$ to the interval $[0,b)$ is continuous, and for each $0 \leq t < b$, the function $AT(t-s)g(s,y_s)$, $s \in [0,t)$, is integrable and y is the solution of the impulsive integral equation

$$\begin{aligned} y(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,y_t) + \int_0^t AT(t-s)g(s,y_s)\,ds \\ + \int_0^t T(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{5.1}$$

The following hypotheses are assumed hereafter.

(B1) (i) There exist constants $c_1, c_2 > 0$ such that

$$\|g(t,u)\| \leq c_1\|u\| + c_2, \quad (t,u) \in [0,b] \times D. \tag{5.2}$$

(ii) The function g is completely continuous such that the operator

$$G : C([-r,b],H) \longrightarrow C([0,b],H) \tag{5.3}$$

defined by $(Gy)(t) = g(t,y_t)$ is compact.

(B2) $A : D(A) \subset H \to H$ is the infinititesimal generator of a compact semigroup $\{T(t)\}$, $t > 0$, such that

$$\|AT(t)\|_{B(H)} \leq M_2, \qquad \text{for some } M_2 > 0. \tag{5.4}$$

(B3) There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$, $p \in L^1([0,b], \mathbb{R}_+)$, and nonnegative numbers $r_1 > 0$ such that

$$M\|\phi\| + (M + 1 + bM_2)(c_1r_1 + c_2) + M\psi(r_1)\|p\|_{L^1} + M\sum_{k=1}^m c_k < r_1. \tag{5.5}$$

(B4) There exists $R_1 > r_1$, such that for each $z \in \Omega$ with $\|z\|_\Omega \leq R_1$, $t \in [0,b]$, and $(x,u) \in H \times D$, we have

$$\begin{aligned} &\Big\langle z(t), T(t)[\phi(0) - g(0,\phi(0))] + \int_0^t AT(t-s)g(s,u)ds \\ &\quad + g(t,u) + \int_0^t T(t-s)f(s,u)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x) \Big\rangle \geq 0, \end{aligned} \tag{5.6}$$

and for all $y \in \Omega$ such that $\|y\|_\Omega = R_1$,

$$\begin{aligned} &\Big| \Big\langle y(t), T(t)[\phi(0) - g(0,\phi(0))] + \int_0^t AT(t-s)g(s,y_s)ds \\ &\quad + g(t,y_t) + \int_0^t T(t-s)f(s,y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)) \Big\rangle \Big| \geq R_1^2. \end{aligned} \tag{5.7}$$

Theorem 5.2. *Assume that hypotheses (H1), (H2), and (B1)–(B4) hold. Then the problem (1.3) has least two mild solutions.*

Proof. Consider the operator $N_2 : \Omega \to \Omega$ defined by

$$N_2(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)[\phi(0) - g(0,\phi)] + g(t,y_t) + \displaystyle\int_0^t AT(t-s)g(s,y_s)ds & \\ + \displaystyle\int_0^t T(t-s)f(s,y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), & \text{if } t \in [0,b]. \end{cases} \tag{5.8}$$

We will show that N_2 is completely continuous. The proof will be given in several steps.
Step 1. N_2 sends bounded sets into bounded sets in Ω.

Indeed, it is enough to show that for any $q > 0$ there exists a positive constant l such that for each $y \in B_q := \{y \in \Omega : \|y\|_\Omega \leq q\}$ one has $\|N_2(y)\|_\Omega \leq l$. Let $y \in B_q$, then

$$\begin{aligned} N_2(y)(t) &= T(t)[\phi(0) - g(0,\phi(0))] + g(t,y_t) + \int_0^t AT(t-s)g(s,y_s)ds \\ &\quad + \int_0^t T(t-s)f(s,y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)). \end{aligned} \tag{5.9}$$

From (H1), (H2), (B1), (B2), and (B3) we have for each $t \in [0,b]$,

$$\begin{aligned} \|N_2(y)(t)\| &\leq \|T(t)\|[\|\phi\| + \|g(0,\phi(0))\|] + \|g(t,y_t)\| + \int_0^t \|AT(t-s)\|\|g(s,y_s)\|ds \\ &\quad + \int_0^t \|T(t-s)\|\|f(s,y_s)\|ds + \sum_{0<t_k<t} \|T(t-t_k)\|\|I_k(y(t_k^-))\| \\ &\leq M\|\phi\| + (M+1+bM_2)(c_1q+c_2) + M\|\varphi_q\|_{L^1} + M\sum_{k=1}^m c_k := l. \end{aligned} \tag{5.10}$$

Step 2. N_2 sends bounded sets in Ω into equicontinuous sets.

Using (B2), it suffices to show that the operator $\overline{N}_2 : \Omega \to \Omega$ defined by

$$\overline{N}_2(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ T(t)\phi(0) + \int_0^t AT(t-s)g(s,y_s)ds & \\ + \int_0^t T(t-s)f(s,y_s)ds + \sum_{0<t_k<t} T(t-t_k)I_k(y(t_k^-)), & \text{if } t \in [0,b], \end{cases} \tag{5.11}$$

maps bounded sets into equicontinuous sets of Ω. Let $u_1,u_2 \in J$ and $\epsilon > 0$ with $0 < \epsilon < u_1 < u_2$, $B_q := \{y \in \Omega : \|y\|_\Omega \leq q\}$ be a bounded set in Ω and $y \in B_q$. Then we have

$$\begin{aligned} \|\overline{N}_2(y)(u_2) - \overline{N}_2(y)(u_1)\| &\leq \|T(u_1)\phi(0) - T(u_2)\phi(0)\| \\ &\quad + (c_1q+c_2)\int_0^{u_1-\epsilon} \|AT(u_1-s) - AT(u_2-s)\|ds \\ &\quad + (c_1q+c_2)\int_{u_1-\epsilon}^{u_1} \|AT(u_1-s) - AT(u_2-s)\|ds \\ &\quad + M_2(c_1q+c_2)(u_2-u_1) \\ &\quad + \int_0^{u_1-\epsilon} \|T(u_1-s) - T(u_2-s)\|\varphi_q(s)ds \\ &\quad + \int_{u_1}^{u_1-\epsilon} \|T(u_1-s) - T(u_2-s)\|\varphi_q(s)ds \\ &\quad + M_1\int_{u_1}^{u_2} \varphi_q(s)ds + M\sum_{0<t_k<\tau_2-\tau_1} c_k \\ &\quad + \sum_{0<t_k<\tau_1} c_k\|T(u_2-t_k) - T(u_1-t_k)\|. \end{aligned} \tag{5.12}$$

As $u_2 \to u_1$ the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator, and the compactness of $T(t)$ for $t > 0$ implies the continuity in the uniform operator topology. As a consequence of the Arzelá-Ascoli theorem, it suffices to show that $\overline{N}_2$ multivalued maps B_q into a precompact set in H. Let $0 < t \le b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ we define

$$\begin{aligned}\overline{N}_2^{\epsilon}(y)(t) = T(t)\phi(0) + \int_0^{t-\epsilon} AT(t-s)g(s,y_s)\,ds \\ + \int_0^{t-\epsilon} T(t-s-\epsilon)f(s,y_s)\,ds + T(t-\epsilon)\sum_{0<t_k<t-\epsilon} T(t-t_k-\epsilon)I_k(y(t_k^-)).\end{aligned} \tag{5.13}$$

Since $T(t)$ is a compact operator, the set $\{\overline{N}_2^{\epsilon}(y)(t) : y \in B_q\}$ is precompact in H for every $0 < \epsilon < t$. Moreover, we have

$$\begin{aligned}||\overline{N}_2^{\epsilon}(y)(t) - \overline{N}_2(y)(t)|| \le \int_{t-\epsilon}^{t} ||AT(t-s)g(s,y_s)||\,ds \\ + \int_{t-\epsilon}^{t} ||T(t-s)||\varphi_q(s)\,ds \\ + \sum_{t-\epsilon<t_k<t} ||T(t-t_k)||c_k.\end{aligned} \tag{5.14}$$

Therefore there are precompact sets arbitrarily close to the set $\{\overline{N}_2^{\epsilon}(y)(t) : y \in B_q\}$. Hence the set $\{\overline{N}_2(y)(t) : y \in B_q\}$ is precompact in E. The equicontinuity for the cases $u_1 < u_2 \le 0$ and $u_1 \le 0 \le u_2$ is obvious.

Step 3. $\overline{N}_2$ is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in Ω. Then

$$\begin{aligned}||\overline{N}_2(y_n)(t) - \overline{N}_2(y)(t)|| \le M_2 \int_0^t ||g(s,y_{n,s}) - g(s,y_s)||\,ds \\ + M\int_0^b ||f(s,y_{n,s}) - f(s,y_s)||\,ds \\ + M\sum_{0<t_k<t} M||I_k(y_n(t_k^-)) - I_k(y(t_k^-))||.\end{aligned} \tag{5.15}$$

Since the functions I_k, $k = 1,\dots,m$, are continuous, g completely continuous, then

$$\begin{aligned}&||\overline{N}_2(y_n) - \overline{N}_2(y)||_{\Omega} \\ &\quad \le M_3 \sup_{t\in J}\int_0^t |g(s,y_{ns}) - g(s,y_s)|\,ds \\ &\qquad + M||f(\cdot,y_n) - f(\cdot,y)||_{L^1} + M\sum_{k=1}^{m} ||I_k(y_n(t_k^-)) - I_k(y(t_k^-))|| \longrightarrow 0.\end{aligned} \tag{5.16}$$

As a consequence of steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N_2 : \Omega \to \Omega$ is completely continuous.

Let $C = \{y \in \Omega : \langle y(t), z(t)\rangle \geq 0$ for $t \in [0,b]$, for all $z \in \Omega, \|z\|_\Omega \leq R_1\}$ be a cone in Ω. Let $y, z \in C_{R_1}$. From (B4) we have

$$\begin{aligned}\Big\langle y(t), T(t)[\phi(0) - g(0,\phi(0))] &+ \int_0^t AT(t-s)g(s,u)ds \\ &+ g(t,u) + \int_0^t T(t-s)f(s,u)ds + \sum_{0<t_k<t} T(t-t_k)I_k(x)\Big\rangle \geq 0.\end{aligned} \tag{5.17}$$

Thus $N(C_{R_1}) \subset C$ and $N : C_{R_1} \to C$ is a completely continuous map. Now it remains to show that the hypotheses of Theorem 2.2 are satisfied. First notice that Theorem 2.2(A1) holds since for $y \in \partial C_{r_1}$, we have from (H1), (H2), and (B1)–(B3)

$$\begin{aligned}||N_2(y)(t)|| &\leq ||T(t)||[\|\phi\| + ||g(0,\phi(0))||] + ||g(t,y_t)|| + \int_0^t ||AT(t-s)||\,||g(s,y_s)||ds \\ &\quad + \int_0^t ||T(t-s)||\,||f(s,y_s)||ds + \sum_{0<t_k<t} ||T(t-t_k)||\,||I_k(y(t_k^-))|| \\ &\leq M\|\phi\| + (M+1+bM_2)(c_1r_1+c_2) + M\psi(r_1)\|p\|_{L^1} + M\sum_{k=1}^m c_k \\ &< r_1.\end{aligned} \tag{5.18}$$

Let $y \in \partial C_{R_1}$, that is, $\|y\|_\Omega = R_1$. Then from (B4) and the Schwarz inequality we get as in the previous Theorems that $\|N_2(y)\|_\Omega \geq R_1$ and hence condition (A2) of Theorem 2.2 holds. The Krasnoselskii twin fixed point theorem implies that N_2 has at least two fixed points y_1, y_2 which are solutions to problem (1.3). Furthermore, we have

$$y_1 \in C_{r_1}, \qquad y_2 \in C_{R_1} - C_{r_1}. \tag{5.19}$$

□

6. Second-order impulsive neutral functional differential inclusions

In this section, we study the initial value problem (1.4). We give first the definition of a mild solution of the IVP (1.4).

Definition 6.1. A function $y \in \Omega$ is said to be a mild solution of (1.4) if $y(t) = \phi(t)$, $t \in [-r,0]$, $y'(0) = \eta$, and y is the solution of impulsive integral equation

$$\begin{aligned}y(t) &= C(t)\phi(0) + S(t)[\eta - g(0,\phi(0))] + \int_0^t C(t-s)g(s,y_s)ds \\ &\quad + \int_0^t S(t-s)f(s,y_s)ds + \sum_{0<t_k<t} [C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\bar{I}_k(y(t_k^-))].\end{aligned} \tag{6.1}$$

Theorem 6.2. *Assume that hypotheses (H2), (H5), (H6), (B1), and the following hypotheses are satisfied.*

(B5) *There exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$, $p \in L^1([0,b], \mathbb{R}_+)$, *and nonnegative numbers* $r_2 > 0$ *such that*

$$M_1\|\phi\| + 2bM_1(c_1r_2 + c_2) + bM_1\|\eta\| + bM_1\psi(r_2)\|p\|_{L^1} + \sum_{k=1}^{m} M_1[c_k + bd_k] < r_2. \tag{6.2}$$

(B6) *There exists* $R_2 > r_2$ *such that for each* $z \in \Omega$ *with* $\|z\|_\Omega \le R_2$, $t \in [0,b]$ *and for all* $(x,u) \in H \times D$,

$$\begin{aligned} &\Big\langle z(t), C(t)\phi(0) + S(t)[\eta - g(0,\phi(0))] + \int_0^t C(t-s)g(s,u)ds \\ &\quad + \int_0^t S(t-s)f(s,u)ds + \sum_{0<t_k<t}[C(t-t_k)I_k(x) + S(t-t_k)\overline{I}_k(x)]\Big\rangle \ge 0, \end{aligned} \tag{6.3}$$

and for all $y \in \Omega$ *such that* $\|y\|_\Omega = R_2$,

$$\begin{aligned} &\Big|\Big\langle y(t), C(t)\phi(0) + S(t)[\eta - g(0,\phi(0))] + \int_0^t C(t-s)g(s,y_s)ds \\ &\quad + \int_0^t S(t-s)f(s,y_s)ds + \sum_{0<t_k<t}[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k))]\Big\rangle\Big| \ge R_2^2. \end{aligned} \tag{6.4}$$

Then the IVP (1.4) has two mild solutions.

Proof. Transform the problem (1.4) into a fixed point problem. Consider the operator $N_3 : \Omega \to \Omega$ defined by

$$N_3(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [-r,0], \\ C(t)\phi(0) + S(t)[\eta - g(0,\phi(0))] & \\ +\displaystyle\int_0^t C(t-s)g(s,y_s)ds + \int_0^t S(t-s)f(s,y_s)ds & \\ +\displaystyle\sum_{0<t_k<t}[I_k(y(t_k^-)) + (t-t_k)\overline{I}_k(y(t_k^-))], & \text{if } t \in [0,b]. \end{cases} \tag{6.5}$$

As in Theorem 5.2 we can show that N_3 is completely continuous.

Let $C = \{y \in \Omega : \langle y(t), z(t)\rangle \ge 0$ for $t \in [0,b]$, for all $z \in \Omega, \|z\|_\Omega \le R_2\}$ be a cone in Ω. Let $y,z \in C_{R_2}$. Then by (B6) we have

$$\begin{aligned} &\Big\langle y(t), C(t)\phi(0) + S(t)[\eta - g(0,\phi(0))] + \int_0^t C(t-s)g(s,y_s)ds \\ &\quad + \int_0^t S(t-s)f(s,y_s)ds + \sum_{0<t_k<t}[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k))]\Big\rangle \ge 0. \end{aligned} \tag{6.6}$$

Thus $N(C_{R_2}) \subset C$ and $N : C_{R_2} \to C$ is a completely continuous map. Now it remains to show that the hypotheses of Theorem 2.2 are satisfied. First notice that condition (A1) of Theorem 2.2 holds since for $y \in \partial C_{r_2}$, we have from (H2), (H5), and (B5)

$$\begin{aligned}||N_3(y)(t)|| \leq M_1\|\phi\| + 2bM_1(c_1r_2+c_2) + bM_1\|\eta\| + bM_1\psi(r_2)\|p\|_{L^1} \\ + \sum_{k=1}^{m} M_1[c_k + bd_k] \leq r_2.\end{aligned} \tag{6.7}$$

Let $y \in \partial C_{R_2}$, then

$$\begin{aligned}N_3(y)(t) = C(t)\phi(0) + S(t)[\eta - g(0,\phi(0))] + \int_0^t C(t-s)g(s,y_s)\,ds \\ + \int_0^t S(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t}[C(t-t_k)I_k(y(t_k^-)) + S(t-t_k)\overline{I}_k(y(t_k))].\end{aligned} \tag{6.8}$$

Similarly to the previous argument, we can show that condition (A2) of Theorem 2.2 holds.

The Krasnoselskii twin fixed point theorem implies that N has at least two fixed points y_1, y_2 which are solutions to problem (1.4). Furthermore, we have

$$y_1 \in C_{r_2}, \qquad y_2 \in C_{R_2} - C_{r_2}. \tag{6.9}$$

□

References

[1] N. U. Ahmed, *Optimal impulse control for impulsive systems in Banach spaces*, Int. J. Differ. Equ. Appl. **1** (2000), no. 1, 37–52.

[2] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood Limited, Chichester, 1989.

[3] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive neutral functional differential equations in Banach spaces*, Appl. Anal. **80** (2001), no. 3-4, 353–365.

[4] ———, *Existence results for impulsive semilinear neutral functional differential equations in Banach spaces*, Mem. Differential Equations Math. Phys. **25** (2002), 105–120.

[5] M. Benchohra, S. K. Ntouyas, and A. Ouahabi, *Existence results for impulsive seminlinear damped differential equations*, Int. J. Appl. Math. **11** (2002), no. 1, 77–93.

[6] M. Benchohra and A. Ouahabi, *Some uniqueness results for impulsive semilinear neutral functional differential equations*, Georgian Math. J. **9** (2002), no. 3, 423–430.

[7] L. H. Erbe, H. I. Freedman, X. Liu, and J. H. Wu, *Comparison principles for impulsive parabolic equations with applications to models of single species growth*, J. Austral. Math. Soc. Ser. B **32** (1991), no. 4, 382–400.

[8] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, vol. 108, North-Holland Publishing, Amsterdam, 1985.

[9] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1985.

[10] D. J. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering, vol. 5, Academic Press, Massachusetts, 1988.

[11] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, Series in Modern Applied Mathematics, vol. 6, World Scientific Publishing, New Jersey, 1989.

[12] J. H. Liu, *Nonlinear impulsive evolution equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **6** (1999), no. 1, 77–85.

[13] X. Liu and S. Zhang, *A cell population model described by impulsive PDEs—existence and numerical approximation*, Comput. Math. Appl. **36** (1998), no. 8, 1–11.

[14] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer, New York, 1983.

[15] Y. V. Rogovchenko, *Impulsive evolution systems: main results and new trends*, Dynam. Contin. Discrete Impuls. Systems **3** (1997), no. 1, 57–88.

[16] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, vol. 14, World Scientific Publishing, New Jersey, 1995.

[17] C. C. Travis and G. F. Webb, *Cosine families and abstract nonlinear second order differential equations*, Acta Math. Acad. Sci. Hungar. **32** (1978), no. 1-2, 75–96.

[18] ———, *Second order differential equations in Banach space*, Nonlinear Equations in Abstract Spaces (Proc. Internat. Sympos., Univ. Texas, Arlington, Tex., 1977), Academic Press, New York, 1978, pp. 331–361.

[19] K. Yosida, *Functional Analysis*, Fundamental Principles of Mathematical Sciences, vol. 123, Springer, Berlin, 1980.

M. Benchohra: Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algeria
E-mail address: benchohra@univ-sba.dz

J. Henderson: Department of Mathematics, Baylor University, Waco, TX 76798-7328, USA
E-mail address: johnny_henderson@baylor.edu

S. K. Ntouyas: Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece
E-mail address: sntouyas@cc.uoi.gr

A. Ouahab: Laboratoire de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algeria
E-mail address: ouahab@univ-sda.dz