TRIPLE FIXED-SIGN SOLUTIONS IN MODELLING A SYSTEM WITH HERMITE BOUNDARY CONDITIONS

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Received 21 August 2003 and in revised form 2 January 2004

We consider the following system of differential equations $u_i^{(m)}(t) = P_i(t, u_1(t), u_2(t), \ldots, u_n(t))$, $t \in [0,1]$, $1 \le i \le n$ together with Hermite boundary conditions $u_i^{(j)}(t_k) = 0$, $j = 0, \ldots, m_k - 1$, $k = 1, \ldots, r$, $1 \le i \le n$, where $0 = t_1 < t_2 < \cdots < t_r = 1$, $m_k \ge 1$ for $k = 1, \ldots, r$, and $\sum_{k=1}^r m_k = m$. By using different fixed point theorems, we offer criteria for the existence of three solutions of the system which are of "prescribed signs" on the interval $[0,1]$.

1. Introduction

Let t_k, $k = 1, 2, \ldots, r$ be given such that $0 = t_1 < t_2 < \cdots < t_r = 1$. In this paper, we will consider a model comprising a *system* of differential equations subject to Hermite type boundary conditions at *multipoints* t_k, $k = 1, 2, \ldots, r$. To be exact, our model is

$$\begin{aligned} u_i^{(m)}(t) &= P_i(t, u_1(t), u_2(t), \ldots, u_n(t)), \quad t \in [0,1], \\ u_i^{(j)}(t_k) &= 0, \quad j = 0, \ldots, m_k - 1,\ k = 1, \ldots, r, \\ & \qquad i = 1, 2, \ldots, n, \end{aligned} \tag{H}$$

where $m_k \ge 1$ for $k = 1, \ldots, r$ and $\sum_{k=1}^r m_k = m$. Assume that for each $1 \le i \le n$, $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function (see Definition 2.6 later).

A solution $u = (u_1, u_2, \ldots, u_n)$ of (H) will be sought in $(C[0,1])^n = C[0,1] \times \cdots \times C[0,1]$ (n times). We say that $u = (u_1, u_2, \ldots, u_n)$ is a solution of *fixed sign* if for each $1 \le i \le n$, we have

$$(-1)^{\delta_k} \theta_i u_i(t) \ge 0 \quad \text{for } t \in [t_k, t_{k+1}],\ 1 \le k \le r-1, \tag{1.1}$$

where $\delta_k = m_{k+1} + m_{k+2} + \cdots + m_r$ and $\theta_i \in \{1, -1\}$ is fixed. Note that in the practical situation, with $\delta_k, 1 \le k \le r-1$ already known, we can choose θ_i so that $(-1)^{\delta_k}\theta_i = 1$. In this way our fixed-sign solution u becomes a *positive* solution, that is,

$$u_i(t) \ge 0 \quad \text{for } t \in [0,1],\ 1 \le i \le n. \tag{1.2}$$

Journal of Inequalities and Applications 2005:4 (2005) 363–385
DOI: 10.1155/JIA.2005.363

We remark that in most practical problems, it is only meaningful to have *positive* solutions. Nonetheless our definition of *fixed-sign* solution gives extra *flexibility*.

We will establish criteria so that the system (H) has at least *triple fixed-sign* solutions. In addition, the criteria developed will also provide estimates on the norms of these solutions.

The present work is motivated by the fact that multipoint boundary value problem of the type (H) models various dynamic systems with m degrees of freedom in which m states are observed at m times, see Meyer [25]. In fact, when $m = r = 2$ the boundary value problem (H) describes a vast spectrum of nonlinear phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, for example, see [8, 14, 15, 16, 21, 22, 27]. It is important to note that in most of these models, only *positive* solutions are meaningful. As boundary value problems model many physical phenomena, it is not surprising that they have received almost all the attention in the recent literature, see the monographs [1, 2, 4, 5] and the references cited therein. Many papers have discussed single, double and triple positive solutions of boundary value problems [3, 6, 7, 10, 9, 12, 13, 17, 18, 19, 20, 24, 28, 29, 31, 32, 33]. In dealing with single and double solutions, the main tool has been Krasnosel'skii's fixed point theorem, whereas in the case of triple solutions, *either* fixed point theorem from Leggett and Williams [23] *or* that from Avery [11] has been employed.

In the present work, *both* fixed point theorems of Leggett and Williams as well as Avery are used to derive criteria for the existence of *triple fixed-sign* solutions. Not only that *new* results have been obtained, we have also generalized a single-dependent-variable boundary value problem, the usual consideration in the literature, to a *system* of boundary value problems, which is a much more appropriate model for many physical phenomena.

2. Preliminaries

In this section, we will state some necessary definitions and the relevant fixed point theorems. Let B be a Banach space equipped with the norm $\|\cdot\|$.

Definition 2.1. Let C $(\subset B)$ be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

(a) if $u \in C$ and $\alpha \geq 0$, then $\alpha u \in C$;

(b) if $u \in C$ and $-u \in C$, then $u = 0$.

The cone C induces an ordering $\leq$ on B. For $y,z \in B$, we write $y \leq z$ if and only if $z - y \in C$. If $y,z \in B$ with $y \leq z$, we let $\langle y,z\rangle$ denote the closed order interval given by

$$\langle y,z\rangle = \{u \in B \mid y \leq u \leq z\}. \tag{2.1}$$

Definition 2.2. Let C $(\subset B)$ be a cone. A map ψ is a *nonnegative continuous concave functional* on C if the following conditions are satisfied:

(a) $\psi : C \to [0,\infty)$ is continuous;

(b) $\psi(ty + (1-t)z) \geq t\psi(y) + (1-t)\psi(z)$ for all $y,z \in C$ and $0 \leq t \leq 1$.

Definition 2.3. Let C $(\subset B)$ be a cone. A map β is a *nonnegative continuous convex functional* on C if the following conditions are satisfied:
(a) $\beta : C \to [0,\infty)$ is continuous;
(b) $\beta(ty+(1-t)z) \le t\beta(y)+(1-t)\beta(z)$ for all $y,z \in C$ and $0 \le t \le 1$.

Let γ,β,Θ be nonnegative continuous convex functionals on C and α,ψ be nonnegative continuous concave functionals on C. For nonnegative numbers w_i, $1 \le i \le 3$, we will introduce the following notations:

$$\begin{aligned}
C(w_1) &= \{u \in C \mid \|u\| < w_1\},\\
C(\psi,w_1,w_2) &= \{u \in C \mid \psi(u) \ge w_1 \text{ and } \|u\| \le w_2\},\\
P(\gamma,w_1) &= \{u \in C \mid \gamma(u) < w_1\},\\
P(\gamma,\alpha,w_1,w_2) &= \{u \in C \mid \alpha(u) \ge w_1 \text{ and } \gamma(u) \le w_2\},\\
Q(\gamma,\beta,w_1,w_2) &= \{u \in C \mid \beta(u) \le w_1 \text{ and } \gamma(u) \le w_2\},\\
P(\gamma,\Theta,\alpha,w_1,w_2,w_3) &= \{u \in C \mid \alpha(u) \ge w_1,\ \Theta(u) \le w_2 \text{ and } \gamma(u) \le w_3\},\\
Q(\gamma,\beta,\psi,w_1,w_2,w_3) &= \{u \in C \mid \psi(u) \ge w_1,\ \beta(u) \le w_2 \text{ and } \gamma(u) \le w_3\}.
\end{aligned} \tag{2.2}$$

The following fixed point theorems are needed later. The first is usually called *Leggett-Williams' fixed point theorem*, and the second is known as the *five-functional fixed point theorem*.

Theorem 2.4 [23]. *Let C $(\subset B)$ be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(u) \le \|u\|$ for all $u \in \overline{C}(w_4)$, and let $S : \overline{C}(w_4) \to \overline{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1,w_2,w_3 where $0 < w_1 < w_2 < w_3 \le w_4$ such that*
(a) *$\{u \in C(\psi,w_2,w_3) \mid \psi(u) > w_2\} \ne \emptyset$, and $\psi(Su) > w_2$ for all $u \in C(\psi,w_2,w_3)$;*
(b) *$\|Su\| < w_1$ for all $u \in \overline{C}(w_1)$;*
(c) *$\psi(Su) > w_2$ for all $u \in C(\psi,w_2,w_4)$ with $\|Su\| > w_3$.*
Then, S has (at least) three fixed points u^1, u^2, and u^3 in $\overline{C}(w_4)$. Furthermore, we have

$$\begin{gathered}
u^1 \in C(w_1), \qquad u^2 \in \{u \in C(\psi,w_2,w_4) \mid \psi(u) > w_2\},\\
u^3 \in \overline{C}(w_4)\setminus\left(C(\psi,w_2,w_4) \cup \overline{C}(w_1)\right).
\end{gathered} \tag{2.3}$$

Theorem 2.5 [11]. *Let C $(\subset B)$ be a cone. Assume that there exist positive numbers w_5, M, nonnegative continuous convex functionals γ,β,Θ on C, and nonnegative continuous concave functionals α,ψ on C, with*

$$\alpha(u) \le \beta(u), \qquad \|u\| \le M\gamma(u) \tag{2.4}$$

for all $u \in \overline{P}(\gamma,w_5)$. Let $S : \overline{P}(\gamma,w_5) \to \overline{P}(\gamma,w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers w_i, $1 \le i \le 4$ with $0 < w_2 < w_3$ such that
(a) *$\{u \in P(\gamma,\Theta,\alpha,w_3,w_4,w_5) \mid \alpha(u) > w_3\} \ne \emptyset$, and $\alpha(Su) > w_3$ for all $u \in P(\gamma,\Theta,\alpha,$ $w_3,w_4,w_5)$;*

(b) $\{u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(u) < w_2\} \neq \emptyset$, *and* $\beta(Su) < w_2$ *for all* $u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$*;*

(c) $\alpha(Su) > w_3$ *for all* $u \in P(\gamma,\alpha,w_3,w_5)$ *with* $\Theta(Su) > w_4$*;*

(d) $\beta(Su) < w_2$ *for all* $u \in Q(\gamma,\beta,w_2,w_5)$ *with* $\psi(Su) < w_1$.

Then, S has (at least) three fixed points u^1, u^2 *and* u^3 *in* $\overline{P}(\gamma,w_5)$*. Furthermore, we have*

$$\beta(u^1) < w_2, \quad \alpha(u^2) > w_3, \quad \beta(u^3) > w_2 \quad \text{with } \alpha(u^3) < w_3. \tag{2.5}$$

We also require the definition of a L^q-*Carathéodory function*.

Definition 2.6 [26]. A function $P : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^q-*Carathéodory function* if the following conditions hold:

(a) The map $t \to P(t,u)$ is measurable for all $u \in \mathbb{R}^n$.

(b) The map $u \to P(t,u)$ is continuous for almost all $t \in [0,1]$.

(c) For any $r > 0$, there exists $\mu_r \in L^q[0,1]$ such that $|u| \leq r$ implies that $|P(t,u)| \leq \mu_r(t)$ for almost all $t \in [0,1]$.

3. Main results

For each $k = 1,\dots,r-1$, define the constant δ_k and the interval I_k by

$$\delta_k = \sum_{j=k+1}^{r} m_j, \qquad I_k = \left[\frac{3t_k + t_{k+1}}{4}, \frac{t_k + 3t_{k+1}}{4}\right]. \tag{3.1}$$

Throughout we will denote $u = (u_1,u_2,\dots,u_n)$. Let the Banach space

$$B = \left\{u \mid u \in (C[0,1])^n\right\} \tag{3.2}$$

be equipped with norm

$$\|u\| = \max_{1\leq i\leq n} \sup_{t\in[0,1]} |u_i(t)| = \max_{1\leq i\leq n} |u_i|_0, \tag{3.3}$$

where we let

$$|u_i|_0 = \sup_{t\in[0,1]} |u_i(t)| = \max_{1\leq k\leq r-1} \sup_{t\in[t_k,t_{k+1}]} |u_i(t)|, \quad 1 \leq i \leq n. \tag{3.4}$$

To apply the fixed point theorems in Section 2, we need to define an operator $S : B \to B$ so that a solution u of the system (H) is a fixed point of S, that is, $u = Su$. For this, let $G(t,s)$ be the Green's function of the boundary value problem

$$\begin{gathered} y^{(m)}(t) = 0, \quad t \in [0,1] \\ y^{(j)}(t_k) = 0, \quad j = 0,\dots,m_k - 1,\ k = 1,\dots,r. \end{gathered} \tag{3.5}$$

If u is a solution of (H), then it can be represented as

$$u_i(t) = \int_0^1 G(t,s)P_i(s,u(s))\,ds, \quad t \in [0,1],\ 1 \leq i \leq n. \tag{3.6}$$

Hence, we will define the operator $S : B \to B$ by

$$Su(t) = \left(Su_1(t), Su_2(t), \ldots, Su_n(t)\right), \quad t \in [0,1], \tag{3.7}$$

where

$$Su_i(t) = \int_0^1 G(t,s)P_i\left(s,u(s)\right)ds, \quad t \in [0,1],\ 1 \le i \le n. \tag{3.8}$$

It is clear that a fixed point of the operator S is a solution of the system (H).

Our first lemma gives the properties of the Green's function $G(t,s)$ which will be used later.

Lemma 3.1 [30]. *It is known that*

(a) $G(t,s) \in C[0,1]$, $t \in [0,1]$ *and the map* $t \to G(t,s)$ *is continuous from* $[0,1]$ *to* $C[0,1]$*;*

(b) $(-1)^{\delta_k} G(t,s) \ge 0$, $(t,s) \in [t_k, t_{k+1}] \times [0,1]$, $k = 1, \ldots, r-1$*;*

(c) $(-1)^{\delta_k} G(t,s) > 0$, $(t,s) \in (t_k, t_{k+1}) \times (0,1)$, $k = 1, \ldots, r-1$*;*

(d) *for each* $k = 1, \ldots, r-1$,

$$(-1)^{\delta_k} G(t,s) \ge L_k \|G(\cdot,s)\|, \quad (t,s) \in I_k \times [0,1], \tag{3.9}$$

where

$$\|G(\cdot,s)\| = \sup_{t\in[0,1]} |G(t,s)| = \max_{1\le j\le r-1} \sup_{t\in[t_j,t_{j+1}]} (-1)^{\delta_j} G(t,s), \tag{3.10}$$

the constant $0 < L_k < 1$ *is given by*

$$\begin{aligned} L_k = \min\Bigg\{ & \min\left\{R\left(\frac{3t_k + t_{k+1}}{4}\right), R\left(\frac{t_k + 3t_{k+1}}{4}\right)\right\} \Big/ \max_{t\in[0,1]} R(t), \\ & \min\left\{Q\left(\frac{3t_k + t_{k+1}}{4}\right), Q\left(\frac{t_k + 3t_{k+1}}{4}\right)\right\} \Big/ \max_{t\in[0,1]} Q(t) \Bigg\}, \end{aligned} \tag{3.11}$$

and the functions R *and* Q *are defined as*

$$R(t) = \prod_{j=1}^{r-1} |t - t_j|^{m_j} (1-t)^{m_r - 1}, \qquad Q(t) = t^{m_1 - 1} \prod_{j=2}^{r} |t - t_j|^{m_j}; \tag{3.12}$$

(e) $(-1)^{\delta_k} G(t,s) \le \|G(\cdot,s)\|$, $(t,s) \in [t_k, t_{k+1}] \times [0,1]$, $k = 1, \ldots, r-1$.

Lemma 3.2. *The operator S defined in (3.7) is continuous and completely continuous.*

Proof. From Lemma 3.1(a), we have $G(t,s) \in C[0,1] \subseteq L^\infty[0,1]$, $t \in [0,1]$ and the map $t \to G(t,s)$ is continuous from $[0,1]$ to $C[0,1]$. This together with $P_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is a L^1-Carathéodory function ensures (as in [26, Theorem 4.2.2]) that S is continuous and completely continuous. □

For clarity, we will list the conditions that are needed later. Note that in these conditions $\theta_i \in \{1,-1\}$, $1 \le i \le n$ are fixed, and the sets $\tilde{K}$ and K are given by

$$\tilde{K} = \Big\{u \in B \mid \text{for each } 1 \le i \le n,\ (-1)^{\delta_k}\theta_i u_i(t) \ge 0 \text{ for } t \in [t_k,t_{k+1}],\ k = 1,2,\dots,r-1\Big\},$$
$$K = \Big\{u \in \tilde{K} \mid \text{for some } j \in \{1,2,\dots,n\},\ (-1)^{\delta_k}\theta_j u_j(t) > 0 \text{ for some } t \in [0,1]\Big\} = \tilde{K}\backslash\{0\}. \tag{3.13}$$

(C1) For each $1 \le i \le n$, assume that

$$\begin{aligned} &\theta_i P_i(t,u) \ge 0, \quad u \in \tilde{K},\ \text{a.e. } t \in (0,1), \\ &\theta_i P_i(t,u) > 0, \quad u \in K,\ \text{a.e. } t \in (0,1). \end{aligned} \tag{3.14}$$

(C2) There exist continuous functions f, b and a_i, $1 \le i \le n$ with $f : \mathbb{R}^n \to [0,\infty)$ and $b, a_i : (0,1) \to [0,\infty)$ such that for each $1 \le i \le n$,

$$a_i(t) \le \frac{\theta_i P_i(t,u)}{f(u)} \le b(t), \quad u \in \tilde{K},\ \text{a.e. } t \in (0,1). \tag{3.15}$$

(C3) For each $1 \le i \le n$, there exists a number $0 < \rho_i \le 1$ such that

$$a_i(t) \ge \rho_i b(t), \quad \text{a.e. } t \in (0,1). \tag{3.16}$$

Next, we define a cone in B as

$$\begin{aligned} C = \Big\{u \in B \mid &\text{for each } 1 \le i \le n,\ (-1)^{\delta_k}\theta_i u_i(t) \ge 0 \text{ for } t \in [t_k,t_{k+1}],\ k = 1,2,\dots,r-1 \\ &\text{and } \min_{t\in I_k}(-1)^{\delta_k}\theta_i u_i(t) \ge L_k\rho_i |u_i|_0,\ k = 1,2,\dots,r-1\Big\}, \end{aligned} \tag{3.17}$$

where L_k and ρ_i are defined in Lemma 3.1(d) and (C3), respectively. Note that $C \subseteq \tilde{K}$. Moreover, a fixed point of S obtained in C will be a *fixed-sign solution* of the system (H).

If (C1) and (C2) hold, then it follows from (3.8) and Lemma 3.1(b) that for $u \in \tilde{K}$ and $t \in [t_k,t_{k+1}]$, $1 \le k \le r-1$,

$$\begin{aligned} &\int_0^1 (-1)^{\delta_k} G(t,s)a_i(s)f(u(s))\,ds \\ &\quad \le (-1)^{\delta_k}\theta_i S u_i(t) \le \int_0^1 (-1)^{\delta_k} G(t,s)b(s)f(u(s))\,ds, \quad 1 \le i \le n. \end{aligned} \tag{3.18}$$

Lemma 3.3. *Let (C1)–(C3) hold. Then, the operator S maps C into itself.*

Proof. Let $u \in C$. From (3.18) we have for $1 \le i \le n$ and $t \in [t_k,t_{k+1}]$, $1 \le k \le r-1$,

$$(-1)^{\delta_k}\theta_i S u_i(t) \ge \int_0^1 (-1)^{\delta_k} G(t,s)a_i(s)f(u(s))\,ds \ge 0. \tag{3.19}$$

Next, using (3.19), (3.18) and Lemma 3.1(e), we obtain for $1 \le i \le n$ and $t \in [t_k, t_{k+1}]$, $1 \le k \le r-1$,

$$|Su_i(t)| = (-1)^{\delta_k}\theta_i Su_i(t) \le \int_0^1 (-1)^{\delta_k} G(t,s)b(s)f(u(s))\,ds \le \int_0^1 \|G(\cdot,s)\| b(s)f(u(s))\,ds. \tag{3.20}$$

Therefore, we have

$$|Su_i|_0 = \max_{1\le k\le r-1} \sup_{t\in[t_k,t_{k+1}]} |Su_i(t)| \le \int_0^1 \|G(\cdot,s)\| b(s)f(u(s))\,ds, \quad 1 \le i \le n \tag{3.21}$$

which immediately gives

$$\|Su\| = \max_{1\le i\le n} |Su_i|_0 \le \int_0^1 \|G(\cdot,s)\| b(s)f(u(s))\,ds. \tag{3.22}$$

Now, applying (3.18), Lemma 3.1(d), (C3) and (3.21), we find for $1 \le i \le n$ and $t \in I_k$, $1 \le k \le r-1$,

$$\begin{aligned}(-1)^{\delta_k}\theta_i Su_i(t) &\ge \int_0^1 (-1)^{\delta_k} G(t,s)a_i(s)f(u(s))\,ds \\ &\ge \int_0^1 L_k\|G(\cdot,s)\|\rho_i b(s)f(u(s))\,ds \\ &\ge L_k\rho_i |Su_i|_0.\end{aligned} \tag{3.23}$$

This leads to

$$\min_{t\in I_k}(-1)^{\delta_k}\theta_i Su_i(t) \ge L_k\rho_i |Su_i|_0, \quad 1 \le i \le n,\ 1 \le k \le r-1. \tag{3.24}$$

With (3.19) and (3.24) established, we have shown that $Su \in C$. □

Remark 3.4. From the proof of Lemma 3.3, we see that it is possible to use another cone C' $(\subset C)$ given by

$$\begin{aligned}C' = \Big\{ &u \in B \mid \text{for each } 1 \le i \le n,\ (-1)^{\delta_k}\theta_i u_i(t) \ge 0 \text{ for } t \in [t_k, t_{k+1}],\ k = 1,2,\ldots,r-1 \\ &\text{and } \min_{t\in I_k}(-1)^{\delta_k}\theta_i u_i(t) \ge L_k\rho_i\|u\|,\ k = 1,2,\ldots,r-1\Big\}.\end{aligned} \tag{3.25}$$

The arguments used will be similar.

For subsequent results, we define the following constants for each $1 \le i \le n$ and fixed numbers $\tau_{1,k}, \tau_{2,k}, \tau_{3,k}, \tau_{4,k} \in [0,1]$, $1 \le k \le r-1$:

$$
\begin{aligned}
q &= \max_{1\le k\le r-1} \max_{t\in[t_k,t_{k+1}]} \int_0^1 (-1)^{\delta_k} G(t,s)b(s)ds,\\
r_i &= \min_{1\le k\le r-1} \min_{t\in I_k} \int_{I_k} (-1)^{\delta_k} G(t,s)a_i(s)ds,\\
d_{1,i} &= \min_{1\le k\le r-1} \min_{t\in[\tau_{2,k},\tau_{3,k}]} \int_{\tau_{2,k}}^{\tau_{3,k}} (-1)^{\delta_k} G(t,s)a_i(s)ds,\\
d_2 &= \max_{1\le k\le r-1} \max_{1\le j\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]} \int_{\tau_{1,k}}^{\tau_{4,k}} (-1)^{\delta_j} G(t,s)b(s)ds,\\
d_3 &= \max_{1\le k\le r-1} \max_{1\le j\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]} \left[\int_0^{\tau_{1,k}} (-1)^{\delta_j} G(t,s)b(s)ds + \int_{\tau_{4,k}}^1 (-1)^{\delta_j} G(t,s)b(s)ds\right],\\
d_4 &= \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} \int_{\tau_{1,k}}^{\tau_{4,k}} (-1)^{\delta_k} G(t,s)b(s)ds,\\
d_5 &= \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} \left[\int_0^{\tau_{1,k}} (-1)^{\delta_k} G(t,s)b(s)ds + \int_{\tau_{4,k}}^1 (-1)^{\delta_k} G(t,s)b(s)ds\right],\\
A &= \sup_{t\in[0,1]} \left|\prod_{j=1}^r (t-t_j)^{m_j}\right|, \qquad A_k = \max_{t\in[\tau_{1,k},\tau_{4,k}]} \left|\prod_{j=1}^r (t-t_j)^{m_j}\right|,\\
B_k &= \max_{t\in[\tau_{2,k},\tau_{3,k}]} \left|\prod_{j=1}^r (t-t_j)^{m_j}\right|.
\end{aligned}
\tag{3.26}
$$

Lemma 3.5. *Let (C1)–(C3) hold, and assume*

(C4) *for each $1 \le k \le r-1$ and each $t \in [t_k, t_{k+1}]$, the function $G(t,s)b(s)$ is nonzero on a subset of $[0,1]$ of positive measure.*

Suppose that there exists a number $d > 0$ such that for $|u_j| \in [0,d]$, $1 \le j \le n$,

$$f(u_1,u_2,\ldots,u_n) < \frac{d}{q}. \tag{3.27}$$

Then,

$$S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d). \tag{3.28}$$

Proof. Let $u \in \overline{C}(d)$. Then, it is clear that $|u_j| \in [0,d]$, $1 \le j \le n$. Applying (3.18), (C4), (3.27) and (3.26), we find for $1 \le i \le n$ and $t \in [t_k, t_{k+1}]$, $1 \le k \le r-1$,

$$
\begin{aligned}
|Su_i(t)| = (-1)^{\delta_k}\theta_i Su_i(t) &\le \int_0^1 (-1)^{\delta_k} G(t,s)b(s)f(u(s))ds\\
&< \int_0^1 (-1)^{\delta_k} G(t,s)b(s)\frac{d}{q}ds\\
&\le q\frac{d}{q} = d.
\end{aligned}
\tag{3.29}
$$

This implies $|Su_i|_0 < d, 1 \le i \le n$ and so $\|Su\| < d$. Coupling with the fact that $Su \in C$ (Lemma 3.3), we have $Su \in C(d)$. The conclusion (3.28) is now immediate. □

The next lemma is similar to Lemma 3.5 and its proof is omitted.

Lemma 3.6. *Let (C1)–(C3) hold. Suppose that there exists a number $d > 0$ such that for $|u_j| \in [0,d], 1 \le j \le n$,*

$$f(u_1,u_2,\dots,u_n) \le \frac{d}{q}. \tag{3.30}$$

Then,

$$S(\overline{C}(d)) \subseteq \overline{C}(d). \tag{3.31}$$

We are now ready to establish existence criteria for three fixed-sign solutions. Our first result employs Theorem 2.4.

Theorem 3.7. *Let (C1)–(C4) hold, and assume*

(C5) *for each $1 \le i \le n$, each $1 \le k \le r-1$, and each $t \in I_k$, the function $G(t,s)a_i(s)$ is nonzero on a subset of I_k of positive measure.*

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 < \frac{w_2}{\min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i} < w_3 \tag{3.32}$$

such that the following hold:

(P) *$f(u_1,u_2,\dots,u_n) < w_1/q$ for $|u_j| \in [0,w_1]$, $1 \le j \le n$;*

(Q) *one of the following holds:*

 (Q1) *$\limsup_{|u_1|,|u_2|,\dots,|u_n|\to\infty} f(u_1,u_2,\dots,u_n)/|u_j| < 1/q$ for some $j \in \{1,2,\dots,n\}$;*

 (Q2) *there exists a number η ($\ge w_3$) such that $f(u_1,u_2,\dots,u_n) \le \eta/q$ for $|u_j| \in [0,\eta]$, $1 \le j \le n$;*

(R) *for each $1 \le i \le n$, $f(u_1,u_2,\dots,u_n) > w_2/r_i$ for $|u_j| \in [w_2,w_3]$, $1 \le j \le n$.*

Then, the system (H) has (at least) three fixed-sign solutions $u^1,u^2,u^3 \in C$ such that

$$\begin{gathered} ||u^1|| < w_1; \qquad |u_i^2(t)| > w_2, \quad t \in I_k,\ 1 \le k \le r-1,\ 1 \le i \le n; \\ ||u^3|| > w_1; \qquad \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in I_k} |u_i^3(t)| < w_2. \end{gathered} \tag{3.33}$$

Proof. We will employ Theorem 2.4. First, we will prove that condition (Q) implies the existence of a number w_4 where $w_4 \ge w_3$ such that

$$S(\overline{C}(w_4)) \subseteq \overline{C}(w_4). \tag{3.34}$$

Suppose that (Q2) holds. Then, by Lemma 3.6 we immediately have (3.34) where we pick $w_4 = \eta$. Suppose now that (Q1) is satisfied. Then, there exist $N > 0$ and $\epsilon < 1/q$ such that for some $j \in \{1,2,\dots,n\}$,

$$\frac{f(u_1,u_2,\dots,u_n)}{|u_j|} < \epsilon, \quad |u_1|,|u_2|,\dots,|u_n| > N. \tag{3.35}$$

Define

$$M = \max_{|u_i|\in[0,N],\,1\le i\le n} f(u_1,u_2,\dots,u_n). \tag{3.36}$$

In view of (3.35), it is clear that for some $j \in \{1,2,\dots,n\}$, the following holds for all $(u_1,u_2,\dots,u_n) \in \mathbb{R}^n$,

$$f(u_1,u_2,\dots,u_n) \le M + \epsilon\,|\,u_j\,|. \tag{3.37}$$

Now, pick the number w_4 so that

$$w_4 > \max\left\{w_3, M\left(\frac{1}{q}-\epsilon\right)^{-1}\right\}. \tag{3.38}$$

Let $u \in \overline{C}(w_4)$. Then, using (3.18), (3.37) and (3.38) we find for $1 \le i \le n$ and $t \in [t_k,t_{k+1}]$, $1 \le k \le r-1$,

$$\begin{aligned}
|\,Su_i(t)\,| &= (-1)^{\delta_k}\theta_i Su_i(t) \\
&\le \int_0^1 (-1)^{\delta_k} G(t,s)b(s)f(u(s))\,ds \\
&\le \int_0^1 (-1)^{\delta_k} G(t,s)b(s)[M+\epsilon\,|\,u_j(s)\,|\,]\,ds \\
&\le \int_0^1 (-1)^{\delta_k} G(t,s)b(s)(M+\epsilon w_4)\,ds \\
&\le q(M+\epsilon w_4) < q\left[w_4\left(\frac{1}{q}-\epsilon\right)+\epsilon w_4\right] = w_4.
\end{aligned} \tag{3.39}$$

This leads to $|Su_i|_0 < w_4$, $1 \le i \le n$. Hence, $\|Su\| < w_4$ and so $Su \in C(w_4) \subset \overline{C}(w_4)$. Thus, (3.34) follows immediately.

Let $\psi : C \to [0,\infty)$ be defined by

$$\psi(u) = \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in I_k}(-1)^{\delta_k}\theta_i u_i(t). \tag{3.40}$$

Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(u) \le \|u\|$ for all $u \in C$.

We will verify that condition (a) of Theorem 2.4 is satisfied. First, we claim that

$$u^* = (u_1^*,u_2^*,\dots,u_n^*) \in \{u \in C(\psi,w_2,w_3)\,|\,\psi(u) > w_2\}, \tag{3.41}$$

where

$$u_i^*(t) = \theta_i \frac{w_2+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\prod_{j=1}^{r}(t-t_j)^{m_j}, \quad 1 \le i \le n \tag{3.42}$$

and $\epsilon > 0$ is chosen such that

$$\frac{w_2+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell} \le w_3. \tag{3.43}$$

Note that it is possible to pick such an ϵ since $w_2\ (\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell)^{-1} < w_3$. From [30] it is known that for $k = 1,2,\dots,r-1$,

$$(-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j} \ge 0, \quad t\in[t_k,t_{k+1}], \qquad \min_{t\in I_k}(-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j} \ge L_kA. \tag{3.44}$$

Hence, it can be easily seen that $u^*(t)\in C$. We also have

$$\begin{aligned} \|u^*\| &= A\frac{w_2+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A} = \frac{w_2+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell} \le w_3,\\ \psi(u^*) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}(-1)^{\delta_k}\theta_i u_i^*(t)\\ &= \frac{w_2+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\min_{1\le k\le r-1}\min_{t\in I_k}(-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\ &\ge \frac{w_2+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\min_{1\le k\le r-1} L_kA > w_2. \end{aligned} \tag{3.45}$$

Thus, $u^* \in \{u\in C(\psi,w_2,w_3)\mid \psi(u)>w_2\}\ne\emptyset$. Next, let $u\in C(\psi,w_2,w_3)$. Then, $w_2\le \psi(u)\le \|u\|\le w_3$ provides

$$(-1)^{\delta_k}\theta_j u_j(s) = |u_j(s)| \in [w_2,w_3], \quad s\in I_k,\ 1\le k\le r-1,\ 1\le j\le n. \tag{3.46}$$

Applying (3.18), (3.46), (C5), (R) and (3.26), we find

$$\begin{aligned} \psi(Su) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}(-1)^{\delta_k}\theta_i(Su_i)(t)\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}\int_0^1(-1)^{\delta_k}G(t,s)a_i(s)f(u(s))\,ds\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}\int_{I_k}(-1)^{\delta_k}G(t,s)a_i(s)f(u(s))\,ds\\ &> \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}\int_{I_k}(-1)^{\delta_k}G(t,s)a_i(s)\frac{w_2}{r_i}\,ds\\ &= \min_{1\le i\le n} r_i\frac{w_2}{r_i} = w_2. \end{aligned} \tag{3.47}$$

Therefore, we have shown that $\psi(Su)>w_2$ for all $u\in C(\psi,w_2,w_3)$.

Next, condition (b) of Theorem 2.4 is fulfilled since by Lemma 3.5 and condition (P), we have $S(\overline{C}(w_1))\subseteq C(w_1)$.

Finally, we will show that condition (c) of Theorem 2.4 holds. Let $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$. Using (3.18), Lemma 3.1(d), (C3) and (3.22), we get

$$\begin{aligned}\psi(Su) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}(-1)^{\delta_k}\theta_i(Su_i)(t)\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}\int_0^1(-1)^{\delta_k}G(t,s)a_i(s)f(u(s))\,ds\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}\int_0^1 L_k\|G(\cdot,s)\|\rho_i b(s)f(u(s))\,ds\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i\|Su\| > \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i w_3\\ &> \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i\frac{w_2}{\min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i} = w_2.\end{aligned} \tag{3.48}$$

Hence, we have proved that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.

It now follows from Theorem 2.4 that the system (H) has (at least) three *fixed-sign* solutions $u^1, u^2, u^3 \in \overline{C}(w_4)$ satisfying (2.3). It is easy to see that here (2.3) reduces to (3.33). □

We will now employ Theorem 2.5 to give other existence criteria. In applying Theorem 2.5 it is possible to choose the functionals and constants in many different ways. We will present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.7.

Theorem 3.8. *Let (C1)–(C3) hold. Assume there exist numbers $\tau_{1,k}, \tau_{2,k}, \tau_{3,k}, \tau_{4,k}$, $1 \le k \le r-1$, with*

$$0 \le \tau_{1,k} \le \frac{3t_k + t_{k+1}}{4} \le \tau_{2,k} < \tau_{3,k} \le \frac{t_k + 3t_{k+1}}{4} \le \tau_{4,k} \le 1 \tag{3.49}$$

such that

(C6) *for each $1 \le i \le n$, each $1 \le k \le r-1$, and each $t \in [\tau_{2,k}, \tau_{3,k}]$, the function $G(t,s)a_i(s)$ is nonzero on a subset of $[\tau_{2,k}, \tau_{3,k}]$ of positive measure;*

(C7) *for each $j,k \in \{1,2,\dots,r-1\}$ such that $[\tau_{1,k}, \tau_{4,k}] \cap [t_j, t_{j+1}] \ne \emptyset$, and each $t \in [\tau_{1,k}, \tau_{4,k}] \cap [t_j, t_{j+1}]$, the function $G(t,s)b(s)$ is nonzero on a subset of $[\tau_{1,k}, \tau_{4,k}]$ of positive measure.*

Suppose that there exist numbers w_i, $2 \le i \le 5$, with

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i} < w_4 \le w_5 \tag{3.50}$$

such that the following hold:

(P) *$f(u_1, u_2, \dots, u_n) < (1/d_2)(w_2 - w_5 d_3/q)$ for $|u_j| \in [0, w_2]$, $1 \le j \le n$;*

(Q) *$f(u_1, u_2, \dots, u_n) \le w_5/q$ for $|u_j| \in [0, w_5]$, $1 \le j \le n$;*

(R) *for each $1 \le i \le n$, $f(u_1, u_2, \dots, u_n) > w_3/d_{1,i}$ for $|u_j| \in [w_3, w_4]$, $1 \le j \le n$.*

Then, the system (H) has (at least) three fixed-sign solutions $u^1, u^2, u^3 \in \overline{C}(w_5)$ *such that*

$$\begin{aligned} &|u_i^1(t)| < w_2, \quad t \in [\tau_{1,k}, \tau_{4,k}],\ 1 \le k \le r-1,\ 1 \le i \le n; \\ &|u_i^2(t)| > w_3, \quad t \in [\tau_{2,k}, \tau_{3,k}],\ 1 \le k \le r-1,\ 1 \le i \le n; \\ \max_{1\le i\le n} \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} &|u_i^3(t)| > w_2; \qquad \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in[\tau_{2,k},\tau_{3,k}]} |u_i^3(t)| < w_3. \end{aligned} \tag{3.51}$$

Proof. In the context of Theorem 2.5, we define the following functionals on C:

$$\begin{aligned} \gamma(u) &= \|u\|, \\ \psi(u) &= \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in I_k} |u_i(t)|, \\ \beta(u) = \Theta(u) &= \max_{1\le i\le n} \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} |u_i(t)|, \\ \alpha(u) &= \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in[\tau_{2,k},\tau_{3,k}]} |u_i(t)|. \end{aligned} \tag{3.52}$$

First, we will show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$. Let $u \in \overline{P}(\gamma, w_5)$. Then, we have $|u_j| \in [0, w_5]$, $1 \le j \le n$. Using (3.18), (Q) and (3.26), we find for $1 \le i \le n$ and $t \in [t_k, t_{k+1}]$, $1 \le k \le r-1$,

$$\begin{aligned} |Su_i(t)| = (-1)^{\delta_k}\theta_i Su_i(t) &\le \int_0^1 (-1)^{\delta_k} G(t,s) b(s) f(u(s))\,ds \\ &\le \int_0^1 (-1)^{\delta_k} G(t,s) b(s) \frac{w_5}{q}\,ds \\ &\le q\frac{w_5}{q} = w_5. \end{aligned} \tag{3.53}$$

This implies $|Su_i|_0 \le w_5$, $1 \le i \le n$ and so $\gamma(Su) = \|Su\| \le w_5$. Coupling with $Su \in C$ (Lemma 3.3), it follows that $Su \in \overline{P}(\gamma, w_5)$. Hence, we have shown that $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$.

Next, we will prove that condition (a) of Theorem 2.5 is fulfilled. We claim that

$$u^* = (u_1^*, u_2^*, \ldots, u_n^*) \in \{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\}, \tag{3.54}$$

where

$$u_i^*(t) = \theta_i \frac{w_3 + \epsilon}{\min_{1\le \ell\le n} \min_{1\le z\le r-1} L_z \rho_\ell A} \prod_{j=1}^{r} (t - t_j)^{m_j}, \quad 1 \le i \le n, \tag{3.55}$$

and $\epsilon > 0$ is chosen such that

$$\frac{w_3 + \epsilon}{\min_{1\le \ell\le n} \min_{1\le z\le r-1} L_z \rho_\ell} \le w_4. \tag{3.56}$$

Such an ϵ exists since $w_3\ (\min_{1\le \ell\le n} \min_{1\le z\le r-1} L_z \rho_\ell)^{-1} < w_4$. It is clear from (3.44) that $u^* \in C$. Further, in view of the assumptions on w_3, w_4, w_5, and the fact that $[\tau_{2,k}, \tau_{3,k}] \subseteq I_k$,

$1 \le k \le r-1$, we obtain the following:

$$\begin{aligned}
\gamma(u^*) &= A\frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A} = \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell} \le w_5,\\
\alpha(u^*) &= \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} (-1)^{\delta_k}\theta_i u_i^*(t)\\
&= \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} (-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\
&\ge \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\ \min_{1\le k\le r-1}\ \min_{t\in I_k} (-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\
&\ge \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\ \min_{1\le k\le r-1} L_k A > w_3, \qquad (3.57)\\
\Theta(u^*) &= \max_{1\le i\le n}\ \max_{1\le k\le r-1}\ \max_{t\in[\tau_{1,k},\tau_{4,k}]} (-1)^{\delta_k}\theta_i u_i^*(t)\\
&= \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\ \max_{1\le k\le r-1}\ \max_{t\in[\tau_{1,k},\tau_{4,k}]} (-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\
&= \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\ \max_{1\le k\le r-1} A_k\\
&\le \frac{w_3+\epsilon}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell} \le w_4.
\end{aligned}$$

Hence, $u^* \in \{u \in P(\gamma,\Theta,\alpha,w_3,w_4,w_5) \mid \alpha(u) > w_3\} \ne \emptyset$. Now, let $u \in P(\gamma,\Theta,\alpha,w_3,w_4,w_5)$. Then, by definition we have $\alpha(u) \ge w_3$ and $\Theta(u) \le w_4$ which imply

$$(-1)^{\delta_k}\theta_j u_j(s) = |u_j(s)| \in [w_3,w_4], \quad s \in [\tau_{2,k},\tau_{3,k}],\ 1 \le k \le r-1,\ 1 \le j \le n. \qquad (3.58)$$

Applying (3.18), (3.58), (C6), (R) and (3.26), we obtain

$$\begin{aligned}
\alpha(Su) &= \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} (-1)^{\delta_k}\theta_i (Su_i)(t)\\
&\ge \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} \int_0^1 (-1)^{\delta_k} G(t,s)a_i(s)f(u(s))\,ds\\
&\ge \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} \int_{\tau_{2,k}}^{\tau_{3,k}} (-1)^{\delta_k} G(t,s)a_i(s)f(u(s))\,ds \qquad (3.59)\\
&> \min_{1\le i\le n}\ \min_{1\le k\le r-1}\ \min_{t\in[\tau_{2,k},\tau_{3,k}]} \int_{\tau_{2,k}}^{\tau_{3,k}} (-1)^{\delta_k} G(t,s)a_i(s)\frac{w_3}{d_{1,i}}\,ds\\
&= \min_{1\le i\le n} d_{1,i}\frac{w_3}{d_{1,i}} = w_3.
\end{aligned}$$

Hence, $\alpha(Su) > w_3$ for all $u \in P(\gamma,\Theta,\alpha,w_3,w_4,w_5)$.

We will now verify that condition (b) of Theorem 2.5 is satisfied. Let w_1 be such that

$$0 < w_1 < w_2 \cdot \min_{1\le \ell \le n} \min_{1\le z\le r-1} L_z\rho_\ell. \tag{3.60}$$

We claim that

$$u^* = (u_1^*, u_2^*, \dots, u_n^*) \in \{u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(u) < w_2\}, \tag{3.61}$$

where

$$u_i^*(t) = \theta_i \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\prod_{j=1}^{r}(t-t_j)^{m_j}, \quad 1 \le i \le n. \tag{3.62}$$

As before, we have $u^* \in C$. Moreover, using (3.44) and the assumptions on w_1, w_2, w_5, we obtain the following:

$$\begin{aligned}
\gamma(u^*) &= A\frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A} = \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell} \le w_5,\\
\psi(u^*) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}(-1)^{\delta_k}\theta_i u_i^*(t)\\
&= \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\min_{1\le k\le r-1}\min_{t\in I_k}(-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\
&\ge \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\min_{1\le k\le r-1} L_k A \ge w_1,\\
\beta(u^*) &= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]}(-1)^{\delta_k}\theta_i u_i^*(t)\\
&= \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\max_{1\le k\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]}(-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\
&= \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell A}\max_{1\le k\le r-1} A_k\\
&\le \frac{w_1}{\min_{1\le \ell\le n}\min_{1\le z\le r-1} L_z\rho_\ell} < w_2.
\end{aligned} \tag{3.63}$$

Hence, $u^* \in \{u \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(u) < w_2\} \neq \emptyset$. Next, let $u \in Q(\gamma,\beta,\psi,w_1,$ $w_2,w_5)$. Then, we have $\beta(u) \le w_2$ and $\gamma(u) \le w_5$ which imply

$$\begin{aligned}
&(-1)^{\delta_k}\theta_j u_j(s) = |u_j(s)| \in [0,w_2], \quad s \in [\tau_{1,k},\tau_{4,k}],\ 1 \le k \le r-1,\ 1 \le j \le n;\\
&(-1)^{\delta_k}\theta_j u_j(s) = |u_j(s)| \in [0,w_5], \quad s \in [0,1],\ 1 \le j \le n.
\end{aligned} \tag{3.64}$$

Noting (3.18), (3.64), (C7), (P), (Q) and (3.26), we find

$$\begin{aligned}
\beta(Su) &= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]}\left|(Su_i)(t)\right| \\
&= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}(-1)^{\delta_j}\theta_i(Su_i)(t) \\
&\le \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}\int_0^1(-1)^{\delta_j}G(t,s)b(s)f(u(s))\,ds \\
&= \max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}\left[\int_{\tau_{1,k}}^{\tau_{4,k}}(-1)^{\delta_j}G(t,s)b(s)f(u(s))\,ds\right. \\
&\qquad+\int_0^{\tau_{1,k}}(-1)^{\delta_j}G(t,s)b(s)f(u(s))\,ds \\
&\qquad\left.+\int_{\tau_{4,k}}^1(-1)^{\delta_j}G(t,s)b(s)f(u(s))\,ds\right] \\
&< \max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}\int_{\tau_{1,k}}^{\tau_{4,k}}(-1)^{\delta_j}G(t,s)b(s)\,ds\cdot\frac{1}{d_2}\left(w_2-\frac{w_5d_3}{q}\right) \\
&\quad+\max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}\left[\int_0^{\tau_{1,k}}(-1)^{\delta_j}G(t,s)b(s)\,ds\right. \\
&\qquad\left.+\int_{\tau_{4,k}}^1(-1)^{\delta_j}G(t,s)b(s)\,ds\right]\frac{w_5}{q} \\
&= d_2\frac{1}{d_2}\left(w_2-\frac{w_5d_3}{q}\right)+d_3\frac{w_5}{q}=w_2.
\end{aligned}\tag{3.65}$$

Therefore, $\beta(Su)<w_2$ for all $u\in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$.

Next, we will show that condition (c) of Theorem 2.5 is met. Using Lemma 3.1(e), we observe that for $u\in C$,

$$\begin{aligned}
\Theta(Su) &= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]}\left|(Su_i)(t)\right| \\
&= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}(-1)^{\delta_j}\theta_i(Su_i)(t) \\
&\le \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}\int_0^1(-1)^{\delta_j}G(t,s)b(s)f(u(s))\,ds \\
&\le \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{1\le j\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]\cap[t_j,t_{j+1}]}\int_0^1\|G(\cdot,s)\|b(s)f(u(s))\,ds \\
&= \int_0^1\|G(\cdot,s)\|b(s)f(u(s))\,ds.
\end{aligned}\tag{3.66}$$

Moreover, (C3) and Lemma 3.1(d) yield for $u \in C$,

$$\begin{aligned}\alpha(Su) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in[\tau_{2,k},\tau_{3,k}]}(-1)^{\delta_k}\theta_i(Su_i)(t)\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in[\tau_{2,k},\tau_{3,k}]}\int_0^1(-1)^{\delta_k}G(t,s)a_i(s)f(u(s))\,ds\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k}\int_0^1(-1)^{\delta_k}G(t,s)\rho_i b(s)f(u(s))\,ds\\ &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i\int_0^1\|G(\cdot,s)\|b(s)f(u(s))\,ds.\end{aligned} \tag{3.67}$$

A combination of (3.66) and (3.67) gives

$$\alpha(Su) \ge \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i\Theta(Su), \quad u\in C. \tag{3.68}$$

Let $u \in P(\gamma,\alpha,w_3,w_5)$ with $\Theta(Su) > w_4$. Then, it follows from (3.68) that

$$\begin{aligned}\alpha(Su) &\ge \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i\Theta(Su) > \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i w_4\\ &> \min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i\frac{w_3}{\min_{1\le i\le n}\min_{1\le k\le r-1}L_k\rho_i} = w_3.\end{aligned} \tag{3.69}$$

Thus, $\alpha(Su) > w_3$ for all $u \in P(\gamma,\alpha,w_3,w_5)$ with $\Theta(Su) > w_4$.

Finally, we will prove that condition (d) of Theorem 2.5 is fulfilled. Let $u \in Q(\gamma,\beta, w_2,w_5)$ with $\psi(Su) < w_1$. Then, we have $\beta(u) \le w_2$ and $\gamma(u) \le w_5$ which give (3.64). Using (3.18), (3.64), (C7), (P), (Q) and (3.26), we get as in an earlier part $\beta(Su) < w_2$ for all $u \in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Su) < w_1$.

It now follows from Theorem 2.5 that the system (H) has (at least) three *fixed-sign* solutions $u^1,u^2,u^3 \in \overline{P}(\gamma,w_5) = \overline{C}(w_5)$ satisfying (2.5). It is clear that (2.5) reduces to (3.51) immediately. □

For each $1 \le k \le r-1$, if

$$\tau_{1,k}=0, \qquad \tau_{4,k}=1, \qquad \tau_{2,k}=\frac{3t_k+t_{k+1}}{4}, \qquad \tau_{3,k}=\frac{t_k+3t_{k+1}}{4}, \tag{3.70}$$

then

$$d_{1,i}=r_i, \quad 1\le i\le n, \qquad d_2=q, \qquad d_3=0. \tag{3.71}$$

In this case Theorem 3.8 yields the following corollary.

Corollary 3.9. *Let (C1)–(C3) hold, and assume*

(C6)′ *for each $1 \le i \le n$, each $1 \le k \le r-1$, and each $t \in I_k$, the function $G(t,s)a_i(s)$ is nonzero on a subset of I_k of positive measure;*

(C7)′ *for each $1 \le k \le r-1$ and each $t \in [t_k,t_{k+1}]$, the function $G(t,s)b(s)$ is nonzero on a subset of $[0,1]$ of positive measure.*

Suppose that there exist numbers w_i, $2 \le i \le 5$ with

$$0 < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i} < w_4 \le w_5 \tag{3.72}$$

such that the following hold:

(P) $f(u_1,u_2,\dots,u_n) < w_2/q$ *for* $|u_j| \in [0,w_2]$, $1 \le j \le n$;

(Q) $f(u_1,u_2,\dots,u_n) \le w_5/q$ *for* $|u_j| \in [0,w_5]$, $1 \le j \le n$;

(R) *for each* $1 \le i \le n$, $f(u_1,u_2,\dots,u_n) > w_3/r_i$ *for* $|u_j| \in [w_3,w_4]$, $1 \le j \le n$.

Then, the system (H) has (at least) three fixed-sign solutions $u^1,u^2,u^3 \in \overline{C}(w_5)$ *such that*

$$\begin{gathered} \|u^1\| < w_2; \qquad |u_i^2(t)| > w_3, \quad t \in I_k,\ 1 \le k \le r-1,\ 1 \le i \le n; \\ \|u^3\| > w_2; \qquad \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in I_k} |u_i^3(t)| < w_3. \end{gathered} \tag{3.73}$$

Remark 3.10. Corollary 3.9 is actually Theorem 3.7. Hence, Theorem 3.8 is more general than Theorem 3.7.

The next result illustrates another application of Theorem 2.5.

Theorem 3.11. *Let (C1)–(C3) hold. Assume there exist numbers* $\tau_{1,k},\tau_{2,k},\tau_{3,k},\tau_{4,k}$ $1 \le k \le r-1$ *with*

$$\frac{3t_k + t_{k+1}}{4} \le \tau_{1,k} < \tau_{2,k} < \tau_{3,k} < \tau_{4,k} \le \frac{t_k + 3t_{k+1}}{4} \tag{3.74}$$

such that (C6) holds and

(C8) *for each* $1 \le k \le r-1$ *and each* $t \in [\tau_{1,k},\tau_{4,k}]$, *the function* $G(t,s)b(s)$ *is nonzero on a subset of* $[\tau_{1,k},\tau_{4,k}]$ *of positive measure.*

Suppose that there exist numbers w_i, $1 \le i \le 5$ with

$$0 < w_1 < w_2 \cdot \min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i < w_2 < w_3 < \frac{w_3}{\min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i} < w_4 \le w_5 \tag{3.75}$$

such that the following hold:

(P) $f(u_1,u_2,\dots,u_n) < (1/d_4)(w_2 - w_5 d_5/q)$ *for* $|u_j| \in [w_1,w_2]$, $1 \le j \le n$;

(Q) $f(u_1,u_2,\dots,u_n) \le w_5/q$ *for* $|u_j| \in [0,w_5]$, $1 \le j \le n$;

(R) *for each* $1 \le i \le n$, $f(u_1,u_2,\dots,u_n) > w_3/d_{1,i}$ *for* $|u_j| \in [w_3,w_4]$, $1 \le j \le n$.

Then, the system (H) has (at least) three fixed-sign solutions $u^1,u^2,u^3 \in \overline{C}(w_5)$ *such that*

$$\begin{gathered} |u_i^1(t)| < w_2, \quad t \in [\tau_{1,k},\tau_{4,k}],\ 1 \le k \le r-1,\ 1 \le i \le n; \\ |u_i^2(t)| > w_3, \quad t \in [\tau_{2,k},\tau_{3,k}],\ 1 \le k \le r-1,\ 1 \le i \le n; \\ \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]} |u_i^3(t)| > w_2; \qquad \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in[\tau_{2,k},\tau_{3,k}]} |u_i^3(t)| < w_3. \end{gathered} \tag{3.76}$$

Proof. In the context of Theorem 2.5, we define the following functionals on C:

$$\begin{aligned}\gamma(u) &= \|u\|,\\ \psi(u) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in[\tau_{1,k},\tau_{4,k}]}(-1)^{\delta_k}\theta_i u_i(t),\\ \beta(u) &= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{1,k},\tau_{4,k}]}(-1)^{\delta_k}\theta_i u_i(t),\\ \alpha(u) &= \min_{1\le i\le n}\min_{1\le k\le r-1}\min_{t\in[\tau_{2,k},\tau_{3,k}]}(-1)^{\delta_k}\theta_i u_i(t),\\ \Theta(u) &= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{2,k},\tau_{3,k}]}(-1)^{\delta_k}\theta_i u_i(t).\end{aligned} \tag{3.77}$$

First, using (Q), as in the proof of Theorem 3.8, we can show that $S:\overline{P}(\gamma,w_5)\to\overline{P}(\gamma,w_5)$. Next, we will verify that condition (a) of Theorem 2.5 is fulfilled. We claim that

$$u^* = (u_1^*,u_2^*,\ldots,u_n^*)\in\{u\in P(\gamma,\Theta,\alpha,w_3,w_4,w_5)\mid\alpha(u)>w_3\}, \tag{3.78}$$

where

$$u_i^*(t) = \theta_i\frac{w_3+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1}L_z\rho_\ell A}\prod_{j=1}^{r}(t-t_j)^{m_j},\quad 1\le i\le n, \tag{3.79}$$

and $\epsilon>0$ is chosen such that

$$\frac{w_3+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1}L_z\rho_\ell}\le w_4. \tag{3.80}$$

Such an ϵ exists since $w_3\,(\min_{1\le\ell\le n}\min_{1\le z\le r-1}L_z\rho_\ell)^{-1}<w_4$. As in the proof of Theorem 3.8, we can show that $u^*\in C$, $\gamma(u^*)\le w_5$ and $\alpha(u^*)>w_3$. Further, in view of the assumptions on w_3 and w_4, we have

$$\begin{aligned}\Theta(u^*) &= \max_{1\le i\le n}\max_{1\le k\le r-1}\max_{t\in[\tau_{2,k},\tau_{3,k}]}(-1)^{\delta_k}\theta_i u_i^*(t)\\ &= \frac{w_3+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1}L_z\rho_\ell A}\max_{1\le k\le r-1}\max_{t\in[\tau_{2,k},\tau_{3,k}]}(-1)^{\delta_k}\prod_{j=1}^{r}(t-t_j)^{m_j}\\ &= \frac{w_3+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1}L_z\rho_\ell A}\max_{1\le k\le r-1}B_k\\ &\le \frac{w_3+\epsilon}{\min_{1\le\ell\le n}\min_{1\le z\le r-1}L_z\rho_\ell}\le w_4.\end{aligned} \tag{3.81}$$

Hence, $u^*\in\{u\in P(\gamma,\Theta,\alpha,w_3,w_4,w_5)\mid\alpha(u)>w_3\}\ne\emptyset$. Using (R) and a similar argument as in the proof of Theorem 3.8, we can show that $\alpha(Su)>w_3$ for all $u\in P(\gamma,\Theta,\alpha,w_3,w_4,w_5)$.

Now, we will check that condition (b) of Theorem 2.5 is satisfied. We claim that

$$u^* = (u_1^*, u_2^*, \dots, u_n^*) \in \{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\}, \tag{3.82}$$

where

$$u_i^*(t) = \theta_i \frac{w_1}{\min_{1\le \ell \le n} \min_{1\le z\le r-1} L_z \rho_\ell A} \prod_{j=1}^{r} (t - t_j)^{m_j}, \quad 1 \le i \le n. \tag{3.83}$$

As in the proof of Theorem 3.8, we see that $u^* \in C$, $\gamma(u^*) \le w_5$ and $\beta(u^*) < w_2$. Further, we have

$$\begin{aligned}
\psi(u^*) &= \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in[\tau_{1,k},\tau_{4,k}]} (-1)^{\delta_k} \theta_i u_i^*(t) \\
&= \frac{w_1}{\min_{1\le \ell \le n} \min_{1\le z\le r-1} L_z \rho_\ell A} \min_{1\le k\le r-1} \min_{t\in[\tau_{1,k},\tau_{4,k}]} (-1)^{\delta_k} \prod_{j=1}^{r} (t - t_j)^{m_j} \\
&\ge \frac{w_1}{\min_{1\le \ell \le n} \min_{1\le z\le r-1} L_z \rho_\ell A} \min_{1\le k\le r-1} \min_{t\in I_k} (-1)^{\delta_k} \prod_{j=1}^{r} (t - t_j)^{m_j} \\
&\ge \frac{w_1}{\min_{1\le \ell \le n} \min_{1\le z\le r-1} L_z \rho_\ell A} \min_{1\le k\le r-1} L_k A \ge w_1.
\end{aligned} \tag{3.84}$$

Thus, $u^* \in \{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \ne \emptyset$. Let $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have $\psi(u) \ge w_1$, $\beta(u) \le w_2$ and $\gamma(u) \le w_5$ which imply

$$\begin{gathered}
(-1)^{\delta_k} \theta_j u_j(s) = |u_j(s)| \in [w_1, w_2], \quad s \in [\tau_{1,k}, \tau_{4,k}],\ 1 \le k \le r-1,\ 1 \le j \le n; \\
(-1)^{\delta_k} \theta_j u_j(s) = |u_j(s)| \in [0, w_5], \quad s \in [0,1],\ 1 \le j \le n.
\end{gathered} \tag{3.85}$$

Using (3.18), (3.85), (C8), (P), (Q), and (3.26), we find by a similar technique as in the proof of Theorem 3.8 that $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$.

Next, we will show that condition (c) of Theorem 2.5 is met. We observe that, by (3.18) and Lemma 3.1(e), for $u \in C$,

$$\begin{aligned}
\Theta(Su) &= \max_{1\le i\le n} \max_{1\le k\le r-1} \max_{t\in[\tau_{2,k},\tau_{3,k}]} (-1)^{\delta_k} \theta_i (Su_i)(t) \\
&\le \max_{1\le i\le n} \max_{1\le k\le r-1} \max_{t\in[\tau_{2,k},\tau_{3,k}]} \int_0^1 (-1)^{\delta_k} G(t,s) b(s) f(u(s))\, ds \\
&\le \max_{1\le i\le n} \max_{1\le k\le r-1} \max_{t\in[\tau_{2,k},\tau_{3,k}]} \int_0^1 \|G(\cdot,s)\| b(s) f(u(s))\, ds \\
&= \int_0^1 \|G(\cdot,s)\| b(s) f(u(s))\, ds.
\end{aligned} \tag{3.86}$$

Moreover, using (3.18), (C3) and Lemma 3.1(d), we obtain (3.67) for $u \in C$. A combination of (3.67) and (3.86) yields (3.68). Following a similar argument as in the proof of Theorem 3.8, we get $\alpha(Su) > w_3$ for all $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$.

Finally, we will prove that condition (d) of Theorem 2.5 is fulfilled. As in (3.86), by (3.18) and Lemma 3.1(e), we see that for $u \in C$,

$$\beta(Su) = \max_{1\le i\le n} \max_{1\le k\le r-1} \max_{t\in[\tau_{1,k},\tau_{4,k}]} (-1)^{\delta_k}\theta_i(Su_i)(t) \le \int_0^1 \|G(\cdot,s)\| b(s) f(u(s))\,ds. \tag{3.87}$$

On the other hand, it follows from (3.18), (C3) and Lemma 3.1(d) that for $u \in C$,

$$\begin{aligned}\psi(Su) &= \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in[\tau_{1,k},\tau_{4,k}]} (-1)^{\delta_k}\theta_i(Su_i)(t)\\ &\ge \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in[\tau_{1,k},\tau_{4,k}]} \int_0^1 (-1)^{\delta_k} G(t,s) a_i(s) f(u(s))\,ds\\ &\ge \min_{1\le i\le n} \min_{1\le k\le r-1} \min_{t\in I_k} \int_0^1 (-1)^{\delta_k} G(t,s)\rho_i b(s) f(u(s))\,ds\\ &\ge \min_{1\le i\le n} \min_{1\le k\le r-1} L_k\rho_i \int_0^1 \|G(\cdot,s)\| b(s) f(u(s))\,ds.\end{aligned} \tag{3.88}$$

A combination of (3.87) and (3.88) gives

$$\psi(Su) \ge \min_{1\le i\le n} \min_{1\le k\le r-1} L_k\rho_i\beta(Su), \quad u \in C. \tag{3.89}$$

Let $u \in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Su) < w_1$. Then, (3.89) leads to

$$\begin{aligned}\beta(Su) &\le \frac{1}{\min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i}\psi(Su) < \frac{1}{\min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i} w_1\\ &< \frac{1}{\min_{1\le i\le n}\min_{1\le k\le r-1} L_k\rho_i} w_2 \cdot \min_{1\le i\le n} \min_{1\le k\le r-1} L_k\rho_i = w_2.\end{aligned} \tag{3.90}$$

Thus, $\beta(Su) < w_2$ for all $u \in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Su) < w_1$.

It now follows from Theorem 2.5 that the system (H) has (at least) three *fixed-sign* solutions $u^1, u^2, u^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ satisfying (2.5). Furthermore, (2.5) reduces to (3.76) immediately. □

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