GENERALIZED NONLINEAR IMPLICIT QUASIVARIATIONAL INCLUSIONS

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Received 10 August 2003

We introduce and study a new class of generalized nonlinear implicit quasivariational inclusions involving relaxed Lipschitzian mappings. We prove the existence of solution for the generalized nonlinear implicit quasivariational inclusions and construct some new stable perturbed iterative algorithms with errors. We also give an application to a class of generalized nonlinear implicit variational inequalities.

1. Introduction

Variational inequality theory and complementarity problem theory are very powerful tools of the current mathematical technology. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide class of problems generated in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, and so forth. A useful and important generalization of variational inequalities is a variational inclusion. Using the resolvent operator technique, many authors have studied various variational inequalities and inclusions with applications (see [1, 2, 5, 6, 8, 9, 10, 11, 12, 13, 14, 16, 18, 19, 20, 21] and the references therein).

In 1997, Verma [19] studied the solvability, based on an iterative algorithm, of a class of generalized nonlinear variational inequalities involving relaxed Lipschitz and relaxed monotone operators. Recently, Huang [9, 10] introduced and studied the Mann- and Ishikawa-type perturbed iterative sequence with errors for the generalized nonlinear implicit quasivariational inequalities and inclusions. On the other hand, Huang et al. [12] and Shim et al. [16] proved some existence theorems of solutions for the generalized nonlinear mixed quasivariational inequalities (inclusions) and convergence theorems of the iterative sequences generated by the perturbed algorithms with errors.

Inspired and motivated by the recent papers [1, 9, 10, 11, 12, 16, 19], in this paper, we introduce and study a new class of generalized nonlinear implicit quasivariational inclusions involving relaxed Lipschitz mappings and construct some new perturbed iterative algorithms with errors. We discuss the convergence and stability of perturbed iterative sequences with errors generated by the algorithms for solving the generalized nonlinear

Journal of Inequalities and Applications 2005:3 (2005) 261–275
DOI: 10.1155/JIA.2005.261

implicit quasivariational inclusions. We also give an application to a class of generalized nonlinear implicit variational inequalities.

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, respectively. For given mappings $f,g,p : H \to H$, and $N : H\times H \to H$. Let $M : H\times H \to 2^H$ be a set-valued mapping such that, for each fixed $t \in H$, $M(\cdot,t) : H \to 2^H$ is a maximal monotone mapping and $\text{Range}(p)\bigcap \text{Dom}(M(\cdot,t)) \neq \emptyset$. We consider the following problem. Find $u \in H$ such that

$$\begin{gathered} p(u) \in \text{Dom}\left(M(\cdot,g(u))\right), \\ 0 \in f(u) - N(u,u) + M\left(p(u),g(u)\right), \end{gathered} \tag{2.1}$$

which is called the *generalized nonlinear implicit quasivariational inclusion*.

Some special cases of the problem (2.1) are as follows.

(1) If $M(x,t) = M(x)$ for all $x,t \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{gathered} p(u) \in \text{Dom}(M), \\ 0 \in f(u) - N(u,u) + M\left(p(u)\right), \end{gathered} \tag{2.2}$$

where $M : H \to 2^H$ is a maximal monotone mapping. The problem (2.2) was considered by Adly [1] and Huang [10], respectively.

(2) If $M(\cdot,t) = \partial\varphi(\cdot,t)$ for each $t \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{gathered} p(u) \in \text{Dom}\left(\partial\varphi(\cdot,g(u))\right), \\ \langle f(u) - N(u,u), v - p(u)\rangle \geq \varphi\left(p(u),g(u)\right) - \varphi\left(v,g(u)\right) \end{gathered} \tag{2.3}$$

for all $v \in H$, where $\varphi : H\times H \to \mathbb{R}\bigcup\{+\infty\}$ such that for each $t\in H$, $\varphi(\cdot,t) : H \to \mathbb{R}\bigcup\{+\infty\}$ is a proper convex lower semicontinuous function with

$$\text{Range}(p)\bigcap \text{Dom}\left(\partial\varphi(\cdot,t)\right) \neq \emptyset. \tag{2.4}$$

When g is the identity mapping, the problem (2.3) was considered by Ding [6].

(3) If $f = 0$ and g is the identity mapping, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{gathered} p(u) \in \text{Dom}\left(M(\cdot,u)\right), \\ 0 \in -N(u,u) + M\left(p(u),u\right) \end{gathered} \tag{2.5}$$

which is called the *generalized strongly nonlinear implicit quasivariational inclusion* considered by Shim et al. [16].

(4) If $f = p$ and $M(\cdot,t) = \partial\varphi$ for all $t \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{gathered} p(u) \in \operatorname{Dom}(\partial\varphi), \\ \langle p(u) - N(u,u), v - p(u)\rangle \geq \varphi(p(u)) - \varphi(v) \end{gathered} \tag{2.6}$$

for all $v \in H$, where $\partial\varphi$ denotes the subdifferential of a proper convex lower semicontinuous function $\varphi : H \to \mathbb{R}\bigcup\{+\infty\}$.

(5) If $f = p$, $N(x,y) = Sx - Ty$ for all $x, y \in H$ and $M(\cdot,t) = \delta_{K(t)}$ for all $t \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$\begin{gathered} p(u) \in K(g(u)), \\ \langle p(u) - (Su - Tu), v - p(u)\rangle \geq 0 \end{gathered} \tag{2.7}$$

for all $v \in K(g(u))$, where $S, T : H \to H$ are two single-valued mappings, $K : H \to 2^H$ is a set-valued mapping with nonempty closed convex values, and $\delta_{K(t)}$ denotes the indicator function of $K(t)$ for each fixed $t \in H$.

Remark 2.1. For a suitable choice of f, p, g, N, M, and the space H, a number of classes of variational inequalities, complementarity problems, and variational inclusions can be obtained as special cases of the generalized nonlinear implicit quasivariational inclusion (2.1).

In the sequel, we give some concepts and lemmas.

Definition 2.2. A mapping $f : H \to H$ is said to be

(i) *strongly monotone* if there exists a constant $r > 0$ such that

$$\langle f(u) - f(v), u - v\rangle \geq r\|u - v\|^2 \tag{2.8}$$

for all $u, v \in H$,

(ii) *Lipschitzian continuous* if there exists a constant $s > 0$ such that

$$\|f(u) - f(v)\| \leq s\|u - v\| \tag{2.9}$$

for all $u, v \in H$.

Definition 2.3. A mapping $N : H \times H \to H$ is said to be *relaxed Lipschitzian* with respect to the first argument if there exists a constant $t > 0$ such that

$$\langle N(u,\cdot) - N(v,\cdot), u - v\rangle \leq -t\|u - v\|^2 \tag{2.10}$$

for all $u, v \in H$.

Definition 2.4. A mapping $N : H \times H \to H$ is said to be Lipschitzian continuous with respect to the first argument if there exists a constant $\alpha > 0$ such that

$$||N(u,\cdot) - N(v,\cdot)|| \le \alpha \|u - v\| \tag{2.11}$$

for all $u, v \in H$.

In a similar way, we can define the Lipschitzian continuity of the mapping $N(\cdot,\cdot)$ with respect to the second argument.

Definition 2.5. Let $\{M^n\}$ and M be maximal monotone mappings for $n = 0,1,2,\ldots$. The sequence $\{M^n\}$ is said to be *graph-convergence* to M (write $M^n \xrightarrow{G} M$) if, for every $(x, y) \in \text{Graph}(M)$, there exists a sequence $(x_n, y_n) \in \text{Graph}(M^n)$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Lemma 2.6 [4]. *Let $\{M^n\}$ and M be maximal monotone mappings for $n = 0,1,2,\ldots$. Then $M^n \xrightarrow{G} M$ if and only if*

$$J_\lambda^{M^n}(x) \longrightarrow J_\lambda^{M}(x) \tag{2.12}$$

for every $x \in H$ and $\lambda > 0$, where $J_\lambda^M = (I + \lambda M)^{-1}$.

Lemma 2.7. *Let $\{a_n\}, \{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative numbers satisfying the following conditions: there exists a positive integer n_0 such that*

$$a_{n+1} \le (1 - t_n)a_n + b_n t_n + c_n \tag{2.13}$$

for $n \ge n_0$, where

$$t_n \in [0,1], \qquad \sum_{n=0}^{\infty} t_n = +\infty, \qquad \lim_{n\to\infty} b_n = 0, \qquad \sum_{n=0}^{\infty} c_n < +\infty. \tag{2.14}$$

Then $a_n \to 0$ as $n \to +\infty$.

Proof. Let $\sigma = \inf\{a_n : n \ge n_0\}$. Then $\sigma \ge 0$. Suppose that $\sigma > 0$. Then $a_n \ge \sigma > 0$ for all $n \ge n_0$. It follows from (2.7) that

$$a_{n+1} \le a_n - \sigma t_n + t_n b_n + c_n = a_n - \left(\frac{1}{2}\sigma - b_n\right)t_n - \frac{1}{2}\sigma t_n + c_n \tag{2.15}$$

for all $n \ge n_0$. Since $b_n \to 0$ as $n \to \infty$, there exists $n_1 \ge n_0$ such that

$$\frac{1}{2}\sigma \ge b_n \tag{2.16}$$

for all $n \ge n_1$. Combining (2.13) and (2.15), we have

$$a_{n+1} \le a_n - \frac{1}{2}\sigma t_n + c_n \tag{2.17}$$

for all $n \geq n_1$, which implies that

$$\frac{1}{2}\sigma \sum_{n=n_1}^{\infty} t_n \leq a_{n_1} + \sum_{n=n_1}^{\infty} c_n < +\infty. \tag{2.18}$$

This is a contradiction. Therefore, $\sigma = 0$ and so there exists a subsequence $\{a_{n_j}\} \subset \{a_n\}$ such that $a_{n_j} \to 0$ as $j \to \infty$. It follows from (2.7) that

$$a_{n_j+1} \leq a_{n_j} + b_{n_j} t_{n_j} + c_{n_j}, \tag{2.19}$$

and so $a_{n_j+1} \to 0$ as $j \to \infty$. A simple induction leads to $a_{n_j+k} \to 0$ as $j \to \infty$ for all $k \geq 1$ and this means that $a_n \to 0$ as $n \to \infty$. This completes the proof. □

Lemma 2.8 (see [11, 12]). *$u \in H$ is a solution of the problem (2.1) if and only if*

$$p(u) = J_\rho^{M(\cdot,g(u))}[p(u) - \rho f(u) + \rho N(u,u)], \tag{2.20}$$

where $\rho > 0$ is a constant and

$$J_\rho^{M(\cdot,g(u))} = (I + \rho M(\cdot,g(u)))^{-1}. \tag{2.21}$$

3. Existence and uniqueness theorems

In this section, we show the existence and uniqueness of solution for the generalized nonlinear implicit quasivariational inclusion problem (2.1) in terms of Lemma 2.8.

Theorem 3.1. *Let $N : H \times H \to H$ be Lipschitzian continuous with respect to the first and second arguments with constants α, β, respectively, and let it be relaxed Lipschitzian with respect to the first argument with a constant $t > 0$. Let $f, p, g : H \to H$ be Lipschitzian continuous with constants σ, s, and l, respectively, and let p be strongly monotone with a constant $r > 0$. Suppose that there exist constants $\lambda > 0$ and $\rho > 0$ such that for each $x, y, z \in H$,*

$$\left\| J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z) \right\| \leq \lambda \|x - y\|, \tag{3.1}$$

$$2\sqrt{1 - 2r + s^2} + \sqrt{1 - 2\rho t + \rho^2\alpha^2} + (\rho\beta + \rho\sigma + \lambda l) < 1. \tag{3.2}$$

Then the problem (2.1) has a unique solution $u^* \in H$.

Proof. By Lemma 2.8, it is enough to show that the mapping $F : H \to H$ has a unique fixed point $u^* \in H$, where F is defined as follows:

$$F(u) = u - p(u) + J_\rho^{M(\cdot,g(u))}[p(u) - \rho f(u) + \rho N(u,u)] \tag{3.3}$$

for all $u \in H$. From (3.1) and (3.3), we have

$$\begin{aligned}
&\|F(u)-F(v)\| \\
&\quad = \Big\| u-p(u)+J_\rho^{M(\cdot,g(u))}[p(u)-\rho f(u)+\rho N(u,u)] \\
&\qquad -\Big\{v-p(v)+J_\rho^{M(\cdot,g(v))}[p(v)-\rho f(v)+\rho N(v,v)]\Big\}\Big\| \\
&\quad \le \|u-v-(p(u)-p(v))\| \\
&\qquad +\Big\|J_\rho^{M(\cdot,g(u))}[p(u)-\rho f(u)+\rho N(u,u)]-J_\rho^{M(\cdot,g(v))}[p(v)-\rho f(v)+\rho N(v,v)]\Big\| \\
&\quad \le \|u-v-(p(u)-p(v))\| \\
&\qquad +\Big\|J_\rho^{M(\cdot,g(u))}[p(u)-\rho f(u)+\rho N(u,u)]-J_\rho^{M(\cdot,g(u))}[p(v)-\rho f(v)+\rho N(v,v)]\Big\| \\
&\qquad +\Big\|J_\rho^{M(\cdot,g(u))}[p(v)-\rho f(v)+\rho N(v,v)]-J_\rho^{M(\cdot,g(v))}[p(v)-\rho f(v)+\rho N(v,v)]\Big\| \\
&\quad \le \|u-v-(p(u)-p(v))\|+\lambda\|g(u)-g(v)\| \\
&\qquad +\|p(u)-\rho f(u)+\rho N(u,u)-[p(v)-\rho f(v)+\rho N(v,v)]\| \\
&\quad \le \|u-v+\rho(N(u,u)-N(v,v))\| \\
&\qquad +2\|u-v-(p(u)-p(v))\|+\rho\|f(u)-f(v)\|+\lambda l\|u-v\| \\
&\quad \le 2\|u-v-(p(u)-p(v))\|+\|u-v+\rho(N(u,u)-N(v,u))\| \\
&\qquad +\rho\|N(v,u)-N(v,v)\|+(\rho\sigma+\lambda l)\|u-v\|.
\end{aligned} \tag{3.4}$$

By the Lipschitzian continuity and strong monotonicity of p, we have

$$\begin{aligned}
&\|u-v-(p(u)-p(v))\|^2 \\
&\quad = \|u-v\|^2-2\langle u-v,p(u)-p(v)\rangle+\|p(u)-p(v)\|^2 \\
&\quad \le (1-2r+s^2)\|u-v\|^2.
\end{aligned} \tag{3.5}$$

Since N is Lipschitzian continuous with respect to the first and second arguments and relaxed Lipschitzian with respect to the first argument, we obtain

$$\begin{aligned}
&\|u-v+\rho(N(u,u)-N(v,u))\|^2 \\
&\quad = \|u-v\|^2+2\rho\langle u-v,N(u,u)-N(v,u)\rangle+\rho^2\|N(u,u)-N(v,u)\|^2 \\
&\quad \le (1-2\rho t+\rho^2\alpha^2)\|u-v\|^2, \\
&\|N(v,u)-N(v,v)\| \le \beta\|u-v\|.
\end{aligned} \tag{3.6}$$

From (3.4), (3.5), and (3.6), we have

$$\|F(u)-F(v)\| \le h\|u-v\|, \tag{3.7}$$

where

$$h = 2\sqrt{1-2r+s^2}+\sqrt{1-2\rho t+\rho^2\alpha^2}+(\rho\beta+\rho\sigma+\lambda l). \tag{3.8}$$

From (3.2), we know that $0 < h < 1$. Therefore, there exists a unique $u^* \in H$ such that $F(u^*) = u^*$. This completes the proof. □

Remark 3.2. If there is a constant $\rho > 0$ such that

$$\left|\rho - \frac{t-(1-k)(\beta+\sigma)}{\alpha^2-(\beta+\sigma)^2}\right| < \frac{\sqrt{\left(t+(k-1)(\beta+\sigma)\right)^2 - k(2-k)\left(\alpha^2-(\beta+\sigma)^2\right)}}{\alpha^2-(\beta+\sigma)^2},$$
$$t > (1-k)(\beta+\sigma)+\sqrt{k(2-k)\left(\alpha^2-(\beta+\sigma)^2\right)}, \quad \alpha > \beta+\sigma,$$
$$\rho(\beta+\sigma) < 1-k, \quad k = 2\sqrt{1-2r+s^2}+\lambda l, \quad k < 1, \tag{3.9}$$

then it is easy to check that condition (3.2) is satisfied.

From Theorem 3.1, we can obtain the following theorem.

Theorem 3.3. *Let N, p, and f be the same as in Theorem 3.1. Suppose that there exists a constant $\rho > 0$ such that (3.2) holds for $k = 2\sqrt{1-2r+s^2}$. Then the problem (2.2) has a unique solution $u^* \in H$.*

4. Perturbed algorithms and stability

In this section, we construct some new perturbed iterative algorithms with errors for solving the generalized nonlinear implicit quasivariational inclusion problem (2.1) and prove the convergence and stability of the iterative sequences generated by the perturbed iterative algorithms with errors.

Definition 4.1. Let T be a self-mapping of H, $x_0 \in H$ and $x_{n+1} = f(T,x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}$ in H. Suppose that $\{x \in H : Tx = x\} \neq \emptyset$ and $\{x_n\}$ converges to a fixed point x^* of T. Let $\{y_n\} \subset H$ and let $\epsilon_n = \|y_{n+1} - f(T,y_n)\|$.

(i) If $\lim_{n\to\infty}\epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $\{x_n\}$ defined by $x_{n+1} = f(T,x_n)$ is said to be T-*stable* or *stable* with respect to T.

(ii) If $\sum_{n=0}^{\infty}\epsilon_n < +\infty$ implies that $\lim_{n\to\infty} y_n = x^*$, then the iteration procedure $\{x_n\}$ is said to be *almost T-stable*.

Some stability results of iteration algorithms have been established by several authors (see [3, 7, 12, 15]). As was shown by Harder and Hicks [7], the study on the stability is both of theoretical and of numerical interest.

Remark 4.2. An iteration procedure $\{x_n\}$ which is T-stable is almost T-stable and an iteration procedure $\{x_n\}$ which is almost T-stable need not be T-stable [15].

Now, we give the perturbed iterative algorithms with errors for the generalized nonlinear implicit quasivariational inclusion problem (2.1) as follows.

Algorithm 4.3. Let $f,p,g : H \to H$ and $N : H\times H \to H$ be four single-valued mappings. Let $\{M^n\}$ and M be set-valued mappings from $H \times H$ into the power of H such that for each $t \in H$, $M^n(\cdot,t)$ and $M(\cdot,t)$ are maximal monotone mappings and $M^n(\cdot,t) \xrightarrow{G} M(\cdot,t)$. For any given $u_0 \in H$, the perturbed iterative sequence $\{u_n\}$ with errors is defined as follows:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n \\ &\quad + \alpha_n\Big[v_n - p(v_n) + J_\rho^{M^n(\cdot,g(v_n))}\big(p(v_n) - \rho f(v_n) + \rho N(v_n,v_n)\big)\Big] + \alpha_n e_n + l_n, \\ v_n &= (1-\beta_n)u_n \\ &\quad + \beta_n\Big[u_n - p(u_n) + J_\rho^{M^n(\cdot,g(u_n))}\big(p(u_n) - \rho f(u_n) + \rho N(u_n,u_n)\big)\Big] + f_n \end{aligned} \tag{4.1}$$

for $n = 0,1,2,\ldots$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$, $\{e_n\}, \{l_n\}$, and $\{f_n\}$ are three sequences in H satisfying the following conditions:

$$\lim_{n\to\infty}\|e_n\| = \lim_{n\to\infty}\|f_n\| = 0, \qquad \sum_{n=0}^{\infty}\alpha_n = +\infty, \qquad \sum_{n=0}^{\infty}\|l_n\| < +\infty. \tag{4.2}$$

From Algorithm 4.3, we obtain the following algorithm for the problem (2.2) as follows.

Algorithm 4.4. Let $f,p : H \to H$ and $N : H\times H \to H$ be three single-valued mappings. Let $\{M^n\}$ and M be maximal monotone mappings from H into the power of H such that $M^n \xrightarrow{G} M$. For any given $u_0 \in H$, define the perturbed iterative sequence $\{u_n\}$ with errors as follows:

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n \\ &\quad + \alpha_n\Big[v_n - p(v_n) + J_\rho^{M^n}\big(p(v_n) - \rho f(v_n) + \rho N(v_n,v_n)\big)\Big] + \alpha_n e_n + l_n, \\ v_n &= (1-\beta_n)u_n \\ &\quad + \beta_n\Big[u_n - p(u_n) + J_\rho^{M^n}\big(p(u_n) - \rho f(u_n) + \rho N(u_n,u_n)\big)\Big] + f_n \end{aligned} \tag{4.3}$$

for $n = 0,1,2,\ldots$, where $\{\alpha_n\}, \{\beta_n\}, \{e_n\}, \{l_n\}$, and $\{f_n\}$ are the same as in Algorithm 4.3.

Remark 4.5. For a suitable choice of f, p, g, N, M^n, and M, Algorithm 4.3 includes several known algorithms in [1, 5, 6, 9, 10, 12, 16, 18] as special cases.

THEOREM 4.6. *Let f, p, g, and N be the same as in Theorem 3.1. Suppose that $\{M^n\}$ and M are set-valued mappings from $H\times H$ into the power of H such that for each $t \in H$, $M^n(\cdot,t)$ and $M(\cdot,t)$ are maximal monotone mappings and $M^n(\cdot,t) \xrightarrow{G} M(\cdot,t)$. Assume that there exist constants $\rho > 0$ and $\lambda > 0$ such that for each $x,y,z \in H$ and $n \geq 0$,*

$$\begin{aligned} \Big\|J_\rho^{M(\cdot,x)}(z) - J_\rho^{M(\cdot,y)}(z)\Big\| &\leq \lambda\|x-y\|, \\ \Big\|J_\rho^{M^n(\cdot,x)}(z) - J_\rho^{M^n(\cdot,y)}(z)\Big\| &\leq \lambda\|x-y\|, \end{aligned} \tag{4.4}$$

and the condition (3.2) holds. Let $\{y_n\}$ be a sequence in H and define a sequence $\{\epsilon_n\}$ of real numbers as follows:

$$\epsilon_n = \Big\|y_{n+1} - \Big\{(1-\alpha_n)y_n + \alpha_n\Big[x_n - p(x_n) + J_\rho^{M^n(\cdot,g(x_n))}\big(p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)\big)\Big] + \alpha_n e_n + l_n\Big\}\Big\|,$$
$$x_n = (1-\beta_n)y_n + \beta_n\Big[y_n - p(y_n) + J_\rho^{M^n(\cdot,g(y_n))}\big(p(y_n) - \rho f(y_n) + \rho N(y_n,y_n)\big)\Big] + f_n, \tag{4.5}$$

where $\{\alpha_n\},\{\beta_n\},\{e_n\},\{l_n\}$, and $\{f_n\}$ are the same as in Algorithm 4.3. Then the following hold.

(1) *The sequence $\{u_n\}$ defined by Algorithm 4.3 converges strongly to the unique solution u^* of the problem (2.1).*
(2) *If $\epsilon_n = \alpha_n\Delta_n + \gamma_n$ with $\sum_{n=0}^{\infty}\gamma_n < +\infty$ and $\lim_{n\to\infty}\Delta_n = 0$, then $\lim_{n\to\infty} y_n = u^*$.*
(3) *$\lim_{n\to\infty} y_n = u^*$ implies that $\lim_{n\to\infty}\epsilon_n = 0$.*

Proof. Let $u^* \in H$ be the unique solution of the problem (2.1). It is easy to see that the conclusion (1) follows from the conclusion (2). Now, we prove that (2) is true. It follows from Lemma 2.8 that

$$u^* = (1-\alpha_n)u^* + \alpha_n\Big[u^* - p(u^*) + J_\rho^{M(\cdot,g(u^*))}\big(p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)\big)\Big]. \tag{4.6}$$

From (4.1), (4.5), and (4.6), we have

$$\begin{aligned}
&\|y_{n+1} - u^*\| \\
&= \Big\|y_{n+1} - \Big\{(1-\alpha_n)u^* + \alpha_n\Big[u^* - p(u^*) + J_\rho^{M(\cdot,g(u^*))}\big(p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)\big)\Big]\Big\}\Big\| \\
&\le \Big\|y_{n+1} - \Big\{(1-\alpha_n)y_n + \alpha_n\Big[x_n - p(x_n) + J_\rho^{M^n(\cdot,g(x_n))}\big(p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)\big)\Big] \\
&\qquad\qquad + \alpha_n e_n + l_n\Big\}\Big\| \\
&\quad + \Big\|(1-\alpha_n)y_n + \alpha_n\Big[x_n - p(x_n) + J_\rho^{M^n(\cdot,g(x_n))}\big(p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)\big)\Big] \\
&\qquad - \Big\{(1-\alpha_n)u^* + \alpha_n\Big[u^* - p(u^*) + J_\rho^{M(\cdot,g(u^*))}\big(p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)\big)\Big]\Big\}\Big\| \\
&\quad + \alpha_n\|e_n\| + \|l_n\| \\
&\le \epsilon_n + (1-\alpha_n)\|y_n - u^*\| + \alpha_n\|x_n - u^* - (p(x_n) - p(u^*))\| \\
&\quad + \alpha_n\Big\|J_\rho^{M^n(\cdot,g(x_n))}[p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)] \\
&\qquad - J_\rho^{M(\cdot,g(u^*))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)]\Big\| \\
&\quad + \alpha_n\|e_n\| + \|l_n\|
\end{aligned}$$

$$\begin{aligned}
&\le \epsilon_n + (1-\alpha_n)\|y_n - u^*\| + \alpha_n\|x_n - u^* - (p(x_n) - p(u^*))\| \\
&\quad + \alpha_n\Big\|J_\rho^{M^n(\cdot,g(x_n))}[p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)] \\
&\qquad - J_\rho^{M^n(\cdot,g(x_n))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)]\Big\| \\
&\quad + \alpha_n\Big\|J_\rho^{M^n(\cdot,g(x_n))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)] \\
&\qquad - J_\rho^{M^n(\cdot,g(u^*))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)]\Big\| \\
&\quad + \alpha_n\Big\|J_\rho^{M^n(\cdot,g(u^*))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)] \\
&\qquad - J_\rho^{M(\cdot,g(u^*))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)]\Big\| + \alpha_n\|e_n\| + \|l_n\| \\
&\le \epsilon_n + (1-\alpha_n)\|y_n - u^*\| + \alpha_n\|x_n - u^* - (p(x_n) - p(u^*))\| \\
&\quad + \alpha_n\|p(x_n) - p(u^*) - \rho(f(x_n) - f(u^*)) + \rho(N(x_n,x_n) - N(u^*,u^*))\| \\
&\quad + \alpha_n\lambda\|g(x_n) - g(u^*)\| + \alpha_n g_n + \alpha_n\|e_n\| + \|l_n\| \\
&\le \epsilon_n + (1-\alpha_n)\|y_n - u^*\| + 2\alpha_n\|x_n - u^* - (p(x_n) - p(u^*))\| \\
&\quad + \alpha_n\|x_n - u^* + \rho(N(x_n,x_n) - N(u^*,x_n))\| + \alpha_n\rho\|N(u^*,x_n) - N(u^*,u^*)\| \\
&\quad + \alpha_n\rho\|f(x_n) - f(u^*)\| + \alpha_n\lambda l\|x_n - u^*\| + \alpha_n g_n + \alpha_n\|e_n\| + \|l_n\| \\
&\le (1-\alpha_n)\|y_n - u^*\| + 2\alpha_n\|x_n - u^* - (p(x_n) - p(u^*))\| \\
&\quad + \alpha_n\|x_n - u^* + \rho(N(x_n,x_n) - N(u^*,x_n))\| + \alpha_n\rho\|N(u^*,x_n) - N(u^*,u^*)\| \\
&\quad + \alpha_n(\rho\sigma + \lambda l)\|x_n - u^*\| + \alpha_n(g_n + \|e_n\| + \Delta_n) + (\|l_n\| + \gamma_n),
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
g_n = \Big\|&J_\rho^{M^n(\cdot,g(u^*))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)] \\
&- J_\rho^{M(\cdot,g(u^*))}[p(u^*) - \rho f(u^*) + \rho N(u^*,u^*)]\Big\| \longrightarrow 0.
\end{aligned} \tag{4.8}$$

It follows from (3.5) and (3.6) that

$$\begin{gathered}
\|x_n - u^* - (p(x_n) - p(u^*))\|^2 \le (1 - 2r + s^2)\|x_n - u^*\|^2, \\
\|x_n - u^* + \rho(N(x_n,x_n) - N(u^*,x_n))\|^2 \le (1 - 2\rho t + \rho^2\alpha^2)\|x_n - u^*\|^2, \\
\|N(u^*,x_n) - N(u^*,u^*)\| \le \beta\|x_n - u^*\|.
\end{gathered} \tag{4.9}$$

Substituting (4.9) into (4.7), we have

$$\begin{aligned}
\|y_{n+1} - u^*\| \le (1-\alpha_n)\|y_n - u^*\| &+ \alpha_n h\|x_n - u^*\| \\
&+ \alpha_n(g_n + \|e_n\| + \Delta_n) + (\|l_n\| + \gamma_n),
\end{aligned} \tag{4.10}$$

where

$$h = 2\sqrt{1-2r+s^2} + \sqrt{1-2\rho t+\rho^2\alpha^2} + \rho(\beta+\sigma) + \lambda l. \tag{4.11}$$

From (3.2), we know that $0 < h < 1$.

Similarly, we have

$$\|x_n - u^*\| \le (1-\beta_n)\|y_n - u^*\| + \beta_n h\|y_n - u^*\| + \beta_n g_n + \|f_n\|. \tag{4.12}$$

Combining (4.10) and (4.12), we have

$$\begin{aligned}
&\| y_{n+1} - u^* \| \\
&\quad \le (1-\alpha_n)\|y_n - u^*\| + \alpha_n h(1-\beta_n)\|y_n - u^*\| + \alpha_n h^2\beta_n\|y_n - u^*\| \\
&\qquad + \alpha_n h\beta_n g_n + \alpha_n h\|f_n\| + \alpha_n(g_n + \|e_n\| + \Delta_n) + (\|l_n\| + \gamma_n) \\
&\quad \le (1-(1-h)\alpha_n)\|y_n - u^*\| \\
&\qquad + (1-h)\alpha_n \cdot \frac{1}{1-h}(h\beta_n g_n + h\|f_n\| + g_n + \|e_n\| + \Delta_n) + (\|l_n\| + \gamma_n).
\end{aligned} \tag{4.13}$$

Let

$$\begin{aligned}
&a_n = \|y_n - u^*\|, \qquad c_n = \|l_n\| + \gamma_n, \qquad t_n = (1-h)\alpha_n, \\
&b_n = \frac{1}{1-h}(h\beta_n g_n + h\|f_n\| + g_n + \|e_n\| + \Delta_n).
\end{aligned} \tag{4.14}$$

We can rewrite (4.13) as follows:

$$a_{n+1} \le (1-t_n)a_n + b_n t_n + c_n. \tag{4.15}$$

From the assumptions, we know that $\{a_n\}, \{b_n\}, \{c_n\}$, and $\{t_n\}$ satisfy the conditions of Lemma 2.7. This implies that $a_n \to 0$ and so $y_n \to u^*$.

Next, we prove the conclusion (3). Suppose that $\lim_{n\to\infty} y_n = u^*$. It follows from (4.2) and (4.12) that $x_n \to u^*$. From (4.5), we have

$$\begin{aligned}
\epsilon_n &= \Big\| y_{n+1} - \Big\{(1-\alpha_n)y_n + \alpha_n\Big[x_n - p(x_n) + J_\rho^{M^n(\cdot,g(x_n))}(p(x_n) - \rho f(x_n) + \rho N(x_n,x_n))\Big] \\
&\qquad\qquad + \alpha_n e_n + l_n\Big\}\Big\| \\
&\le \|y_{n+1} - u^*\| + \alpha_n\|e_n\| + \|l_n\| \\
&\quad + \Big\|(1-\alpha_n)y_n + \alpha_n\Big[x_n - p(x_n) + J_\rho^{M^n(\cdot,g(x_n))}[p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)]\Big] - u^*\Big\|.
\end{aligned} \tag{4.16}$$

As in the proof of (4.10), we have

$$\begin{aligned} &\Big\|(1-\alpha_n)y_n+\alpha_n\Big[x_n-p(x_n)+J_\rho^{M^n(\cdot,g(x_n))}(p(x_n)-\rho f(x_n)+\rho N(x_n,x_n))\Big]-u^*\Big\| \\ &\qquad\le (1-\alpha_n)\|y_n-u^*\|+\alpha_n h\|x_n-u^*\|+\alpha_n g_n. \end{aligned} \tag{4.17}$$

It follows from (4.16) and (4.17) that

$$\epsilon_n \le \|y_{n+1}-u^*\|+\alpha_n\|e_n\|+\|l_n\|+(1-\alpha_n)\|x_n-u^*\|+\alpha_n h\|x_n-u^*\|+\alpha_n g_n. \tag{4.18}$$

This implies that $\lim_{n\to\infty}\epsilon_n=0$. This completes the proof. □

From Theorem 4.6, we have the following theorem.

THEOREM 4.7. *Let f, p, and N be the same as in Theorem 4.6. Let $\{M^n\}$ and M be maximal monotone mappings from H into the power of H such that $M^n \xrightarrow{G} M$. Assume that there exists a constant $\rho>0$ such that (3.2) holds for $k=2\sqrt{1-2r+s^2}$. Let $\{y_n\}$ be a sequence in H and define $\{\epsilon_n\}$ as follows:*

$$\begin{aligned} \epsilon_n &= \Big\|y_{n+1}-\Big\{(1-\alpha_n)y_n+\alpha_n\Big[x_n-p(x_n)+J_\rho^{M^n}(p(x_n)-\rho f(x_n)+\rho N(x_n,x_n))\Big] \\ &\qquad\qquad +\alpha_n e_n+l_n\Big\}\Big\|, \\ x_n &= (1-\beta_n)y_n+\beta_n\Big[y_n-p(y_n)+J_\rho^{M^n}(p(y_n)-\rho f(y_n)+\rho N(y_n,y_n))\Big]+f_n, \end{aligned} \tag{4.19}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$, $\{l_n\}$, and $\{f_n\}$ are the same as in Algorithm 4.4. Then

(1) *the sequence $\{u_n\}$ defined by Algorithm 4.4 converges strongly to the unique solution u^* of the problem (2.2),*
(2) *if $\epsilon_n=\alpha_n\Delta_n+\gamma_n$ with $\sum_{n=0}^{\infty}\gamma_n<+\infty$ and $\lim_{n\to\infty}\Delta_n=0$, then $\lim_{n\to\infty}y_n=u^*$,*
(3) *$\lim_{n\to\infty}y_n=u^*$ implies that $\lim_{n\to\infty}\epsilon_n=0$.*

5. An application

In this section, we give an application to a class of generalized nonlinear implicit variational inequalities.

Definition 5.1. Let A be a single-valued mapping from H to H. The mapping A is said to be *hemicontinuous* if the mapping from $[0,1]$ into $(-\infty,+\infty)$ defined by

$$t \longmapsto \langle A((1-t)u+tv),w\rangle \tag{5.1}$$

is continuous for all $u,v,w\in H$.

LEMMA 5.2 [17]. *Let $\varphi: H\to R\bigcup\{+\infty\}$ be a proper convex lower semicontinuous function and let $A: H\to H$ be a single-valued monotone mapping such that $\mathrm{CL}(\mathrm{Dom}(\varphi))\subset \mathrm{Dom}(A)$, where $\mathrm{CL}(\mathrm{Dom}(\varphi))$ denotes the closure of $\mathrm{Dom}(\varphi)$. If A is hemicontinuous on $\mathrm{CL}(\mathrm{Dom}(\varphi))$ and if $\mathrm{Dom}(\partial\varphi)$ is closed, then $\partial\varphi+A$ is a maximal monotone mapping on H.*

Theorem 5.3. *Let $\varphi : H \to R\bigcup\{+\infty\}$ be a proper convex lower semicontinuous function and $A : H \to H$ is a single-valued monotone mapping such that* $\mathrm{CL}(\mathrm{Dom}(\varphi)) \subset \mathrm{Dom}(A)$*, S is hemicontinuous on* $\mathrm{CL}(\mathrm{Dom}(\varphi))$*, and* $\mathrm{Dom}(\partial\varphi)$ *is closed. Let N, f, and p be the same as in Theorem 3.3. If there exists a constant $\rho > 0$ such that (3.2) holds for $k = 2\sqrt{1-2r+s^2}$, then there exists a unique $u^* \in H$ such that*

$$\begin{gathered} p(u) \in \mathrm{Dom}(\partial\varphi), \\ \langle A(p(u)) + f(u) - N(u,u), v - p(u)\rangle \geq \varphi(p(u)) - \varphi(v) \end{gathered} \tag{5.2}$$

for all $v \in H$, which is called the generalized nonlinear implicit variational inequality. Moreover, let $\{y_n\}$ be a sequence in H and define a sequence $\{\epsilon_n\}$ of positive real numbers as follows:

$$\begin{aligned} \epsilon_n &= \Big\| y_{n+1} - \Big\{(1-\alpha_n)y_n + \alpha_n\Big[x_n - p(x_n) + J_\rho^{A+\partial\varphi}\big(p(x_n) - \rho f(x_n) + \rho N(x_n,x_n)\big)\Big] \\ &\qquad\qquad + \alpha_n e_n + l_n\Big\}\Big\|, \\ x_n &= (1-\beta_n)y_n + \beta_n\Big[y_n - p(y_n) + J_\rho^{A+\partial\varphi}\big(p(y_n) - \rho f(y_n) + \rho N(y_n,y_n)\big)\Big] + f_n, \end{aligned} \tag{5.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{e_n\}, \{l_n\}$, and $\{f_n\}$ are the same as in Algorithm 4.4. Then

(1) *if $\epsilon_n = \alpha_n\Delta_n + \gamma_n$ with $\sum_{n=0}^{\infty}\gamma_n < +\infty$ and $\lim_{n\to\infty}\Delta_n = 0$, then $\lim_{n\to\infty} y_n = u^*$,*

(2) *$\lim_{n\to\infty} y_n = u^*$ implies that $\lim_{n\to\infty}\epsilon_n = 0$.*

Proof. The monotone mapping $A + \partial\varphi$ is maximal monotone by Lemma 5.2. Let $M = A + \partial\varphi$. From Theorem 3.3, we know that there exists a unique $u^* \in H$ such that $p(u) \in \mathrm{Dom}(A + \partial\varphi)$ and

$$N(u,u) - f(u) - A(p(u)) \in \partial\varphi(p(u)). \tag{5.4}$$

Hence $p(u) \in \mathrm{Dom}(\partial\varphi)$ and

$$\langle A(p(u)) + f(u) - N(u,u), v - p(u)\rangle \geq \varphi(p(u)) - \varphi(v) \tag{5.5}$$

for all $v \in H$. The conclusions (1) and (2) follow from Theorem 4.7. This completes the proof. □

Acknowledgments

The first and second authors were supported by the National Natural Science Foundation of China, and the fourth author was supported financially from the Korea Research Foundation Grant (KRF 2001-005-D00002).

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