SURFACE INTEGRALS AND HARMONIC FUNCTIONS

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Using the notion of inferior mean due to M. Heins, we establish two inequalities for such a mean relative to a positive harmonic function defined on the open unit ball or half-space in $\mathbb{R}^{n+1}$.

1. Introduction

In connection with E_P spaces, M. Heins proved the following PL-Lemma (unpublished).

Lemma 1.1 (PL-Lemma). *If u is a positive function on the annulus $\{R < |z| < 1\}$ with a subharmonic logarithm, and γ are rectifiable Jordan curves in $\{r < |z| < 1\}$ separating 0 from ∞, then*

$$\lim_{r\to 1}\inf_{\gamma}\int_{\gamma} u(z)|dz| = \lim_{r\to 1}\int_{|z|=r} u(z)|dz|. \tag{1.1}$$

Wu showed in [4] that for a positive harmonic function in the unit disc, one has in most cases inequality, while equality occurs for functions whose boundary measures are absolutely continuous. She also showed that there exists a nonzero lower bound of the lim inf for this class of functions in the disc. The bound is achieved for functions whose boundary measures, for example, are purely singular. We generalize these results to higher dimensions.

Let Ω be the open unit ball or upper half-space in $\mathbb{R}^{n+1}$ and let S denote its boundary. Let u be a positive harmonic function on Ω, which, by Riesz's theorem, is given by a Borel measure μ with the total measure $\|\mu\|$ on S.

Definition 1.2. Let Γ be a piecewise C^1-smooth hypersurface in $A_\delta = \{q \in \Omega : d(q,S) < \delta\}$ separating the two boundaries of A_δ. The *inferior mean* of u is defined by

$$IM(u) = \lim_{\delta\to 0}\inf_{\Gamma\subset A_\delta}\int_{\Gamma} u(q)d\Gamma. \tag{1.2}$$

Let ω_n be the volume of the unit sphere in $\mathbb{R}^{n+1}$, and let $M_n = \omega_{n+1}/\pi\omega_n$. In this paper, we establish the following theorem.

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Theorem 1.3. *For any positive harmonic function u on Ω with boundary measure μ, there exists the following inequality:*

$$IM(u) \le \|\mu\|. \tag{1.3}$$

Equality occurs for those u whose boundary measures μ are absolutely continuous, when the inferior mean is attained along boundaries of A_δ not equal to S as $\delta \to 0$.

Theorem 1.4. *For any positive harmonic function u on Ω, with boundary measure μ, there exists the following inequality:*

$$IM(u) \ge M_n \|\mu\|. \tag{1.4}$$

Equality occurs for u with point-mass boundary measures μ concentrated at p_0, when $IM(u)$ is attained along the boundary of the set

$$\tilde{\Omega} = \{q \in \Omega : d(q,S) < \sigma^2\} \setminus \{q \in \Omega : |q - p_0| < \sigma\} \quad \textit{as } \sigma \to 0. \tag{1.5}$$

The proofs rely on Sard's theorem (see [3]), and inequality (2.5) obtained below.

2. A surface measure lemma

Given spherical angles $\phi_i \in [0,\pi]$, $i < n$, $\phi_n \in [0,2\pi]$, we include $\phi_0 = \pi/2$ and $\phi_{n+1} = 0$. For a point $q \in \mathbb{R}^{n+1}$, the relation between its Cartesian $(x_1,\dots,x_{n+1})$ and spherical $(r,\phi_1,\dots,\phi_n)$ coordinates is given by

$$x_j = X_j \cos\phi_j, \quad X_j = r\prod_{i=0}^{j-1}\sin\phi_i, \ r = |q|. \tag{2.1}$$

From [2, Section 676], we know that on a sphere $r = \text{const}$, the Jacobian of this relation satisfies

$$I_n = \frac{D(x_1,\dots,x_{n+1})}{D(r,\phi_1,\dots,\phi_n)} = X_n I_{n-1} = \cdots = \prod_{k=1}^{n} X_k. \tag{2.2}$$

If S^n is the unit sphere in $\mathbb{R}^{n+1}$ and dS^n is its volume element, then the volume element on $r = \text{const}$ equals

$$r^n dS^n = I_n d\phi_1 \cdots d\phi_n. \tag{2.3}$$

We take $r = 1$ in (2.2) and (2.3) to compute the constant

$$M_n = \int_{S^n\cap\{0<\phi_1<\pi/2\}} \frac{2\cos\phi_1}{\omega_n}\, dS^n = \frac{2\omega_{n-1}}{n\omega_n} = \frac{\omega_{n+1}}{\pi\omega_n}. \tag{2.4}$$

When a hypersurface Γ is given by $r = r(\phi_1,\dots,\phi_n)$, then its volume element satisfies

$$d\Gamma \ge r^n dS^n. \tag{2.5}$$

A nongeometric proof of (2.5) follows from the following lemma.

Lemma 2.1. *If Γ is locally given by $r = r(\phi_1,\dots,\phi_n)$, then*

$$d\Gamma = \sqrt{1+\sum_{k=1}^{n}\frac{r_{\phi_k}^2}{X_k^2}}\,r^n dS^n. \tag{2.6}$$

Proof. Assume that Γ is also given by $x_{n+1} = f(x_1,\dots,x_n)$. We know that $d\Gamma \geq d\mathbb{R}^n$, because in this case,

$$d\Gamma = \sqrt{1+|\operatorname{grad} f|^2}\,dx_1\cdots dx_n. \tag{2.7}$$

We differentiate $x_{n+1} = f$ with respect to $\phi_1,\dots,\phi_n$, solve the system by Cramer's rule for $\partial f/\partial x_i$, and substitute the result into (2.7), thus obtaining

$$d\Gamma = \sqrt{\sum_{i=1}^{n+1} J_i^2(n+1)}\,d\phi_1\cdots d\phi_n \quad \textit{with } J_i(n+1) = \frac{D(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+1})}{D(\phi_1,\dots,\phi_n)}. \tag{2.8}$$

Next, we show by induction on m that

$$\sum_{i=1}^{m} J_i^2(m) = I_{m-1}^2\left(1+\sum_{k=1}^{m-1}\frac{r_{\phi_k}^2}{X_k^2}\right), \quad m = 1,2,\dots, \tag{2.9}$$

which is known to be true for $m = 1,2,3$. Assume that it is also true for $m = 4,\dots,n$. For Jacobians J_i, $i < n$, we obtain a recurrence relation using the product rule

$$\begin{aligned} J_i(n+1) &= \frac{D(\dots,\cos\phi_n X_n,\sin\phi_n X_n)}{D(\phi_1,\dots,\phi_n)} \\ &= 0+0+X_n\cos^2\phi_n J_i(n)+X_n\sin^2\phi_n J_i(n) = X_n J_i(n). \end{aligned} \tag{2.10}$$

In order to obtain a recurrence relation for $J_n^2 + J_{n+1}^2$, we use likewise the product rule in J_n, J_{n+1}. We also apply the chain rule to

$$\frac{D(x_1,\dots,x_{n-1},X_n)}{D(\phi_1,\dots,\phi_n)} = \frac{D(x_1,\dots,x_{n-1},X_n)}{D(r,\phi_1,\dots,\phi_{n-1})}\,\frac{D(r,\phi_1,\dots,\phi_{n-1})}{D(\phi_1,\dots,\phi_n)}, \tag{2.11}$$

noting that this Jacobian depends on ϕ_n only implicitly through the equation for r. Then

$$J_n^2(n+1)+J_{n+1}^2(n+1) = X_n^2 J_n^2(n)+I_{n-1}^2 r_{\phi_n}^2. \tag{2.12}$$

Applying (2.2) and the induction assumption to the sum with $m = n$, we obtain

$$\sum_{i=1}^{n+1} J_i^2(n+1) = X_n^2\sum_{i=1}^{n} J_i^2(n)+I_{n-1}^2 r_{\phi_n}^2 = I_n^2\left(1+\sum_{k=1}^{n-1}\frac{r_{\phi_k}^2}{X_k^2}\right)+\frac{I_n^2 r_{\phi_n}^2}{X_n^2}. \tag{2.13}$$

The asserted equality for $d\Gamma$ is an immediate consequence of this and (2.3). $\square$

3. Poisson kernel

We recall that a positive harmonic function u on Ω has a representation via the Poisson-Stieltjes integral $u(q) = \int_S P(q,p)d\mu(p)$. We write the kernel in the usual half-space coordinates $q = (y,s)$, with $s \in S$ so that $y = \text{dist}(q,S) = \text{dist}(q,s)$. We have (cf. [1, pages 12, 127])

$$P(q,p) = \frac{2y - \kappa y^2}{\omega_n|q-p|^{n+1}} = \frac{2y-\kappa y^2}{\omega_n\left[y^2 + |s-p|^2(1-\kappa y)\right]^{(n+1)/2}}, \quad \kappa = \begin{cases} 0 & \text{for half-space,} \\ 1 & \text{for ball.} \end{cases} \tag{3.1}$$

By direct integration of P over $\Gamma_\delta = \partial A_\delta \neq S$ in the half-space for all positive δ, or by the mean value property of P in the unit ball as a harmonic function of q for $\delta < 1$, we obtain

$$\int_{\Gamma_\delta} P(q,p)d\Gamma_\delta = (1-\kappa\delta)^n. \tag{3.2}$$

4. Proof of Theorem 1.3

The upper bound of IM follows from (3.2) right away:

$$IM(u) \le \lim_{\delta\to 0}\int_{\Gamma_\delta} u\,d\Gamma_\delta = \lim_{\delta\to 0}\int_S\int_{\Gamma_\delta} P(q,p)d\Gamma_\delta\,d\mu(p) = \|\mu\|. \tag{4.1}$$

To prove equality, let u have absolutely continuous boundary measure μ. Sard's theorem and (2.5) allow us to use, just as in [4], the existence of nonzero u^* to show that $IM \ge \|\mu\|$.

Let Γ_j be a C^1-smooth hypersurface separating boundaries of $A_{1/j}$, $j = 3,4,\ldots$. Consider

$$\Gamma'_j = \{q \in \Gamma_j : y = y_j(s) \text{ is defined in some neighborhood of } q\}. \tag{4.2}$$

By Sard's theorem, the image S'_j of Γ'_j under the map $q \to s$ has full measure in S. For each point s, we choose a preimage on Γ'_j nearest to S, and denote this subset of Γ'_j by Γ''_j. It has the same image in S as Γ'_j, moreover, Γ''_j and S'_j are the coordinate charts related via the map $q = (y,s) \to s$. From (2.5) and (2.7), we have $d\Gamma_j \ge (1-\kappa y_j)^n dS$. Thus,

$$\int_{\Gamma_j} u\,d\Gamma_j \ge \int_{\Gamma''_j} u(q)d\Gamma_j \ge \int_{S'_j} u(y_j,s)\left(1-\kappa y_j\right)^n dS = \int_S u(y_j,s)\left(1-\kappa y_j\right)^n dS. \tag{4.3}$$

Then Fatou's lemma and the existence of the nontangential limit of u a.e. yield

$$\lim_{j\to\infty}\int_{\Gamma_j} u\,d\Gamma_j \ge \int_S \liminf_{j\to\infty} u(y_j,s)\left(1-\kappa y_j\right)^n dS = \int_S u^*(s)dS = \|\mu\|. \tag{4.4}$$

5. Proof of Theorem 1.4

We use local spherical coordinates with the origins at $p \in S$ and the x_1-axis orthogonal to S. Thus, $0 \leq \phi_1 < \pi/2$, and the Poisson kernel

$$P(q,p) = \frac{2y - \kappa y^2}{\omega_n |q-p|^{n+1}} = \frac{2\cos\phi_1 - \kappa r}{\omega_n r^n}. \tag{5.1}$$

For $\delta \in (0,1)$, let Γ be a C^1-smooth hypersurface in A_δ separating boundaries of A_δ. We may assume that every $q \in \Gamma$ has a neighborhood in which $r = r(\phi_1,\dots,\phi_n)$ is defined (see the argument using Sard's theorem in the proof of Theorem 1.3). Fubini's theorem, (2.4), (2.5), and plane geometry yield

$$\begin{aligned}\int_\Gamma u\,d\Gamma &= \int_S\int_\Gamma \frac{2\cos\phi_1 - \kappa r}{\omega_n r^n}\,d\Gamma\,d\mu(p)\\ &> \int_S\int_{\{r<\sqrt{2\delta-\delta^2}<\cos\phi_1\}} \frac{2\cos\phi_1 - \kappa r}{\omega_n\, r^n} r^n\,dS^n\,d\mu(p)\\ &= \|\mu\| M_n\left[(1-\delta)^n - \frac{\kappa\sqrt{2\delta-\delta^2}}{M_n}\right].\end{aligned} \tag{5.2}$$

We obtain the lower bound for IM when $\delta \to 0$.

To prove equality, assume that u has the boundary measure μ that is concentrated at point $p_0 \in S$. Then,

$$u(q) = P(q,p_0)\mu(p_0), \quad \mu(p_0) = \|\mu\|. \tag{5.3}$$

Let $\sigma \in (0,\delta)$. Note that the boundary of $\tilde{\Omega} = \{d(q,S) < \sigma^2\} \setminus \{|q-p_0| < \sigma\}$ is formed by two hypersurfaces: Γ_1 consisting of points q on a sphere $|q-p_0| = \sigma$ with the distance y to S larger than σ^2; and Γ_2 consisting of points $q = (y,s)$ on a level hypersurface $y = \sigma^2$ with $|q-p_0| \geq \sigma$.

On Γ_1, we use spherical coordinates with the origin at p_0. We see from (5.1) and (5.3) that

$$u(q) = \frac{2\cos\phi_1 - \kappa\sigma}{\omega_n\sigma^n}\|\mu\|, \tag{5.4}$$

and from (2.3) that $d\Gamma_1 = \sigma^n dS^n$. We use these two facts and (2.4) to estimate the integral of u over Γ_1 as follows:

$$\int_{\Gamma_1} u\,d\Gamma_1 < \|\mu\| \int_{S^n\cap\{0<\phi_1<\pi/2\}} \frac{2\cos\phi_1}{\omega_n\sigma^n}\sigma^n\,dS^n = M_n\|\mu\|. \tag{5.5}$$

Once we show that

$$\int_{\Gamma_2} u\,d\Gamma_2 = O(\sigma), \tag{5.6}$$

and allow $\delta \to 0$, the proof will be complete, since $\sigma \in (0,\delta)$.

Let α be the distance from p_0 to s along a geodesic in S. Then

$$\alpha \geq |s - p_0| \geq \frac{2}{\pi}\alpha, \tag{5.7}$$

and on Γ_2, $|s - p_0| \geq \sigma' = \sqrt{\sigma^2 - \sigma^4}$, where our coordinates are $(y, s) = (\sigma^2, s)$. Also equality (2.3) implies that $d\Gamma_2 = (1 - \kappa\sigma^2)^n dS$. Hence, it follows that

$$\begin{aligned}
\int_{\Gamma_2} u\, d\Gamma_2 &= \frac{\|\mu\|}{\omega_n} \int_{|s-p_0| \geq \sigma'} \frac{2\sigma^2 - \kappa\sigma^4}{\left[\sigma^4 + |s - p_0|^2 (1 - \kappa\sigma^2)\right]^{(n+1)/2}} (1 - \kappa\sigma^2)^n dS \\
&< \frac{\|\mu\|}{\omega_n} \int_{|s-p_0| \geq \sigma'} \frac{2\sigma^2}{|s - p_0|^{n+1}} dS \\
&< \frac{\|\mu\|}{\omega_n} 2\sigma^2 \omega_{n-1} \int_{\sigma'}^{\infty} \frac{\alpha^{n-1} d\alpha}{((2/\pi)\alpha)^{n+1}} = O(\sigma).
\end{aligned} \tag{5.8}$$

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