WEIGHTED INTEGRALS OF HOLOMORPHIC FUNCTIONS IN THE UNIT POLYDISC

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Let f be a measurable function defined on the unit polydisc U^n in $\mathbf{C}^n$ and let $\omega_j(z_j)$, $j = 1,\dots,n$, be admissible weights on the unit disk U, with distortion functions $\psi_j(z_j)$, $\mathcal{L}^{p,q}_{\vec{\omega},N}(U^n) = \{f \mid \|f\|_{\mathcal{L}^{p,q}_{\vec{\omega},N}} < \infty\}$, where $\|f\|^q_{\mathcal{L}^{p,q}_{\vec{\omega},N}} = \int_{[0,1)^n} M^q_p(f,r)\prod_{j=1}^n \omega_j(r_j)dr_j$, and $\mathcal{A}^{p,q}_{\vec{\omega},N}(U^n) = \mathcal{L}^{p,q}_{\vec{\omega},N}(U^n) \cap H(U^n)$. We prove the following result: if $p,q \in [1,\infty)$ and for all $j = 1,\dots,n$, $\psi_j(z_j)(\partial f/\partial z_j)(z) \in \mathcal{L}^{p,q}_{\vec{\omega},N}$, then $f \in \mathcal{A}^{p,q}_{\vec{\omega},N}$ and there is a positive constant $C = C(p,q,\omega_j,n)$ such that $\|f\|_{\mathcal{A}^{p,q}_{\vec{\omega},N}} \le C(|f(0)| + \sum_{j=1}^n \|\psi_j(\partial f/\partial z_j)\|_{\mathcal{L}^{p,q}_{\vec{\omega},N}})$.

1. Introduction

Let $U^1 = U$ be the unit disk in the complex plane, $dm(z) = (1/\pi)dr\,d\theta$ the normalized Lebesgue measure on U, U^n the unit polydisc in complex vector space $\mathbf{C}^n$ and $H(U^n)$ the space of all analytic functions on U^n. For $z,w \in \mathbf{C}^n$ we write $z \cdot w = (z_1w_1,\dots,z_nw_n)$; $e^{i\theta}$ is an abbreviation for $(e^{i\theta_1},\dots,e^{i\theta_n})$; $dt = dt_1\cdots dt_n$; $d\theta = d\theta_1\cdots d\theta_n$ and r, θ are vectors in $\mathbf{C}^n$. If we write $0 \le r < 1$, where $r = (r_1,\dots,r_n)$ it means $0 \le r_j < 1$ for $j = 1,\dots,n$.

For $f \in H(U^n)$ and $p \in (0,\infty)$ we usually write

$$M_p(f,r) = \left(\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} |f(r\cdot e^{i\theta})|^p d\theta\right)^{1/p}, \quad \text{for } 0 \le r < 1 \tag{1.1}$$

for the integral means of f.

Let $\omega(s)$, $0 \le s < 1$, be a weight function which is positive and integrable on $(0,1)$. We extend ω on U by setting $\omega(z) = \omega(|z|)$. We may assume that our weights are normalized so that $\int_0^1 \omega(s)ds = 1$.

Let $\mathcal{L}^p_{\vec{\omega}} = \mathcal{L}^p_{\vec{\omega}}(U^n)$ denotes the class of all measurable functions defined on U^n such that

$$\|f\|^p_{\mathcal{L}^p_{\vec{\omega}}} = \int_{U^n} |f(z)|^p \prod_{j=1}^n \omega_j(z_j)dm(z_j) < \infty, \tag{1.2}$$

where $\omega_j(z_j)$, $j = 1,\dots,n$, are admissible weights on the unit disk U.

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The weighted Bergman space $\mathcal{A}^p_{\vec{\omega}}$ is the intersection of $\mathcal{L}^p_{\vec{\omega}}$ and $H(U^n)$. For $\omega_j(z_j) = (1-|z_j|^2)^{\alpha_j}$, $\alpha_j > -1$, $j = 1,\dots,n$, we obtain the classical Bergman space $\mathcal{A}^p(dV_{\vec{\alpha}})$, see [1, page 33].

Let $\mathcal{L}^{p,q}_{\vec{\omega},N} = \mathcal{L}^{p,q}_{\vec{\omega},N}(U^n)$, $p,q > 0$, denotes the class of all measurable functions defined on U^n such that

$$\|f\|^q_{\mathcal{L}^{p,q}_{\vec{\omega},N}} = \int_{[0,1)^n} M^q_p(f,r)\prod_{j=1}^{n}\omega_j(r_j)\,dr_j < \infty, \tag{1.3}$$

and $\mathcal{A}^{p,q}_{\vec{\omega},N}$ be the intersection of $\mathcal{L}^{p,q}_{\vec{\omega},N}$ and $H(U^n)$. When $p = q$ we denote $\mathcal{A}^{p,q}_{\vec{\omega},N}$ by $\mathcal{A}^p_{\vec{\omega},N}$.

In the case $p = q$, these two norms are equivalent on the space $H(U^n)$, but the later one is more suitable for calculations than the first one. The result is contained in the following lemma.

Lemma 1.1. *The norms $\|\cdot\|_{\mathcal{A}^p_{\vec{\omega}}}$ and $\|\cdot\|_{\mathcal{A}^p_{\vec{\omega},N}}$ are equivalent on the space $H(U^n)$.*

Proof. By the polar coordinates it is easy to see that $\|f\|_{\mathcal{A}^p_{\vec{\omega}}} \le 2^n\|f\|_{\mathcal{A}^p_{\vec{\omega},N}}$ for every $f \in H(U^n)$, moreover $\|f\|_{\mathcal{L}^p_{\vec{\omega}}} \le 2^n\|f\|_{\mathcal{L}^p_{\vec{\omega},N}}$ for every f measurable on U^n.

Now we prove that there is a positive constant C, which is independent of f, such that

$$\|f\|_{\mathcal{A}^p_{\vec{\omega},N}} \le C\|f\|_{\mathcal{A}^p_{\vec{\omega}}}, \tag{1.4}$$

for every $f \in H(U^n)$. Without loss of generality we may assume that $n = 2$. Let $f \in H(U^n)$, then

$$\|f\|^p_{\mathcal{A}^p_{\vec{\omega},N}} = \int_0^{1/2}\int_0^{1/2} + \int_0^{1/2}\int_{1/2}^{1} + \int_{1/2}^{1}\int_0^{1/2} + \int_{1/2}^{1}\int_{1/2}^{1} g(r_1,r_2)\,dr_1\,dr_2, \tag{1.5}$$

where $g(r_1,r_2) = M^p_p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)$. Now we estimate these four integrals, which we denote by I_i, $i = 1,2,3,4$.

Since $f \in H(U^2)$ then the function f is analytic in each variable separately on U and consequently $M^p_p(f,r_1,r_2)$ is nondecreasing function in r_1 and r_2, see, for example, [3]. Let

$$C_{\omega_i} = \int_0^{1/2}\omega_i(r_i)\Big/\int_{1/2}^{1}\omega_i(r_i), \quad i = 1,2. \tag{1.6}$$

Note that C_{ω_i}, $i = 1,2$, are well defined and finite numbers since ω_i are positive integrable functions on $(0,1)$.

Using the above mentioned facts and definitions, we have

$$\begin{aligned}
I_1 &\le M_p^p(f,1/2,1/2)\int_0^{1/2}\int_0^{1/2}\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2 \\
&= C_{\omega_1}C_{\omega_2}M_p^p(f,1/2,1/2)\int_{1/2}^{1}\int_{1/2}^{1}\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2 \\
&\le C_{\omega_1}C_{\omega_2}\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2 \\
&\le 4C_{\omega_1}C_{\omega_2}\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2\,dr_1\,dr_2,
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
I_2 &\le \int_{1/2}^{1}M_p^p(f,1/2,r_2)\int_0^{1/2}\omega_1(r_1)\,dr_1\,\omega_2(r_2)\,dr_2 \\
&= C_{\omega_1}\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,1/2,r_2)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2 \\
&\le C_{\omega_1}\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2 \\
&\le 4C_{\omega_1}\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2\,dr_1\,dr_2.
\end{aligned} \tag{1.8}$$

Similarly

$$I_3 \le 4C_{\omega_2}\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2\,dr_1\,dr_2. \tag{1.9}$$

Finally, it is clear that

$$I_4 \le 4\int_{1/2}^{1}\int_{1/2}^{1}M_p^p(f,r_1,r_2)\omega_1(r_1)\omega_2(r_2)r_1r_2\,dr_1\,dr_2. \tag{1.10}$$

From (1.7)–(1.10), we obtain

$$\|f\|^p_{\mathcal{A}^p_{\vec{\omega},N}} \le (C_{\omega_1}+1)(C_{\omega_2}+1)\|f\|^p_{\mathcal{A}^p_{\vec{\omega}}}, \tag{1.11}$$

as desired.

Following [8], for a given weight ω we define the function

$$\psi(r)=\psi_\omega(r)\overset{\text{def}}{=}\frac{1}{\omega(r)}\int_r^1\omega(u)\,du, \quad 0\le r<1, \tag{1.12}$$

and we call it the *distortion function* of ω. We put $\psi(z)=\psi(|z|)$ for $z\in U$. □

Definition 1.2 [8]. We say that a weight ω is *admissible* if it satisfies the following conditions:

(i) there is a positive constant $A=A(\omega)$ such that

$$\omega(r)\ge\frac{A}{1-r}\int_r^1\omega(u)\,du, \quad \text{for } 0\le r<1; \tag{1.13}$$

(ii) ω is differentiable and there is a positive constant $B = B(\omega)$ such that

$$\omega'(r) \le \frac{B}{1-r}\omega(r), \quad \text{for } 0 \le r < 1; \tag{1.14}$$

(iii) for each sufficiently small positive δ there is a positive constant $C = C(\delta,\omega)$ such that

$$\sup_{0\le r<1} \frac{\omega(r)}{\omega(r+\delta\psi(r))} \le C. \tag{1.15}$$

Observe that (i) implies $A\psi(r) \le 1-r$ thus for sufficiently small positive δ we have $r+\delta\psi(r) < 1$ and the quantity in the denominator of the fraction in (iii) is well defined.

For a list of examples of admissible weights, see [8, pages 660–663]. The following theorem was proved in [8].

Theorem 1.3. *Suppose $1 \le p < \infty$ and ω is an admissible weight with distortion function ψ. Then*

$$\int_U |f(z)|^p \omega(z)dm(z) \asymp |f(0)|^p + \int_U |f'(z)|^p \psi(z)^p \omega(z)dm(z), \tag{1.16}$$

for all analytic functions f on the unit disc U, where $dm(z) = r\,dr\,d\theta/\pi$ denotes the normalized Lebesgue area measure on U.

The above means that there are finite positive constants C and C' independent of f such that the left- and right-hand sides $L(f)$ and $R(f)$ satisfy

$$CR(f) \le L(f) \le C'R(f) \tag{1.17}$$

for all analytic f.

Some generalizations of Theorem 1.3 in many directions can be found in [10, 11]. In [5, 6, 9] was also investigated Bergman space of analytic functions with weights other than classical. Closely related results for the classical weight $\omega(r) = (1-r)^\alpha$, $\alpha > -1$, are presented in [1, 2, 3, 4, 7, 13].

Using a Bergman type projection $\mathbf{B}_\alpha : \mathcal{L}^p(dV_{\vec{\alpha}}) \to \mathcal{A}^p(dV_{\vec{\alpha}})$, in [1] the authors proved the following theorem.

Theorem 1.4. *Let $p \in [1,\infty)$, $\alpha_j > -1$, $j = 1,\dots,n$ and m be a fixed positive integer and let $\mathbf{k} = (k_1,\dots,k_n) \in (\mathbf{Z}_+)^n$. Let f be a holomorphic function defined on the polydisc U^n in $\mathbf{C}^n$. Then for $\vec{\alpha} = (\alpha_1,\dots,\alpha_n)$, $f \in \mathcal{A}^p(dV_{\vec{\alpha}})$ if and only if*

$$\left[\prod_{j=1}^{n}\left(1-|z_j|^2\right)^{k_j}\right]\frac{\partial^{|\mathbf{k}|} f}{\partial z_1^{k_1}\dots\partial z_n^{k_n}}(z) \in \mathcal{L}^p(dV_{\vec{\alpha}}) \quad \forall |\mathbf{k}| = m. \tag{1.18}$$

Moreover,

$$\|f\|_{\mathcal{A}^p(dV_{\vec{\alpha}})} \asymp \left(\sum_{|\mathbf{k}|=0}^{m-1}\left|\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)\right| + \sum_{|\mathbf{k}|=m}\left\|\left[\prod_{j=1}^{n}\left(1-|z_j|^2\right)^{k_j}\right]\frac{\partial^m f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right\|_{\mathcal{L}^p(dV_{\vec{\alpha}})}\right). \tag{1.19}$$

In [11] among other things we proved the following theorem which generalizes Theorem 1.4.

Theorem 1.5. *Let* $\mathbf{k} = (k_1,\dots,k_n) \in (\mathbf{Z}_+)^n$, f *be a holomorphic function defined on the polydisc* U^n *in* $\mathbf{C}^n$ *and* $\omega_j(z_j)$, $j = 1,\dots,n$ *are admissible weights on the unit disk* U, *with distortion functions* $\psi_j(z_j)$. *If* $f \in \mathcal{A}^p_{\vec{\omega}}$ *and* $p > 0$, *then*

$$\left[\prod_{j=1}^{n}\psi_j^{k_j}(z_j)\right]\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(z) \in \mathcal{L}^p_{\vec{\omega}}. \tag{1.20}$$

Moreover, let m *be a fixed positive integer. Then there is a positive constant* $C = C(p,\omega_j,m,n)$ *such that*

$$\|f\|_{\mathcal{A}^p_{\vec{\omega}}} \geq C\left(\sum_{|\mathbf{k}|=0}^{m-1}\left|\frac{\partial^{|\mathbf{k}|}f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}(0)\right| + \sum_{|\mathbf{k}|=m}\left\|\left[\prod_{j=1}^{n}\psi_j^{k_j}(z_j)\right]\frac{\partial^m f}{\partial z_1^{k_1}\cdots\partial z_n^{k_n}}\right\|_{\mathcal{L}^p_{\vec{\omega}}}\right). \tag{1.21}$$

In the same paper, we formulate and give a sketch of a proof of the following partially converse of Theorem 1.5.

Theorem 1.6. *Let* $f \in H(U^n)$ *and* $\omega_j(z_j)$, $j = 1,\dots,n$ *are admissible weights on the unit disk* U, *with distortion functions* $\psi_j(z_j)$. *If* $p \in [1,\infty)$ *and for all* $j = 1,\dots,n$, $\psi_j(z_j)(\partial f/\partial z_j)(z) \in \mathcal{L}^p_{\vec{\omega}}$, *then* $f \in \mathcal{A}^p_{\vec{\omega}}$ *and there is a positive constant* $C = C(p,\omega_j,n)$ *such that*

$$\|f\|_{\mathcal{A}^p_{\vec{\omega}}} \leq C\left(|f(0)| + \sum_{j=1}^{n}\left\|\psi_j\frac{\partial f}{\partial z_j}\right\|_{\mathcal{L}^p_{\vec{\omega}}}\right). \tag{1.22}$$

In communication with other specialists in this field it has turned out that the proof is more complicated than we have expected. Hence in this note we will present a clear detailed proof of Theorem 1.6. Also, we prove the following generalization of Theorem 1.6.

Theorem 1.7. *Let* $f \in H(U^n)$ *and* $\omega_j(z_j)$, $j = 1,\dots,n$ *are admissible weights on the unit disk* U, *with distortion functions* $\psi_j(z_j)$. *If* $p,q \in [1,\infty)$ *and for all* $j = 1,\dots,n$, $\psi_j(z_j)(\partial f/\partial z_j)(z) \in \mathcal{L}^{p,q}_{\vec{\omega},N}$, *then* $f \in \mathcal{A}^{p,q}_{\vec{\omega},N}$ *and there is a positive constant* $C = C(p,q,\omega_j,n)$ *such that*

$$\|f\|_{\mathcal{A}^{p,q}_{\vec{\omega},N}} \leq C\left(|f(0)| + \sum_{j=1}^{n}\left\|\psi_j\frac{\partial f}{\partial z_j}\right\|_{\mathcal{L}^{p,q}_{\vec{\omega},N}}\right). \tag{1.23}$$

2. An auxiliary results

In this section, we prove an auxiliary result which we use in the proof of the main result.

LEMMA 2.1. *Suppose* $1 \le p < \infty$ *and* $f \in H(U^n)$. *Then*

$$\frac{d}{dt}M_p^p(f,tr) \le pM_p^{p-1}(f,tr)\sum_{i=1}^{n} r_i M_p\left(\frac{\partial f}{\partial z_i},tr\right), \tag{2.1}$$

almost everywhere.

Proof. For $f \equiv 0$ the result is obvious. If $f \not\equiv 0$, at points where f is not zero, we have

$$\begin{aligned}\frac{d}{dt}(|f(tr\cdot e^{i\theta})|^p) &= p|f(tr\cdot e^{i\theta})|^{p-1}\frac{d}{dt}|f(tr\cdot e^{i\theta})| \\ &\le p|f(tr\cdot e^{i\theta})|^{p-1}\left|\frac{d}{dt}(f(tr\cdot e^{i\theta}))\right| \\ &= p|f(tr\cdot e^{i\theta})|^{p-1}|\langle\nabla f(tr\cdot e^{i\theta}),r\cdot e^{i\theta}\rangle| \\ &\le p|f(tr\cdot e^{i\theta})|^{p-1}\sum_{i=1}^{n} r_i\left|\frac{\partial f}{\partial z_i}(tr\cdot e^{i\theta})\right|.\end{aligned} \tag{2.2}$$

From (2.2) and by the dominated convergence theorem we obtain

$$\frac{d}{dr}M_p^p(f,tr) \le \frac{p}{(2\pi)^n}\sum_{i=1}^{n} r_i\int_{[0,2\pi]^n}|f(tr\cdot e^{i\theta})|^{p-1}\left|\frac{\partial f}{\partial z_i}(tr\cdot e^{i\theta})\right|d\theta. \tag{2.3}$$

If $p = 1$ the assertion is clear. If $p > 1$, applying in the last integral Hölder's inequality with exponents $p/(p-1)$ and p we obtain the result. □

COROLLARY 2.2. *Suppose* $p,q \in [1,\infty)$ *and* $f \in H(U^n)$. *Then*

$$\frac{d}{dt}M_p^q(f,tr) \le qM_p^{q-1}(f,tr)\sum_{i=1}^{n} r_i M_p\left(\frac{\partial f}{\partial z_i},tr\right), \tag{2.4}$$

almost everywhere.

Proof. Computing $(d/dt)M_p^q(f,tr)$ and then using Lemma 2.1 we prove the corollary. □

3. Proofs of the theorem

In this section, we prove the main results in this paper.

Proof of Theorem 1.6. Without loss of generality, we may assume that $n = 2$, and $f(0,0) = 0$. Also we assume that f is not constant and all integrals are finite. In order to avoid some complicated notations we use $M_p^p(f,r_1t,r_2t)$ instead of $M_p^p(r_1t,r_2t)$. We have

$$
\begin{aligned}
\|f\|^p_{\mathcal{A}^p_{\vec{\omega},N}} &= \int_0^1\int_0^1\left(\int_0^1 \frac{d}{dt}M_p^p(r_1t,r_2t)\,dt\right)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2\\
&\le p\int_0^1\int_0^1\left(\int_0^1 M_p^{p-1}(r_1t,r_2t)\sum_{j=1}^2 M_p\left(\frac{\partial f}{\partial z_i},r_1t,r_2t\right)r_i dt\right)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2\\
&\le p\int_0^1\int_0^1\left(\int_0^1 M_p^{p-1}(r_1t,r_2)M_p\left(\frac{\partial f}{\partial z_1},r_1t,r_2\right)r_1 dt\right)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2\\
&\quad+p\int_0^1\int_0^1\left(\int_0^1 M_p^{p-1}(r_1,r_2t)M_p\left(\frac{\partial f}{\partial z_2},r_1,r_2t\right)r_2 dt\right)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2\\
&\le p\int_0^1\int_0^1\left(\int_0^{r_1} M_p^{p-1}(s,r_2)M_p\left(\frac{\partial f}{\partial z_1},s,r_2\right)ds\right)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2\\
&\quad+p\int_0^1\int_0^1\left(\int_0^{r_2} M_p^{p-1}(r_1,\tau)M_p\left(\frac{\partial f}{\partial z_2},r_1,\tau\right)d\tau\right)\omega_1(r_1)\omega_2(r_2)\,dr_1\,dr_2\\
&\le p\int_0^1\int_0^1\left(M_p^{p-1}(s,r_2)M_p\left(\frac{\partial f}{\partial z_1},s,r_2\right)\int_s^1\omega_1(r_1)\,dr_1\right)\omega_2(r_2)\,ds\,dr_2\\
&\quad+p\int_0^1\int_0^1\left(M_p^{p-1}(r_1,\tau)M_p\left(\frac{\partial f}{\partial z_2},r_1,\tau\right)\int_\tau^1\omega_2(r_2)\,dr_2\right)\omega_1(r_1)\,d\tau\,dr_1\\
&= p\int_0^1\int_0^1 M_p^{p-1}(s,r_2)M_p\left(\frac{\partial f}{\partial z_1},s,r_2\right)\psi_1(s)\omega_1(s)\omega_2(r_2)\,ds\,dr_2\\
&\quad+p\int_0^1\int_0^1 M_p^{p-1}(r_1,\tau)M_p\left(\frac{\partial f}{\partial z_2},r_1,\tau\right)\psi_2(\tau)\omega_2(\tau)\omega_1(r_1)\,d\tau\,dr_1.
\end{aligned}
\tag{3.1}
$$

If $p>1$, by Hölder inequality, we get

$$
\begin{aligned}
&\int_0^1\int_0^1 M_p^{p-1}(s,r_2)M_p\left(\frac{\partial f}{\partial z_1},s,r_2\right)\psi_1(s)\omega_1(s)\omega_2(r_2)\,ds\,dr_2\\
&\qquad\le\left(\int_0^1\int_0^1 M_p^p(s,r_2)\omega_1(s)\omega_2(r_2)\,ds\,dr_2\right)^{(p-1)/p},\\
&\left(\int_0^1\int_0^1 M_p^p\left(\frac{\partial f}{\partial z_1},s,r_2\right)\psi_1^p(s)\omega_1(s)\omega_2(r_2)\,ds\,dr_2\right)^{1/p}\\
&\qquad=\|f\|^{p-1}_{\mathcal{A}^p_{\vec{\omega},N}}\left(\int_0^1\int_0^1 M_p^p\left(\frac{\partial f}{\partial z_1},s,r_2\right)\psi_1^p(s)\omega_1(s)\omega_2(r_2)\,ds\,dr_2\right)^{1/p}.
\end{aligned}
\tag{3.2}
$$

Similarly,

$$
\begin{aligned}
&\int_0^1\int_0^1 M_p^{p-1}(r_1,\tau)M_p\left(\frac{\partial f}{\partial z_2},r_1,\tau\right)\psi_2(\tau)\omega_2(\tau)\omega_1(r_1)\,d\tau\,dr_1\\
&\qquad\le\|f\|^{p-1}_{\mathcal{A}^p_{\vec{\omega},N}}\left(\int_0^1\int_0^1 M_p^p\left(\frac{\partial f}{\partial z_2},r_1,\tau\right)\psi_2^p(\tau)\omega_2(\tau)\omega_1(r_1)\,d\tau\,dr_1\right)^{1/p}.
\end{aligned}
\tag{3.3}
$$

From (3.1)–(3.3) we obtain the result in this case. For $p = 1$ the result it follows from (3.3). If f is constant the result is clear. To remove the restriction of the finiteness of the integrals we consider holomorphic functions $f_\rho(z) = f(\rho z)$, $\rho \in (0,1)$ and use the Monotone Convergence theorem, when $\rho \to 1$. □

Open question 3.1. Does Theorem 1.6 hold in the case $0 < p < 1$?

Remark 3.2. In the case when $\omega_j(z_j)$, $j = 1,\dots,n$, are the classical weights $(1 - |z_j|)^{\alpha_j}$, $j = 1,\dots,n$, a positive answer to Open question 3.1 was given in [12].

Proof of Theorem 1.7. If $f(0,0) = 0$, is not constant and all integrals are finite, then by Corollary 2.2, as in the proof of Theorem 1.6, we obtain

$$\begin{aligned}\|f\|^q_{\mathcal{A}^{p,q}_{\vec{\omega},N}} \le \|f\|^{q-1}_{\mathcal{A}^{p,q}_{\vec{\omega},N}} \left(\int_0^1\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_1}, s, r_2\right)\psi_1^q(s)\omega_1(s)\omega_2(r_2)\,ds\,dr_2\right)^{1/q} \\ + \|f\|^{q-1}_{\mathcal{A}^{p,q}_{\vec{\omega},N}} \left(\int_0^1\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_2}, r_1, \tau\right)\psi_2^q(\tau)\omega_2(\tau)\omega_1(r_1)\,d\tau\,dr_1\right)^{1/q},\end{aligned} \tag{3.4}$$

that is,

$$\|f\|_{\mathcal{A}^{p,q}_{\vec{\omega},N}} \le \sum_{j=1}^{2}\left\|\psi_j\frac{\partial f}{\partial z_j}\right\|_{\mathcal{L}^{p,q}_{\vec{\omega},N}}. \tag{3.5}$$

The rest of the proof is similar to the proof of Theorem 1.6. □

References

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