EMBEDDING OPERATORS AND MAXIMAL REGULAR DIFFERENTIAL-OPERATOR EQUATIONS IN BANACH-VALUED FUNCTION SPACES

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This study focuses on anisotropic Sobolev type spaces associated with Banach spaces E_0, E. Several conditions are found that ensure the continuity and compactness of embedding operators that are optimal regular in these spaces in terms of interpolations of E_0 and E. In particular, the most regular class of interpolation spaces E_α between E_0, E, depending of α and order of spaces are found that mixed derivatives D^α belong with values; the boundedness and compactness of differential operators D^α from this space to E_α-valued L_p spaces are proved. These results are applied to partial differential-operator equations with parameters to obtain conditions that guarantee the maximal L_p regularity uniformly with respect to these parameters.

1. Introduction

Embedding theorems in function spaces have been elaborated in detail by [5, 28]. A comprehensive introduction to the theory of embedding of function spaces and historical references may be also found in [28]. In abstract function spaces embedding theorems have been studied by [3, 18, 22, 23, 24, 25, 26]. Lions-Peetre [18] showed that, if $u \in L_2(0,T;H_0)$, $u^{(m)} \in L_2(0,T;H)$ then $u^{(i)} \in L_2(0,T;[H,H_0]_{i/m})$, $i = 1,2,\ldots,m-1$, where H_0, H are Hilbert spaces, H_0 is continuously and densely embedded in H and $[H_0,H]_\theta$ are interpolation spaces between H_0, H for $0 \le \theta \le 1$. In [22, 23, 24, 25, 26] the similar questions were investigated for anisotropic Sobolev spaces $W_p^l(\Omega;H_0,H)$, $\Omega \subset R^n$. Moreover, boundary value problems for differential-operator equations have been studied in detail by [16, 27, 30, 32]. The solvability and the spectrum of boundary value problems for elliptic differential-operator equations have also been refined by [1, 2, 4, 8, 10, 11, 13, 22, 23, 24, 25, 26]. A comprehensive introduction to the differential-operator equations and historical references may be found in [16, 32]. In these works Hilbert-valued function spaces essentially have been considered. In the present paper, are to be introduced a Banach-valued function spaces $W_p^l(\Omega;E_0,E)$, where $l = (l_1,l_2,\ldots,l_n)$ and E_0, E are Banach spaces such that E_0 is continuously and densely embedded in E. The properties of continuity and compactness of embedding operators in these spaces are obtained. We prove that the generalized derivative operator D^α is continuous from these Banach-valued

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Sobolev spaces to E_α-valued L_p spaces, where E_α are interpolation spaces between E_0 and E depending on the order of differentiations D^α. By applying these results, the maximal L_p-regularity of certain class of anisotropic partial differential-operator equations are derived.

Let $\alpha_1,\alpha_2,\dots,\alpha_n$ be nonnegative integer numbers and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}. \tag{1.1}$$

Under certain assumptions to be stated later, we prove that the operators $u \to D^\alpha u$ are bounded from space $W_p^l(\Omega;E(A),E)$ to space $L_q(\Omega;E(A^{1-\varkappa}))$, that is, embedding

$$D^\alpha W_p^l(\Omega;E(A),E) \subset L_q(\Omega;E(A^{1-\varkappa})) \tag{1.2}$$

is continuous. More precisely for $0 < \mu \le 1-\varkappa$ we prove the estimate

$$||D^\alpha u||_{L_p(\Omega;E(A^{1-\varkappa}))} \le C_\mu \left(h^\mu \|u\|_{W_p^l(\Omega;E(A)E)} + h^{-(1-\mu)} \|u\|_{L_p(\Omega;E)} \right) \tag{1.3}$$

for all $u \in W_p^l(\Omega;E(A),E)$ and $0 \le h \le h_0 < \infty$. The constant C_μ in the above equation is independent of $u \in W_p^l(\Omega;E(A),E)$ and of the choice of h. Further, we prove compactness of this embedding operator. Furthermore, we consider certain applications of these theorems. This kind of embedding theorems arise in the investigation of boundary value problems for anisotropic partial differential-operator equations

$$\sum_{k=1}^n a_k t_k D_k^{l_k} u + Au \sum_{|\alpha:l|<1} \prod_{k=1}^n t_k^{\alpha_k/l_k} A_\alpha(x) D^\alpha u = f, \tag{1.4}$$

depend on parameters $t = (t_1,t_2,\dots,t_n)$, where A is a positive operator on the Banach space E, $A_\alpha(x)$ is an operator such that $A_\alpha(x)A^{-(1-|\alpha:l|)}$ is bounded on E, where $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$, $l = (l_1,l_2,\dots,l_n)$, $|\alpha : l| = \sum_{k=1}^n(\alpha_k/l_k)$. In general, this equations possess different derivatives and different parameters with respect to the various variables. Taking $l_1 = l_2 = \cdots = l_n = 2l$ in the above equations we obtain elliptic equation with parameters

$$\sum_{k=1}^n a_k t_k D_k^{2l} u + Au + \sum_{|\alpha|<2l} \prod_{k=1}^n t_k^{\alpha_k/2l} A_\alpha(x) D^\alpha u = f(x). \tag{1.5}$$

We prove the maximal regularity of this differential-operator equations in $L_p(R^n;E)$ uniformly with respect to parameter t. In this direction we should mention the works [10, 22, 23, 24, 25, 26, 31].

2. Notations and definitions

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{C}$ be the set of complex numbers. Let E and E_0 be Banach spaces and $L(E_0,E)$ denotes the spaces of bounded linear operators acting from E_0 to E. For $E_0 = E$ we denote $L(E,E)$ by $L(E)$, I denotes the identity operator in the Banach space E. Let A be a linear operator in E. We will sometimes use $A+\xi$ or A_ξ instead of $A+\xi I$

for a scalar ξ and $(A-\xi I)^{-1}$ denotes the inverse of the operator $A-\xi I$ or the resolvent of operator A.

Let

$$S_\varphi = \{\xi,\ \xi \in \mathbf{C},\ |\arg\xi - \pi| \le \pi - \varphi\} \cup \{0\}, \quad 0 < \varphi \le \pi. \tag{2.1}$$

A linear operator A is said to be positive in a Banach space E, if $D(A)$ is dense on E and

$$\|(A-\xi I)^{-1}\|_{L(E)} \le M(1+|\xi|)^{-1} \tag{2.2}$$

with $\xi \in S_\varphi$, where M is a positive constant [28].

$$E(A^\theta) = \left\{u,\ u \in D(A^\theta),\ \|u\|_{E(A^\theta)} = \|A^\theta u\|_E + \|u\| < \infty,\ -\infty < \theta < \infty\right\}. \tag{2.3}$$

We denote by $L_p(\Omega;E)$ the space of strongly measurable E-valued functions on $\Omega \subset R^n$ with the norm

$$\|u\|_{L_p} = \|u\|_{L_p(\Omega;E)} = \left(\int_\Omega \|u(x)\|_E^p\,dx\right)^{1/p}, \quad 1 \le p < \infty. \tag{2.4}$$

Let $l = (l_1, l_2, \dots, l_n)$, where l_i, $i = 1,2,\dots,n$ positive integers and $D_k^{l_k} = \partial^{l_k}/\partial x_k^{l_k}$, $k = 1,2,\dots,n$.

We introduce a E_0-valued anisotropic function space $W_p^l(\Omega;E_0,E)$ that consist of functions $u \in L_p(\Omega;E_0)$ such that have the generalized derivatives $D_k^{l_k}u \in L_p(\Omega;E)$ with the norm

$$\|u\|_{W_p^l(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^n \|D_k^{l_k}u\|_{L_p(\Omega;E)} < \infty, \quad 1 \le p < \infty. \tag{2.5}$$

Let be $t = (t_1, t_2, \dots, t_n)$, where t_k, $k = 1,2,\dots,n$ are nonnegative parameters. Let us define in the space $W_p^l(\Omega;E_0,E)$ parameterized norm

$$\|u\|_{W_{p,t}^l(\Omega;E_0,E)} = \|u\|_{L_p(\Omega;E_0)} + \sum_{k=1}^n \|t_k D_k^{l_k}u\|_{L_p(\Omega;E)}. \tag{2.6}$$

The Banach space E is said to be ξ-convex [7] if there exists on $E \times E$ a symmetric function $\xi(u,v)$ which is convex with respect to every one of the variables and satisfies the condition

$$\xi(0,0) > 0, \qquad \xi(u,v) \le \|u+v\| \quad \text{for } \|u\|_E = \|v\|_E = 1. \tag{2.7}$$

It is shown in [7] that a Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy \tag{2.8}$$

is bounded in the space $L_p(R;E)$, $p \in (1,\infty)$, for those and only those Banach spaces E which possess the property of ξ-convexity. The ξ-convex Banach spaces is often called

UMD spaces. UMD spaces contains L_p, l_p spaces and the Lorentz spaces L_{pq}, $p,q \in (1,\infty)$ for instance.

$C^{(l)}(\Omega;E)$ denotes the space of E-valued continuously differentiable functions of lth order. Let E_1 and E_2 be Banach spaces. A function $\Psi \in C^{(l)}(R^n;L(E_1,E_2))$ is called a multiplier from $L_p(R^n;E_1)$ to $L_q(R^n;E_2)$ if there exists a constant $M>0$ such that

$$\|F^{-1}\Psi(\xi)Fu\|_{L_q(R^n;E_2)} \leq C\|u\|_{L_p(R^n;E_1)} \tag{2.9}$$

for all $u \in L_p(R^n;E_1)$, where F and F^{-1} are Fourier and inverse Fourier transformations, respectively.

We denote the set of all multipliers from $L_p(R^n;E_1)$ to $L_q(R^n;E_2)$ by $M_p^q(E_1,E_2)$. For $E_1=E_2=E$ we denote $M_p^q(E_1,E_2)$ by $M_p^q(E)$. Let

$$H_k = \{\Psi_h \in M_p^q(E_1,E_2),\ h=(h_1,h_2,\dots,h_L) \in Q\} \tag{2.10}$$

be a collection of multipliers in $M_{p,\gamma}^{q,\gamma}(E_1,E_2)$. We say that $\Psi_h=\Psi_h(\xi)$ is a uniformly bounded multipliers with respect to h if there exists a constant $C>0$, independent of $h \in B(h)$, such that

$$\|F^{-1}\Psi_h Fu\|_{L_q(R^n,E_2)} \leq C\|u\|_{L_p(R^n,E_1)} \tag{2.11}$$

for all $h \in K$ and $u \in L_p(R^n;E_1)$.

The exposition of the theory of L_p-multipliers of the Fourier transformation, and some related references, can be found in [28, Sections 2.2.1, 2.2.2, 2.2.3, and 2.2.4]. On the other hand, in vector-valued function spaces, Fourier multipliers have been studied, for example, by [3, 6, 12, 15, 20, 21, 29].

A set $K \subset B(E_1,E_2)$ is called R-bounded [6, 29] if there is a constant C such that for all $T_1,T_2,\dots,T_m \in K$ and $u_1,u_2,\dots,u_m \in E_1$, $m \in N$.

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y)T_ju_j \right\|_{E_2} dy \leq C\int_0^1 \left\| \sum_{j=1}^m r_j(y)u_j \right\|_{E_1} dy, \tag{2.12}$$

where $\{r_j\}$ is a sequence of independent symmetric $[-1,1]$-valued random variables on $[0,1]$.

A set $K(h) \subset B(E_1,E_2)$ depending on parameters $h=(h_1,h_2,\dots,h_L) \in B(h) \in R^L$ is called uniformly R-bounded with respect to h if there is a constant C such that for all $T_1(h),T_2(h),\dots,T_m(h) \in K$ and $u_1,u_2,\dots,u_m \in E_1$, $m \in N$.

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y)T_j(h)u_j \right\|_{E_2} dy \leq C\int_0^1 \left\| \sum_{j=1}^m r_j(y)u_j \right\|_{E_1} dy, \tag{2.13}$$

where a positive constant C is independent of parameters h.

Let

$$\begin{gathered} U_n = \{\beta=(\beta_1,\beta_2,\dots,\beta_n),\ \beta_i \in (0,1),\ i=1,2,\dots,n\}, \\ V_n = \{\xi=(\xi_1,\xi_2,\dots,\xi_n) \in R^n,\ \xi_i \neq 0,\ i=1,2,\dots,n\}, \\ \alpha=(\alpha_1,\alpha_2,\dots,\alpha_n), \qquad \xi^\alpha=\xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_n^{\alpha_n}, \qquad |\xi|^\alpha=|\xi|^{\alpha_1}|\xi|^{\alpha_2}\cdots|\xi|^{\alpha_n}. \end{gathered} \tag{2.14}$$

Definition 2.1. The Banach space E is said to be a space satisfying a multiplier condition with respect to $p,q \in (1,\infty)$, $p \leq q$ when for $\Psi \in C^{(n)}(R^n;B(E))$ if the set

$$\Psi(\xi):\left\{\xi^{\beta+1/p-1/q}D_\xi^\beta\Psi(\xi):\xi \in V_n,\ \beta \in U_n\right\} \tag{2.15}$$

are R-bounded, then $\Psi \in M_p^q(E)$.

A Banach space E has a property (α), (see, e.g., [12]) if there exists a constant α such that

$$\left\|\sum_{i,j=1}^{N}\alpha_{ij}\varepsilon_i\varepsilon_j^{'}x_{ij}\right\|_{L_2(\Omega\times\Omega^{'};E)} dy \leq \alpha\left\|\sum_{i,j=1}^{N}\varepsilon_i\varepsilon_j^{'}x_{ij}\right\|_{L_2(\Omega\times\Omega^{'};E)} \tag{2.16}$$

for all $N \in \mathbf{N}$, $x_{i,j} \in E$, $\alpha_{ij} \in \{0,1\}$, $i,j = 1,2,\dots,N$, and all choices of independent, symmetric, $\{-1,1\}$-valued random variables $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_N$, $\varepsilon_1^{'},\varepsilon_2^{'},\dots,\varepsilon_N^{'}$ on probability spaces Ω, $\Omega^{'}$. For example, the spaces $L_p(\Omega)$, $1 \leq p < \infty$ has the property (α).

Remark 2.2. If E is UMD space with property (α) then these spaces are satisfy the multiplier condition with respect to $p \in (1,\infty)$ (see [12]).

Definition 2.3. The φ-positive operator A is said to be a R-positive in the Banach space E if there exists $\varphi \in (0,\pi]$ such that the set

$$L_A = \left\{(1+|\xi|)(A-\xi I)^{-1}:\xi \in S_\varphi\right\} \tag{2.17}$$

is R-bounded.

Note that in the Hilbert spaces every norm bounded set is R-bounded. Therefore, in the Hilbert spaces all positive operators are R-positive. If A is a generator of a contraction semigroup on L_q, $1 \leq q \leq \infty$ [17], A has the bounded imaginary powers with $\|(-A^{it})\|_{B(E)} \leq Ce^{\nu|t|}$, $\nu < \pi/2$ in $E \in$ UMD [8, 9] then those operators are R-positive.

It is well known (see, e.g., [19]) that any Hilbert space satisfies the multiplier condition. By virtue of [21] Mikhlin conditions are not sufficient for operator-valued multiplier theorem. There are however, Banach spaces which are not Hilbert spaces but satisfy the multiplier condition, for example, UMD spaces (see, e.g., [29]).

By $\sigma_\infty(E)$ will be denoted a space of compact operators acting in E.

Example 2.4. If $\gamma \in A_p$, $\delta \in C^\infty(R)$ with $\delta(y) \geq 0$ for all $y \geq 0$, $\delta(y) = 0$ for $|y| \leq 1/2$ and $\delta(-y) = -\delta(y)$ for all y, then $\delta \in M_{p,\gamma}^{p,\gamma}(R)$. Really it clear to see that $\delta(y)$ satisfies multiplier conditions [28, Section 2.3.3].

3. Embedding theorems

Lemma 3.1. *Let A be a positive operator on a Banach space E and $r = (r_1,r_2,\dots,r_n)$ where $r_k \in \{0,b\}$. Let $t = (t_1,t_2,\dots,t_n)$, where t_k, $k = 1,2,\dots,n$ are nonnegative parameters, $0 < t_k \leq t_0 < \infty$, $\alpha = (\alpha_1,\alpha_2,\dots,\alpha_n)$ and $l = (l_1,l_2,\dots,l_n)$, $l_k > 0$ such that $\varkappa = |(\alpha+r):l| \leq 1$.*

Let δ be a multiplier of the form described in Example 2.4. Then for $0 \le h \le h_0 < \infty$ and $0 \le \mu \le 1 - \varkappa$ the operator-function

$$\begin{aligned}\Psi_t(\xi) &= \Psi_{t,r,h,\mu}(\xi) \\ &= \prod_{k=1}^{n} t_k^{(\alpha_k+r)/l_k} \xi^r (i\xi)^\alpha A^{1-\varkappa-\mu} h^{-\mu} \left[A + \sum_{k=1}^{n} t_k (\delta(\xi_k))^{l_k} + h^{-1} \right]^{-1}\end{aligned} \tag{3.1}$$

is bounded operator in E uniformly with respect to $\xi \in R^n$, h and t, that is, there is a constant C_μ such that

$$||\Psi_{t,h,\mu}(\xi)||_{L(E)} \le C_\mu \tag{3.2}$$

for all ξ, t and h.

Proof. Since $-[\sum_{k=1}^{n} t_k (\delta(\xi)\xi_k)^{l_k} + h^{-1}] \in S(\varphi)$ for all $\varphi \in [0, \pi)$ then by virtue of the positiveness of A, operator $B(\xi) = A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} + h^{-1}$ is invertible in the space E. Let $u = h^{-\mu} B^{-1}(\xi) f$. Then

$$\begin{aligned}||\Psi_t(\xi) f||_E &= \prod_{k=1}^{n} t_k^{(\alpha_k+r)/l_k} |\xi|^{r+\alpha} ||A^{1-\varkappa-\mu} u||_E \\ &= ||(hA)^{1-\varkappa-\mu} u||_E h^{-(1-\mu)} \left| (ht_1)^{1/l_1} \xi_1 \right|^{\alpha_1+r_1} \cdots \left| (ht_n)^{1/l_n} \xi_n \right|^{\alpha_n+r_n}.\end{aligned} \tag{3.3}$$

Using the moment inequality for powers of a positive operators, we get a constant C_μ depending only on μ such that

$$||\Psi_t(\xi)||_E \le C_\mu h^{(1-\mu)} \|hAu\|^{1-\varkappa-\mu} \|u\|^{\varkappa+\mu} \left| (ht_1)^{1/l_1} \xi_1 \right|^{\alpha_1+r_1} \cdots \left| (ht_n)^{1/l_n} \xi_n \right|^{\alpha_n+r_n}. \tag{3.4}$$

Now, we apply the Young inequality, which states that $g_1 g_2 \le g_1^{k_1}/k_1 + g_2^{k_2}/k_2$ for any positive real numbers g_1, g_2 and k_1, k_2 with $1/k_1 + 1/k_2 = 1$, to the product

$$\|hAu\|^{1-\varkappa-\mu} \left[\|u\|^{\varkappa+\mu} \left| (ht_1)^{1/l_1} \xi_1 \right|^{\alpha_1+r_1} \cdots \left| (ht_n)^{1/l_n} \xi_n \right|^{\alpha_n+r_n} \right] \tag{3.5}$$

with $k_1 = 1/(1 - \varkappa - \mu)$, $k_2 = 1/(\varkappa + \mu)$ to get

$$\begin{aligned}||\Psi_t(\xi) f||_E \le C_\mu h^{-(1-\mu)} &(1 - \varkappa - \mu) \|hAu\| \\ &+ (\varkappa+\mu)(ht_1|\xi_1|)^{(\alpha_1+r_1)/(\varkappa+\mu)} \cdots (ht_n|\xi_n|)^{(\alpha_n+r_n)/(\varkappa+\mu)}.\end{aligned} \tag{3.6}$$

Since

$$\sum_{i=1}^{n} \frac{\alpha_i + r_i}{(\varkappa+\mu)} = \frac{1}{\varkappa+\mu} \sum_{i=1}^{n} \frac{\alpha_i + r_i}{l_i} = \frac{\varkappa}{\varkappa+\mu} \le 1 \tag{3.7}$$

there exists a positive constant M_0 independent of ξ, such that

$$|\xi_1|^{(\alpha_1+r_1)/(\varkappa+\mu)} \cdots |\xi_n|^{(\alpha_n+r_n)/(\varkappa+\mu)} \le M_0 \left(1 + \sum_{k=1}^{n} |\xi_k|^{l_k} \right) \tag{3.8}$$

for all $\xi \in R^n$. It is clear that $|y|^l \le (\delta(y)y)^l$ for all $|y| > 1/2$. Therefore

$$|\xi_1|^{(\alpha_1+r_1)/(\varkappa+\mu)} \cdots |\xi_n|^{(\alpha_n+r_n)/(\varkappa+\mu)} \le M_1\left[1+\sum_{k=1}^{n}(\delta(\xi_k)\xi_k)^{l_k}\right] \tag{3.9}$$

for a suitable $M_1 > 0$ and all $\xi \in R^n$. Substituting this on the inequality (3.6) and absorbing the constant coefficients in C_μ, we obtain

$$||\psi_t(\xi)f|| \le C_\mu h^\mu\left[\|Au\| + \left(\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k} + h^{-1}\right)\|u\|\right]. \tag{3.10}$$

Substituting the value of u, we get

$$||\psi_t(\xi)f|| \le C_\mu||AB^{-1}(\xi)f|| + \left[\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k} + h^{-1}\right]||B^{-1}(\xi)f||. \tag{3.11}$$

Since A is positive operator in the space E, we have

$$\left|\left|\left[A+\sum_{k=1}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k} + h^{-1}\right]^{-1} f\right|\right| \le M\left[1+\sum_{k=1k}^{n} t_k(\delta(\xi_k)\xi_k)^{l_k} + h^{-1}\right]^{-1}\|f\| \tag{3.12}$$

for all $f \in E$. Combining those with the inequality (3.11) we obtain

$$||\Psi_t(\xi)f||_E \le C_\mu\|f\|_E \tag{3.13}$$

for all $f \in E$, h and t. The inequality (3.13) implies the estimate (3.2). □

Theorem 3.2. *Suppose the following conditions hold:*

(1) *E is a Banach space satisfying the multiplier condition with respect to p and q, where $1 < p \le q < \infty$;*
(2) *$t = (t_1, t_2, \dots, t_n)$, where t_k, $k = 1,2,\dots,n$ are nonnegative parameters $0 < t_k \le t_0 < \infty$ and $0 \le h \le h_0 < \infty$;*
(3) *$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $l = (l_1, l_2, \dots, l_n)$, where l_k are positive and α_k are nonnegative real numbers such that $\varkappa = |(\alpha + 1/p - 1/q) : l| \le 1$, and let $0 \le \mu \le 1 - \varkappa$;*
(4) *A is a R-positive operator on E.*

Then an embedding

$$D^\alpha W_p^l(R^n; E(A), E) \subset L_q(R^n; E(A^{1-\varkappa-\mu})) \tag{3.14}$$

is continuous and there exists a constant $C_\mu > 0$, depending only on μ, such that

$$\prod_{k=1}^{n} t_k^{(\alpha_k+1/p-1/q)/l_k} ||D^\alpha u||_{L_p(R^n; E(A^{1-\varkappa-\mu}))} \le C_\mu\left[h^\mu\|u\|_{W_{p,t}^l(R^n; E(A), E)} + h^{-(1-\mu)}\|u\|_{L_p(R^n; E)}\right] \tag{3.15}$$

for all $u \in W_p^l(R^n; E(A), E)$, t and h.

Proof. We have

$$
\begin{aligned}
\|D^\alpha u\|_{L_q(R^n;E(A^{1-\varkappa-\mu}))} &= \left(\int_{R^n} \|D^\alpha u\|^q_{E(A^{1-\varkappa-\mu})}dx\right)^{1/q} \\
&\backsim \left(\int_{R^n} \|A^{1-\varkappa-\mu}D^\alpha u\|^q_E\, dx\right)^{1/q} \\
&\backsim \|A^{1-\varkappa-\mu}D^\alpha u\|_{L_q(R^n;E)}
\end{aligned}
\tag{3.16}
$$

for all u such that

$$
\|D^\alpha u\|_{L_q(R^n;E(A^{1-\varkappa-\mu}))} < \infty. \tag{3.17}
$$

On the other hand we have

$$
\begin{aligned}
A^{1-\alpha-\mu}D^\alpha u = F^{-1}FA^{1-\varkappa-\mu}D^\alpha u &= F^{-1}A^{1-\varkappa-\mu}FD^\alpha u \\
= F^{-1}A^{1-\varkappa-\mu}(i\xi)^\alpha Fu &= F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}Fu.
\end{aligned}
\tag{3.18}
$$

Hence denoting Fu by $\hat{u}$, we get from relations (3.16) and (3.18)

$$
\|D^\alpha u\|_{L_q(R^n;E(A^{1-\varkappa-\mu}))} \backsim \|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}\|_{L_q(R^n;E)}. \tag{3.19}
$$

Moreover, we have

$$
\begin{aligned}
\|u\|_{W^l_{p,t}(R^n;E(A),E)} &= \|u\|_{L_p(R^n;E(A))} + \sum_{k=1}^n \|t_k D_k^{l_k} u\|_{L_p(R^n;E)} \\
&= \|F^{-1}\hat{u}\|_{L_p(R^n;E(A))} + \sum_{k=1}^n \left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{L_p(R^n;E)} \\
&\backsim \|F^{-1}A\hat{u}\|_{L_p(R^n;E)} + \sum_{k=1}^n \left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{L_p(R^n;E)}
\end{aligned}
\tag{3.20}
$$

for all $u \in W^l_p(R^n;E(A),E)$. Thus proving the inequality (3.15) for some constants C_μ is equivalent to proving

$$
\begin{aligned}
&\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}\|_{L_q(R^n,E)} \\
&\qquad \le C_\mu\left(h^\mu\|F^{-1}A\hat{u}\|_{L_p(R^n,E)} + \sum_{k=1}^n \left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{L_p(R^n,E)} + h^{-(1-\mu)}\|F^{-1}\hat{u}\|_{L_p(R^n,E)}\right)
\end{aligned}
\tag{3.21}
$$

for a suitable C_μ. Now if δ is a multiplier of the form described as in Example 2.4, by virtue of multiplier there is constants $C_k > 0$ for each $k = 1,2,\dots,n$ such that

$$
\left\|F^{-1}\frac{1}{i}\delta(\xi_k)(i\xi_k)^{l_k}\hat{u}\right\|_{L_p(R^n;E)} \le C_k\left\|F^{-1}(i\xi_k)^{l_k}\hat{u}\right\|_{L_p(R^n;E)} \tag{3.22}
$$

for all $\xi \in R^n$. Thus the inequality (3.15) will follow if we prove the following inequality

$$\begin{aligned}
&\prod_{k=1}^{n} t_k^{(\alpha_k+1/p-1/q)/l_k} \|F^{-1}[(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}]\|_{L_p(R^n;E)} \\
&\qquad \le C_\mu \left\| F^{-1}\left[h^\mu \left(A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} \right) + h^{-(1-\mu)} \right] \hat{u} \right\|_{L_p(R^n;E)}
\end{aligned} \tag{3.23}$$

for a suitable $C_\mu > 0$, and for all $u \in W_p^l(R^n; E(A), E)$.

Let us express the left-hand side of (3.23) as follows

$$\begin{aligned}
&\prod_{k=1}^{n} t_k^{(\alpha_k+1/p-1/q)/l_k} \|F^{-1}[(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}]\|_{L_q(R^n;E)} \\
&\qquad = \prod_{k=1}^{n} t_k^{(\alpha_k+1/p-1/q)/l_k} \|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu} Q^{-1}(\xi) Q(\xi)\|_{L_q(R^n;E)},
\end{aligned} \tag{3.24}$$

where

$$Q(\xi) = h^\mu \left(A + \sum_{k=1}^{n} t_k (\delta(\xi_k)\xi_k)^{l_k} \right) + h^{-(1-\mu)}. \tag{3.25}$$

(Since A is the positive operator in E so it is possible.) By virtue of definition of multiplier it is clear that the inequality (3.23) will follow immediately if we can prove that the operator-function $\Psi_{t,h,\mu} = (i\xi)^\alpha A^{1-\varkappa-\mu} Q^{-1}(\xi)$ is a multiplier in $L_p(R^n;E)$, which is uniform with respect to parameters t and h.

Firstly by using Lemma 3.1 we obtain that the operator function $\Psi_{t,h,\mu}(\xi)$ is bounded uniformly with respect to h and t. That is,

$$\|\Psi_{t,h,\mu}(\xi)\|_{B(E)} \le C. \tag{3.26}$$

By virtue of the R-positivity of operator A and by virtue of the homogenous properties of R-bounds with respect to product by scalar and the triangle inequality (see, e.g., [8, Proposition 3.4]) by using (3.26) for $0 < t_k \le T$, $0 < h \le h_0$ and $\xi \in (-\infty, \infty)$ we obtain

$$\begin{gathered}
R(\{\Psi_{t,h,\mu}(\xi) : \xi \in V_n\}) \le M, \\
R\left(\left\{\xi^{\beta+1/p-1/q} D_\xi^\beta \Psi_{t,h,\mu}(\xi) : \beta \in U_n : \xi \in V_n\right\}\right) \le M_\beta.
\end{gathered} \tag{3.27}$$

By virtue of (3.27) we obtain that the operator-valued functions $\Psi_{t,h,\mu}(\xi)$ are uniformly R-bounded multipliers with respect to t, h and R-bounds are independent of t and h. Then in view of Definition 2.1 it follows that the operator-valued function $\Psi_{t,h,\mu}(\xi)$ are uniformly bounded Fourier multipliers from $L_p(R^n;E)$ to $L_q(R^n;E)$. This completes the proof of Theorem 3.2. □

It is possible to state Theorem 3.2 in a more general setting. For this, we use the concept of extension operator.

Condition 3.3. Let be the region $\Omega \subset R^n$ such that there exists a bounded linear extension operator P acting from $L_q(\Omega;E)$ to $L_q(R^n;E)$ also from $W_p^l(\Omega;E(A),E)$ to $W_p^l(R^n;E(A),E)$, for $1 < p \le q < \infty$.

Remark 3.4. If $\Omega \subset R^n$ is a region satisfying the strong l-horn condition (see [5, page 117]) and $l = (l_1,\dots,l_n)$, l_i, $i = 1,2,\dots,n$ are nonnegative integers numbers, $E = R$, $A = I$, then there exists a bounded linear extension operator from $W_p^l(\Omega) = W_p^l(\Omega;R,R)$ to $W_p^l(R^n) = W_p^l(R^n;R,R)$.

Theorem 3.5. *Suppose all conditions of Theorem 3.2 and Condition 3.3 are hold. Then an embedding*

$$D^\alpha W_p^l(\Omega;E(A),E) \subset L_q(\Omega;E(A^{1-\varkappa-\mu})) \tag{3.28}$$

is continuous and there exists a constant C_μ depending only on μ such that

$$\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|D^\alpha u\|_{L_p(\Omega;E(A^{1-\varkappa-\mu}))} \le C_\mu\Big[h^\mu \|u\|_{W_{p,t}^l(\Omega;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(\Omega;E)}\Big] \tag{3.29}$$

for all $u \in W_p^l(\Omega;E(A),E)$, t and h.

Proof. It is suffices to prove the estimate (3.29). Let P is a bounded linear extension operator from $L_p(\Omega;E)$ to $L_p(R^n;E)$ and from $W_p^l(\Omega;E(A),E)$ to $W_p^l(R^n;E(A),E)$, and let P_Ω be the restriction operator from R^n to Ω. Then for any $u \in W_p^l(\Omega;E(A),E)$ we have

$$\begin{aligned}
&\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|D^\alpha u\|_{L_q(\Omega;E(A^{1-\varkappa-\mu}))}\\
&\quad= \prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|D^\alpha P_\Omega P u\|_{L_q(\Omega;E(A^{1-\varkappa-\mu}))}\\
&\quad\le C\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|D^\alpha P u\|_{L_q(R^n;E(A^{1-\varkappa-\mu}))}\\
&\quad\le C_\mu\Big[h^\mu \|Pu\|_{W_{p,t}^l(R^n;E(A),E)} + h^{-(1-\mu)}\|Pu\|_{L_p(R^n;E)}\Big]\\
&\quad\le C\mu\Big[h^\mu \|u\|_{W_{p,t}^l(\Omega;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_p(\Omega;E)}\Big].
\end{aligned} \tag{3.30}$$

□

Result 3.6. Let all conditions of Theorem 3.5 holds. Then for all $u \in W_p^l(\Omega;E(A),E)$ we have multiplicative estimate

$$\|D^\alpha u\|_{L_q(\Omega;E(A^{1-\varkappa-\mu}))} \le C_\mu \|u\|^{1-\mu}_{W_p^l(\Omega;E(A),E)} \|u\|^\mu_{L_p(\Omega;E)}. \tag{3.31}$$

Indeed setting $h = \|u\|_{L_p(\Omega;E)} \cdot \|u\|^{-1}_{W_p^l(\Omega;E(A),E)}$ in estimate (3.29) we obtain (3.31).

Theorem 3.7. *Assume that all conditions of Theorem 3.5 are satisfied. Let Ω be a bounded region on R^n and A^{-1} be compact operator in the space E. Then for $0 < \mu < 1 - \varkappa$ an embedding $D^\alpha W_p^l(\Omega;E(A),E) \subset L_q(\Omega;E)$ is compact.*

Proof. By virtue of [22, Theorem 1] an embedding $W_p^l(\Omega;E(A),E) \subset L_q(\Omega;E)$ is compact. Then by the estimate (3.31) we obtain the assertion of Theorem 3.7. □

THEOREM 3.8. *Suppose all conditions of theorem A_2 are satisfied.*
Then for $0 < \mu < 1 - \varkappa$ an embedding

$$D^\alpha W_p^l(\Omega;E(A),E) \subset L_p(\Omega;(E(A),E)_\varkappa) \tag{3.32}$$

is continuous and there exists a positive constant C_μ such that

$$\begin{aligned}
&\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|D^\alpha u\|_{L_p(\Omega;(E(A),E)_{\varkappa+\mu,p})} \\
&\quad \le C_\mu \left[h^\mu \left(\|Au\|_{L_p(\Omega;E)} + \sum_{k=1}^n \|t_k D_k^{l_k} u\|_{L_p(\Omega;E)} \right) + h^{-(1-\mu)} \|u\|_{L_p(\Omega;E)} \right]
\end{aligned} \tag{3.33}$$

for all $u \in W_p^l(\Omega;E(A),E)$ and $0 < h \le h_0 < \infty$.

Proof. Let at first to show the theorem for the case $\Omega = R^n$. Then it is sufficient to prove the estimate

$$\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|D^\alpha u\|_{L_q(R^n;(E(A),E)_{\varkappa+\mu,p})} \le C_\mu \left[h^\mu \|u\|_{W_p^l(R^n;E(A),E)} + h^{-(1-\mu)} \|u\|_{L_p(R^n;E)} \right] \tag{3.34}$$

for all $u \in W_p^l(R^n;E(A),E)$, t and h. By the definition of interpolation spaces $(E(A),E)_{\varkappa+\mu}$ (see [28, Section 1.14.5]) the estimate (3.34) is equivalent to the inequality

$$\begin{aligned}
&\prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} \|F^{-1} y^{1-\varkappa-\mu-1/p} [A^{\chi+\mu}(A+y)^{-1}] \xi^\alpha \hat{u}\|_{L_q(R^n;L_p(R_+;E))} \\
&\quad \le C_\mu \left\| F^{-1} \left[h^\mu \left(A + \sum_{k=1}^n t_k (\delta(\xi_k)\xi_k)^{l_k} \right) + h^{-(1-\mu)} \right] \hat{u} \right\|_{L_p(R^n;E)}.
\end{aligned} \tag{3.35}$$

By virtue of the definition of the multiplier it is clear that the inequality (3.34) will follow immediately from (3.35) if we can prove that the operator-function

$$\begin{aligned}
\Psi_{t,h,\mu} = (i\xi)^\alpha \prod_{k=1}^n t_k^{(\alpha_k+1/p-1/q)/l_k} & y^{1-\varkappa-\mu-1/p} [A^{\chi+\mu}(A+y)^{-1}] \\
&\times \left[h^\mu \left(A + \sum_{k=1}^n t_k (\delta(\xi_k)\xi_k)^{l_k} \right) + h^{-(1-\mu)} \right]^{-1}
\end{aligned} \tag{3.36}$$

is the multiplier from $L_p(R^n;E)$ to $L_q(R^n;L_p(R_+;E))$. This fact is proved by the same manner as in Theorem 3.2.

Therefore, we get the estimate (3.35). Then by using the extension operator we obtain (3.33). □

Result 3.9. Let all conditions of Theorem 3.5 holds. Then for all

$$u \in W_p^l(\Omega; E(A), E) \tag{3.37}$$

we have a multiplicative estimate

$$\|D^\alpha u\|_{L_q(\Omega;(E(A),E)_{\varkappa+\mu,p})} \le C_\mu \|u\|^{1-\mu}_{W_p^l(\Omega;E(A),E)} \|u\|^{\mu}_{L_p(\Omega;E)}. \tag{3.38}$$

Indeed setting

$$h = \|u\|_{L_p(\Omega;E)} \cdot \|u\|^{-1}_{W_p^l(\Omega;E(A),E)} \tag{3.39}$$

in the estimate (3.33) we obtain (3.38).

Theorem 3.10. *Assume that all the conditions of Theorem 3.8 are satisfied. Let Ω be a bounded region in R^n and A^{-1} is a compact operator in the space E. Then for $0 < \mu < 1 - \varkappa$, $1 < p < \infty$ an embedding*

$$D^\alpha W_p^l(\Omega; E(A), E) \subset L_q\Big(\Omega; (E(A), E)_{\varkappa+\mu,p}\Big) \tag{3.40}$$

is compact.

Proof. By virtue of [22] an embedding

$$W_p^l(\Omega; E(A), E) \subset L_p(\Omega; E) \tag{3.41}$$

is compact. Then by the estimate (3.38) we obtain the assertion of Theorem 3.10. □

Result 3.11. If $l_1 = l_2 = \cdots = l_n = l$ then we obtain continuity of embedding operators in isotropic class

$$W_p^l(\Omega, E(A)E). \tag{3.42}$$

Remark 3.12. If $E = H$ and $p = q = 2$, $\Omega = (0, T)$, $l_1 = l_2 = \cdots = l_n = l$, $A = A^\times \ge cI$ then we obtain the result of Lions-Peetre [18] and even in the one dimensional case the result of Lions-Peetre are improving for in general, non self adjoint positive operators A.

If $E = R$, $A = I$ then we obtain embedding theorems

$$D^\alpha W_p^l(\Omega) \subset L_q(\Omega) \tag{3.43}$$

proved in [5] for numerical Sobolev spaces $W_p^l(\Omega)$.

4. Applications

4.1. Embedding in vector-valued spaces. Let $s \in R$, $s > 0$. Let us consider the space [28, Section 1.18.2]

$$l_\sigma^s = \left\{ u;\ u = \{u_i\},\ i = 1,2,\ldots,\infty,\ u_i \in \mathbf{C},\ \left(\sum_{i=1}^{\infty} 2^{ips} |u_i|^\sigma\right)^{1/\sigma} < \infty \right\}, \tag{4.1}$$

with the norm

$$\|u\|_{l_\sigma^s} = \left(\sum_{i=1}^{\infty} 2^{ips} \left| u_i \right|^{\sigma}\right)^{1/\sigma} < \infty, \quad 1 < \sigma < \infty. \tag{4.2}$$

Note that $l_\sigma^0 = l_\sigma$. Let A is infinite matrix defined in the space l_σ such that $D(A) = l_\sigma^s$, $A = [\delta_{ij} 2^{si}]$, where $\delta_{ij} = 0$, when $i \neq j$, $\delta_{ij} = 1$, when $i = j$, $i, j = 1, 2, \dots, \infty$. It is clear to see that, this operator A is positive in l_p. Then by Theorem 3.5 we obtain that for $0 \le \mu \le 1 - \varkappa$, $\varkappa = \sum_{k=1}^n (\alpha_k + 1/p - 1/q)/l_k$ the embedding $D^\alpha W_p^l(\Omega, l_\sigma^s, l_\sigma) \subset L_q(\Omega, l_\sigma^{s(1-\varkappa-\mu)})$ is continuous and also an estimate of type (3.29) is satisfied.

It should be noted that the above embedding has not been obtained with classical method until now.

4.2. Maximal regular differential-operator equations. Let us consider differential-operator equations

$$Lu = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u + A_\lambda u + \sum_{|\alpha:2l|<1} \prod_{k=1}^{n} t_k^{\alpha_k/2l_k} A_\alpha(x) D^\alpha u = f \tag{4.3}$$

in the space $L_p(R^n; E)$, where, $A_\lambda = A - \lambda I$, A and $A_\alpha(x)$ are in general, unbounded operators in Banach space E, t_k, $k = 1, 2, \dots, n$ parameters, $l = (l_1, l_2, \dots, l_n)$, l_i-positive integers.

Theorem 4.1. *Suppose the following conditions hold:*

(1) $0 < t_k \le t_0 < \infty$, $k = 1, 2, \dots, n$, $0 < \varphi \le \pi$;
(2) *E is a Banach space satisfying multiplier condition with respect to p,* $1 < p < \infty$;
(3) *A is a R-positive operator in E and*

$$A_\alpha(x) A^{-(1-|\alpha:2l|-\mu)} \in L_\infty\left(R^n, L(E)\right), \quad 0 < \mu < 1 - |\alpha : 2l|. \tag{4.4}$$

Then for all $f \in L_p(R^n; E)$ *and for sufficiently large* $|\lambda| > 0$, $\lambda \in S(\varphi)$ *(4.3) has a unique solution* $u(x)$ *that belongs to space* $W_p^{2l}(R^n; E(A), E)$ *and the estimate hold*

$$\sum_{k=1}^{n} t_k \left|\left| D_k^{2l_k} u \right|\right|_{L_p(R^n;E)} + \|Au\|_{L_p(R^n;E)} \le C \|f\|_{L_p(R^n;E)}. \tag{4.5}$$

Proof. At first we will consider principal part of (4.3), that is, differential-operator equation

$$L_0 u = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u + A_\lambda u = f. \tag{4.6}$$

Then we apply Fourier transform to (4.6) with respect to $x = (x_1, \dots, x_n)$ and obtain

$$\sum_{k=1}^{n} t_k \xi_k^{2l_k} \hat{u}(\xi) + A_\lambda \hat{u}(\xi) = \hat{f}(\xi). \tag{4.7}$$

In view of (1) condition, $\sum_{k=1}^{n} t_k\xi_k^{2l_k} \geq 0$ for all $\xi = (\xi_1,\ldots,\xi_n) \in R^n$. Therefore, $\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k} \in S(\pi)$ for all $\xi \in R^n$. That is, an operator $A - [\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k}]I$ is invertible in E. Hence from (4.7) we obtain that the solution of (4.6) can be represented in the form

$$u(x) = F^{-1}\left[A - \left(\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k}\right)I\right]^{-1} \hat{f}. \tag{4.8}$$

It is clear to see that operator-function $\varphi_{\lambda,t}(\xi) = [A - (\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k})I]^{-1}$ is multiplier in $L_p(R^n;E)$ uniformly to λ and t. Actually, by virtue of φ-positiveness of operator A for all $\xi \in R^n$ and $\lambda \in S(\varphi)$ we get

$$\|\varphi_\lambda(\xi)\|_{L(E)} \leq M\left(1 + \left|\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k}\right|\right)^{-1} \leq M_0. \tag{4.9}$$

Moreover, since $D_k\varphi_{\lambda,t}(\xi) = [A - (\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k})]^{-2}2l_kt_k\xi_k^{2l_k-1}$ then

$$\begin{aligned}\|\xi_kD_k\varphi_{\lambda,t}\|_{L(E)} &\leq 2l_kt_k\xi_k^{2l_k}\left\|\left[A - \left(\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k}\right)\right]^{-2}\right\| \\ &\leq 2l_kt_k\xi_k^{2l_k}\left(1 + \left|\lambda - \sum_{k=1}^{n} t_k\xi_k^{2l_k}\right|\right)^{-2} \leq M.\end{aligned} \tag{4.10}$$

Using the estimate (4.10) and the R-positiveness of operator A for operator-functions $\varphi_{k\lambda,t}(\xi) = \xi_k^{2l_k}\varphi_{\lambda,t}$, $k = 1,2,\ldots,n$ and $\varphi_{0\lambda,t} = A\varphi_{\lambda,t}$ we have

$$\begin{aligned}&R(\xi^\beta\varphi_{k,\lambda,t}(\xi),\ \beta \in U_n : \xi \in V_n) \leq C_\beta, \\ &R(\xi^\beta\varphi_{0,\lambda,t}(\xi),\ \beta \in U_n : \xi \in V_n) \leq M_\beta.\end{aligned} \tag{4.11}$$

Then by virtue of condition (2) and estimates (4.11) we obtain that operator-functions $\varphi_{\lambda,t}$, $\varphi_{k\lambda,t}$, $\varphi_{0\lambda,t}$ are multiplier in the space $L_p(R^n;E)$. By using the representation of (4.8) we have

$$\begin{aligned}\|D_k^{2l_k}u\|_{L_p} &= \left\|F^{-1}(i\xi_k)^{2l_k}\varphi_{\lambda,t}(\xi)\hat{f}\right\|_{L_p}, \\ \|Au\|_{L_p} &= \|F^{-1}A\hat{u}\|_{L_p} = \|F^{-1}A\varphi_{\lambda,t}(\xi)\hat{f}\|_{L_p}.\end{aligned} \tag{4.12}$$

By the definition of multiplier we obtain that for all $f \in L_p(R^n;E)$ there is unique solution of (4.6) in the form (4.8) and holds estimate

$$\sum_{k=1}^{n} t_k\|D_k^{2l_k}u\|_{L_p} + \|Au\|_{L_p} \leq C\|f\|_{L_p}. \tag{4.13}$$

In the space $L_p(R^n;E)$, we consider the differential operator $L_0 - \lambda$ that is generated by the problem (4.6), that is

$$D(L_0 - \lambda) = W_p^{2l}(R^n,E(A),E), \qquad (L_0 - \lambda)u = \sum_{k=1}^{n}(-1)^{l_k}t_kD_k^{2l_k}u + A_\lambda u. \tag{4.14}$$

The estimate (4.13) implies that the operator $L_0-\lambda$ has a bounded inverse acting from $L_p(R^n;E)$ into $W_p^{2l}(R^n;E(A),E)$. We denote by $L-\lambda$ the differential operator in $L_p(R^n;E)$ that is generated by the problem (4.3). Namely,

$$D(L-\lambda)=W_p^{2l}(R^n,E(A),E), \qquad (L-\lambda)u=(L_0-\lambda)u+L_1u, \tag{4.15}$$

where

$$L_1u=\sum_{|\alpha:2l|<1}\prod_{k=1}^{n}t_k^{\alpha_k/2l_k}A_\alpha(x)D^\alpha u. \tag{4.16}$$

In view of condition (3) and by virtue of Theorem 3.5 for $u\in W_p^{2l}(R^n;E(A),E)$ we have

$$\begin{aligned}\|L_1u\|_{L_p}&\le\sum_{|\alpha:2l|<1}\prod_{k=1}^{n}t_k^{\alpha_k/2l_k}\|A_\alpha(x)D^\alpha u\|_{L_p}\\&\le\sum_{|\alpha:2l|<1}\prod_{k=1}^{n}t_k^{\alpha_k/2l_k}\|A^{1-|\alpha:2l|-\mu}D^\alpha u\|_{L_p}\\&\le C\left[h^\mu\left(\sum_{k=1}^{n}t_k\|D_k^{2l_k}u\|_{L_p}+\|Au\|_{L_p}\right)+h^{-(1-\mu)}\|u\|_{L_p}\right].\end{aligned}\tag{4.17}$$

Then from estimates (4.13) and (4.17) for $u\in W_p^{2l}(R^n;E(A),E)$ we obtain

$$\|L_1u\|_{L_p}\le C\left[h^\mu\|(L_0-\lambda)u\|_{L_p}+h^{-(1-\mu)}\|u\|_{L_p}\right]. \tag{4.18}$$

Since $\|u\|_{L_p}=(1/\lambda)\|(L_0-\lambda)u+L_0u\|_{L_p}$ for $u\in W_p^{2l}(R^n;E(A),E)$ we get

$$\begin{aligned}\|u\|_{L_p}&\le\frac{1}{|\lambda|}\|(L_0-\lambda)u\|_{L_p}+\|L_0u\|_{L_p}\\&\le\frac{1}{|\lambda|}\|(L_0-\lambda)u\|_{L_p}+\frac{1}{|\lambda|}\left[\sum_{k=1}^{n}t_k\|D_k^{2l_k}u\|_{L_p}+\|Au\|_{L_p}\right].\end{aligned}\tag{4.19}$$

From estimates (4.13) and (4.17), (4.18), and (4.19) for $u\in W_p^{2l}(R^n;E(A),E)$ obtain

$$\|L_1u\|_{L_p}\le Ch^\mu\|(L_0-\lambda)u\|+C_1|\lambda|^{-1}h^{-(1-\mu)}\|(L_0-\lambda)u\|. \tag{4.20}$$

Then choosing h and λ such that $Ch^\mu<1$, $C_1|\lambda|^{-1}h^{-(1-\mu)}<1$ from (4.20) obtain

$$\left\|L_1(L_0-\lambda)^{-1}\right\|_{L(E)}<1. \tag{4.21}$$

Using relation (4.15), estimates (4.13) and (4.21) and perturbation theory of linear operators [14], we establish that the differential operator $L-\lambda$ is invertible from $L_p(R^n;E)$ into $W_p^{2l}(R^n;E(A),E)$. This implies the estimate (4.5). $\square$

Remark 4.2. There are a lot of positive operators in the different concrete Banach spaces. Therefore, putting instead of E, concrete Banach spaces and instead of operator A, concrete R-positive differential, pseudo differential operators, or finite, infinite matrices on the differential-operator equations (4.3), by virtue of Theorem 4.1 we can obtain the different class of maximal regular partial differential equations or system of equations.

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