A RECONSIDERATION OF HUA'S INEQUALITY. PART II

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We give a new interpretation of Hua's inequality and its generalization. From this interpretation, we know the best possibility of those inequalities.

1. Introduction

In 1965, L. Keng Hua discovered the following inequality.

Theorem 1.1 [2]. *If $\delta,\lambda > 0$ and $x_1,\dots,x_n \in \mathbb{R}$, then*

$$\left(\delta - \sum_{i=1}^{n} x_i\right)^2 + \lambda \sum_{i=1}^{n} {x_i}^2 \geq \frac{\lambda\delta^2}{\lambda + n}. \tag{1.1}$$

In (1.1), the equality holds if and only if $x_1 = \cdots = x_n = \delta/(\lambda+n)$.

This inequality played an important role in number theory and has been generalized in several directions [1, 3–6]. One of its generalizations states the following.

Theorem 1.2 [5, Corollary 2.7]. *Let X be a real or complex normed space with dual X^*, and suppose $p,q > 1$ and $1/p + 1/q = 1$. If $\delta,\lambda > 0$, $x \in X$, and $f \in X^*$, then*

$$|\delta - f(x)|^p + \lambda^{p-1}\|x\|^p \geq \left(\frac{\lambda}{\lambda + \|f\|^q}\right)^{p-1}\delta^p. \tag{1.2}$$

In (1.2), the equality holds if and only if $f(x) = \|f\|\|x\|$ and $\|x\| = \delta\|f\|^{q-1}/(\lambda + \|f\|^q)$.

In this paper, we give a new interpretation of the inequality (1.2) and consider whether the coefficients λ^{p-1} and $(\lambda/(\lambda+\|f\|^q))^{p-1}$ are best possible. For this purpose, we divide

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both sides of (1.2) by $(\lambda/(\lambda+\|f\|^q))^{p-1}\delta^p$, and then replace x/δ by x. Thus we obtain a replica of Theorem 1.2.

THEOREM 1.3. *Let X be a real or complex normed space with dual X^*, and suppose $p,q>1$ and $1/p+1/q=1$. If $\lambda>0$, $x\in X$, and $f\in X^*$, then*

$$\left(\frac{\lambda+\|f\|^q}{\lambda}\right)^{p-1}|1-f(x)|^p+(\lambda+\|f\|^q)^{p-1}\|x\|^p\geq 1. \tag{1.3}$$

In (1.3), the equality holds if and only if $f(x)=\|f\|\,\|x\|$ and $\|x\|=\|f\|^{q-1}/(\lambda+\|f\|^q)$.

Clearly, Theorems 1.2 and 1.3 are equivalent. So, we turn our attention to Theorem 1.3, which is more convenient for us. Put

$$\Omega=\{(|1-f(x)|,\|x\|):x\in X\}. \tag{1.4}$$

Then Ω is a subset of $\mathbb{R}^+\times\mathbb{R}^+$, where $\mathbb{R}^+=\{s\in\mathbb{R}:s\geq 0\}$. Moreover, we have

$$\Omega\subset\{(s,t)\in\mathbb{R}^+\times\mathbb{R}^+:s+\|f\|\,t\geq 1\}, \tag{1.5}$$

because $|1-f(x)|+\|f\|\,\|x\|\geq 1-|f(x)|+\|f\|\,\|x\|\geq 1$ for all $x\in X$. While the inequality (1.3) has the form

$$as^p+bt^p\geq 1\quad\forall(s,t)\in\Omega, \tag{1.6}$$

where a and b are nonnegative constants. If we know all the nonnegative constants a and b such that (1.6) holds, then we may determine whether the coefficients $((\lambda+\|f\|^q)/\lambda)^{p-1}$ and $(\lambda+\|f\|^q)^{p-1}$ in (1.3) are best possible.

2. General theory

Let k and ℓ be positive numbers. Let Ω be an index set such that

$$\Omega\subset\{(s,t)\in\mathbb{R}^+\times\mathbb{R}^+:ks+\ell t\geq 1\}. \tag{2.1}$$

For such an index set Ω and any $p>0$, we consider the domain

$$D(p;\Omega)=\{(a,b)\in\mathbb{R}^+\times\mathbb{R}^+:as^p+bt^p\geq 1\ \forall(s,t)\in\Omega\}. \tag{2.2}$$

We wish to identify the domain $D(p;\Omega)$.

We first consider the case $p>1$. We define a function $h_{p,k,\ell}$ on the open interval (k^p,∞) by

$$h_{p,k,\ell}(a)=\frac{\ell^p a}{(a^{q-1}-k^q)^{p-1}}\quad(a>k^p), \tag{2.3}$$

where q is the number satisfying $1/p+1/q=1$. It is easily seen that the function $h_{p,k,\ell}$ is decreasing and strictly convex, and that the graph of $b=h_{p,k,\ell}(a)$ has the asymptotic lines $a=k^p$ and $b=\ell^p$. Next, we put

$$S(k,\ell)=\{(s,t)\in\mathbb{R}^+\times\mathbb{R}^+ : ks+\ell t=1\}. \tag{2.4}$$

In other words, $S(k,\ell)$ is the line segment connecting two points $(1/k,0)$ and $(0,1/\ell)$. Also, we write $\overline{\Omega}$ for the closure of Ω in the Euclidean plane $\mathbb{R}\times\mathbb{R}$.

Theorem 2.1. *Let k and ℓ be positive numbers and let Ω be an index set such that $\Omega\subset\{(s,t)\in\mathbb{R}^+\times\mathbb{R}^+ : ks+\ell t\geq 1\}$. Suppose that $p>1$ and $1/p+1/q=1$. Then the following assertions hold.*

(i) *If $a>k^p$, then*

$$as^p+h_{p,k,\ell}(a)t^p\geq 1\quad \forall(s,t)\in\Omega. \tag{2.5}$$

In (2.5), the equality holds if and only if $(s,t)=((k/a)^{q-1},(a^{q-1}-k^q)/\ell a^{q-1})\in\Omega$. This attaining point (s,t) lies on the line segment $S(k,\ell)$.

(ii) $D(p;\Omega)\supset\{(a,b)\in\mathbb{R}^+\times\mathbb{R}^+ : a>k^p,\ b\geq h_{p,k,\ell}(a)\}$.

(iii) *If $S(k,\ell)\subset\overline{\Omega}$, then*

$$D(p;\Omega)=\{(a,b)\in\mathbb{R}^+\times\mathbb{R}^+ : a>k^p,\ b\geq h_{p,k,\ell}(a)\}. \tag{2.6}$$

The formula (2.6) says that when $a>k^p$ and $b=h_{p,k,\ell}(a)$, the pair (a,b) is one of the best possible constants such that (1.6) holds.

Before proving Theorem 2.1, we make some remarks on the domains which appear in (ii) and (iii). Evidently, the domain $D(p;\Omega)$ has the property that

$$(\alpha,\beta)\in D(p;\Omega),\quad a\geq\alpha,\ b\geq\beta\Longrightarrow(a,b)\in D(p;\Omega). \tag{2.7}$$

Next, for each $(s,t)\in S(k,\ell)$, we put

$$\begin{aligned} L(p;s,t)&=\{(a,b)\in\mathbb{R}\times\mathbb{R} : as^p+bt^p=1\},\\ \Delta(p;s,t)&=\{(a,b)\in\mathbb{R}\times\mathbb{R} : as^p+bt^p\geq 1\}. \end{aligned} \tag{2.8}$$

In the ab-plane, $L(p;s,t)$ denotes a straight line, and $\Delta(p;s,t)$ denotes the closed upper half plane whose boundary is the line $L(p;s,t)$, while the domain $\{(a,b)\in\mathbb{R}^+\times\mathbb{R}^+ : a>k^p,\ b\geq h_{p,k,\ell}(a)\}$ consists of the points above or on the curve $b=h_{p,k,\ell}(a)$ $(a>k^p)$. These domains have a relation in the following sense.

Lemma 2.2. *For positive numbers k and ℓ, the following assertions hold.*

(i) *If $\mathcal{L}$ is a family of the lines $\{L(p;s,t) : (s,t)\in S(k,\ell)\}$, then the envelope of $\mathcal{L}$ is given by $b=h_{p,k,\ell}(a)$ $(a>k^p)$.*

(ii) $\{(a,b)\in\mathbb{R}^+\times\mathbb{R}^+ : a>k^p,\ b\geq h_{p,k,\ell}(a)\}=\bigcap_{(s,t)\in S(k,\ell)}\Delta(p;s,t)$.

Proof. (i) Since $S(k,\ell)=\{(s,(1-ks)/\ell):0\le s\le 1/k\}$, the family of lines $\mathscr{L}$ is represented by

$$s^p a+\left(\frac{1-ks}{\ell}\right)^p b=1 \quad \left(0\le s\le \frac{1}{k}\right). \tag{2.9}$$

We here remark that each line of $\mathscr{L}$ has no singular point. Now, put

$$F(a,b,s)=s^p a+\frac{(1-ks)^p}{\ell^p}b-1. \tag{2.10}$$

Then

$$\frac{\partial F}{\partial s}(a,b,s)=ps^{p-1}a-\frac{pk(1-ks)^{p-1}}{\ell^p}b. \tag{2.11}$$

Consider the simultaneous equations $F(a,b,s)=0$ and $(\partial F/\partial s)(a,b,s)=0$. If $0<s<1/k$, then the equation $(\partial F/\partial s)(a,b,s)=0$ yields

$$b=\frac{\ell^p s^{p-1}}{k(1-ks)^{p-1}}a, \tag{2.12}$$

and so the equation $F(a,b,s)=0$ becomes

$$s^p a+s^{p-1}\frac{1-ks}{k}a-1=0, \tag{2.13}$$

or, equivalently, $a=k/s^{p-1}$, which implies $b=\ell^p/(1-ks)^{p-1}$. Let us delete the letter s in the resulting equations

$$a=\frac{k}{s^{p-1}},\quad b=\frac{\ell^p}{(1-ks)^{p-1}} \quad \left(0<s<\frac{1}{k}\right). \tag{2.14}$$

Since the former equation yields $s=(k/a)^{q-1}$, the latter equation becomes

$$b=\frac{\ell^p}{\left(1-k(k/a)^{q-1}\right)^{p-1}}=\frac{\ell^p a}{\left(a^{q-1}-k^q\right)^{p-1}}=h_{p,k,\ell}(a). \tag{2.15}$$

Also, $0<s<1/k$ if and only if $a=k/s^{p-1}>k^p$. Thus the envelope of $\mathscr{L}$ is given by $b=h_{p,k,\ell}(a)$ $(a>k^p)$.

(ii) Visualize the domains $\{(a,b)\in\mathbb{R}^+\times\mathbb{R}^+ : a>k^p,\ b\ge h_{p,k,\ell}(a)\}$ and $\Delta(p;s,t)$ for $(s,t)\in S(k,\ell)$, in the ab-plane. Next, note that the first domain is strictly convex set. Then we can see (ii) directly from (i). □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. (i) Suppose $\alpha>k^p$ and $\beta=h_{p,k,\ell}(\alpha)$. To see (2.5), we show that $\alpha s^p+\beta t^p\ge 1$ for all $(s,t)\in\Omega$. Choose $(s,t)\in\Omega$ arbitrarily. Then we have $ks+\ell t\ge 1$. So, we can easily find the point (σ,τ) in $S(k,\ell)$ such that $\sigma\le s$ and $\tau\le t$. By Lemma 2.2(ii), we have

$$(\alpha,\beta) \in \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > k^p,\ b \geq h_{p,k,\ell}(a)\} = \bigcap_{(s,t)\in S(k,\ell)} \Delta(p;s,t) \subset \Delta(p;\sigma,\tau). \tag{2.16}$$

Hence $\alpha\sigma^p + \beta\tau^p \geq 1$. Thus we have

$$\alpha s^p + \beta t^p \geq \alpha\sigma^p + \beta\tau^p \geq 1, \tag{2.17}$$

which was to be proved for (2.5).

Let us check the equality condition of (2.5). Suppose that $\alpha s^p + \beta t^p = 1$ for some $(s,t) \in \Omega$. Then two inequalities in (2.17) become the equalities. Hence $(s,t) = (\sigma,\tau) \in S(k,\ell)$ and $\alpha\sigma^p + \beta\tau^p = 1$. The last equation means that the point (α,β) lies on the straight line

$$\sigma^p a + \tau^p b = 1, \tag{2.18}$$

which is a member $L(p;\sigma,\tau)$ of $\mathcal{L}$. Also, the point (α,β) lies on the graph of $b = h_{p,k,\ell}(a)$ $(a > k^p)$, because $\alpha > k^p$ and $\beta = h_{p,k,\ell}(\alpha)$. Here we recall from Lemma 2.2(i) that the curve $b = h_{p,k,\ell}(a)$ $(a > k^p)$ is the envelope of $\mathcal{L}$. These facts and the strict convexity of $h_{p,k,\ell}$ imply that the line (2.18) is tangent to the graph of $b = h_{p,k,\ell}(a)$ $(a > k^p)$ at the point (α,β). Let us find this tangent line. Since a routine computation shows that $h'_{p,k,\ell}(a) = -k^q\ell^p/(a^{q-1} - k^q)^p$, the desired tangent line is formulated as

$$b - \frac{\ell^p \alpha}{(\alpha^{q-1} - k^q)^{p-1}} = -\frac{k^q \ell^p}{(\alpha^{q-1} - k^q)^p}(a - \alpha), \tag{2.19}$$

that is,

$$\left(\frac{k}{\alpha}\right)^q a + \frac{(\alpha^{q-1} - k^q)^p}{\ell^p \alpha^q} b = 1. \tag{2.20}$$

Since this denotes the line (2.18), we have

$$\sigma^p = \left(\frac{k}{\alpha}\right)^q, \qquad \tau^p = \frac{(\alpha^{q-1} - k^q)^p}{\ell^p \alpha^q}, \tag{2.21}$$

and so $\sigma = (k/\alpha)^{q-1}$, $\tau = (\alpha^{q-1} - k^q)/\ell\alpha^{q-1}$. Thus we obtain $(s,t) = (\sigma,\tau) \in S(k,\ell)$ and $(s,t) = ((k/\alpha)^{q-1}, (\alpha^{q-1} - k^q)/\ell\alpha^{q-1})$.

Conversely, if $(s,t) = ((k/a)^{q-1}, (a^{q-1} - k^q)/\ell a^{q-1})$, then

$$\begin{aligned} as^p + h_{p,k,\ell}(a)t^p &= a\left(\frac{k}{a}\right)^q + \frac{\ell^p a}{(a^{q-1} - k^q)^{p-1}} \cdot \frac{(a^{q-1} - k^q)^p}{\ell^p a^q} \\ &= \frac{k^q}{a^{q-1}} + \frac{a^{q-1} - k^q}{a^{q-1}} = 1, \end{aligned} \tag{2.22}$$

which is the equality in (2.5).

(ii) By (i), we see that if $\alpha > k^p$ and $\beta = h_{p,k,\ell}(\alpha)$, then $(\alpha,\beta) \in D(p,\Omega)$. Hence (ii) follows immediately from the property (2.7).

(iii) By (ii), it suffices to show that $D(p;\Omega) \subset \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > k^p,\ b \geq h_{p,k,\ell}(a)\}$. Pick $(\alpha,\beta) \in D(p;\Omega)$. For each $(s,t) \in S(k,\ell)$, there exists a sequence $\{(s_n,t_n)\}$ in Ω such that $s_n \to s$ and $t_n \to t$, because $S(k,\ell) \subset \overline{\Omega}$. Noting that $(\alpha,\beta) \in D(p;\Omega)$ and $(s_n,t_n) \in \Omega$, we have $\alpha {s_n}^p + \beta {t_n}^p \geq 1$. Letting $n \to \infty$, we obtain $\alpha s^p + \beta t^p \geq 1$. Hence $(\alpha,\beta) \in \Delta(p;s,t)$. Since this holds for all $(s,t) \in S(k,\ell)$, it follows that $(\alpha,\beta) \in \bigcap_{(s,t)\in S(k,\ell)} \Delta(p;s,t)$. Hence Lemma 2.2(ii) shows that $(\alpha,\beta) \in \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a > k^p,\ b \geq h_{p,k,\ell}(a)\}$. Thus (iii) is proved. □

Next, we consider the case $0 < p \leq 1$.

Theorem 2.3. *Let k and ℓ be positive numbers and let Ω be an index set such that $\Omega \subset \{(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ : ks + \ell t \geq 1\}$. Suppose that $0 < p \leq 1$. Then the following assertions hold.*

(i) *The inequality $k^p s^p + \ell^p t^p \geq 1$ holds for all $(s,t) \in \Omega$. If $0 < p < 1$, then the equality holds if and only if $(s,t) = (1/k,0)$ or $(0,1/\ell)$.*

(ii) $D(p;\Omega) \supset \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a \geq k^p,\ b \geq \ell^p\}$.

(iii) *If $(1/k,0)$ and $(0,1/\ell)$ belong to $\overline{\Omega}$, then $D(p;\Omega) = \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a \geq k^p,\ b \geq \ell^p\}$.*

Proof. (i) Since the case $p = 1$ is trivial, we assume that $0 < p < 1$. For any $(s,t) \in \Omega$, we have $ks, \ell t \geq 0$ and $ks + \ell t \geq 1$. By Minkowski's inequality, we obtain $k^p s^p + \ell^p t^p \geq 1$. Also, an easy consideration implies that the equality holds precisely when $(ks,\ell t) = (1,0)$ or $(0,1)$, namely $(s,t) = (1/k,0)$ or $(0,1/\ell)$.

(ii) The inequality in (i) implies $(k^p,\ell^p) \in D(p;\Omega)$. Hence (ii) follows from (2.7).

(iii) By (ii), it suffices to show that $D(p;\Omega) \subset \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a \geq k^p,\ b \geq \ell^p\}$. Pick $(\alpha,\beta) \in D(p;\Omega)$. We must show that $\alpha \geq k^p$ and $\beta \geq \ell^p$. Since $(1/k,0) \in \overline{\Omega}$, we can find the sequence $\{(s_n,t_n)\}$ in Ω such that $s_n \to 1/k$ and $t_n \to 0$. Noting that $(\alpha,\beta) \in D(p;\Omega)$ and $(s_n,t_n) \in \Omega$, we see that $\alpha {s_n}^p + \beta {t_n}^p \geq 1$. Letting $n \to \infty$, we have $\alpha/k^p \geq 1$, namely $\alpha \geq k^p$. Similarly, we obtain $\beta \geq \ell^p$. Thus we proved (iii). □

We close the general theory with the opposite inequalities obtained similarly.

Theorem 2.4. *Let k and ℓ be positive numbers and let Ω' be an index set such that $\Omega' \subset \{(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ : ks + \ell t \leq 1\}$. Suppose that $0 < p < 1$ and $1/p + 1/q = 1$. Put*

$$D'(p;\Omega') = \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : as^p + bt^p \leq 1\ \forall (s,t) \in \Omega'\}. \tag{2.23}$$

Define $h_{p,k,\ell}(a) = \ell^p(1 - k^q a^{1-q})^{1-p}$ for $0 \leq a < k^p$. Then the following assertions hold.

(i) *If $0 \leq a < k^p$, then $as^p + h_{p,k,\ell}(a)t^p \leq 1$ for all $(s,t) \in \Omega'$. Here, the equality holds if and only if $(s,t) = ((a/k)^{1-q}, (1 - k^q a^{1-q})/\ell) \in \Omega'$. This attaining point (s,t) lies on the line segment $S(k,\ell)$.*

(ii) $D'(p;\Omega') \supset \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a < k^p,\ b \leq h_{p,k,\ell}(a)\}$.

(iii) *If $S(k,\ell) \subset \overline{\Omega'}$, then $D'(p;\Omega') = \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a < k^p,\ b \leq h_{p,k,\ell}(a)\}$.*

THEOREM 2.5. *Let k and ℓ be positive numbers and let Ω' be an index set such that $\Omega' \subset \{(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ : ks + \ell t \leq 1\}$. Suppose that $p \geq 1$. Define the domain $D'(p;\Omega')$ by (2.23). Then the following assertions hold.*

(i) *The inequality $k^p s^p + \ell^p t^p \leq 1$ holds for all $(s,t) \in \Omega'$. If $p > 1$, then the equality holds if and only if $(s,t) = (1/k, 0)$ or $(0, 1/\ell)$.*

(ii) *$D'(p;\Omega') \supset \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a \leq k^p,\ b \leq \ell^p\}$.*

(iii) *If $(1/k, 0)$ and $(0, 1/\ell)$ belong to $\overline{\Omega'}$, then $D'(p;\Omega') = \{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+ : a \leq k^p,\ b \leq \ell^p\}$.*

3. The best possibility of Hua type inequality

We now return to Theorem 1.3. We give a new proof of Theorem 1.3 by using Theorem 2.1.

Proof of Theorem 1.3. If f is a zero functional on X, then the statements of Theorem 1.3 are trivial. So, we assume that f is nonzero. Set $k = 1$ and $\ell = \|f\|$. Then $k, \ell > 0$. Put $\Omega = \{(|1 - f(x)|, \|x\|) : x \in X\}$. As we saw in Section 1, we have $\Omega \subset \{(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+ : ks + \ell t \geq 1\}$. Since $p > 1$, it follows from Theorem 2.1(i) that if $a > k^p = 1$, then

$$as^p + h_{p,k,\ell}(a)t^p \geq 1 \quad \forall (s,t) \in \Omega. \tag{3.1}$$

In (3.1), the equality holds if and only if $s = (1/a)^{q-1}$ and $t = (a^{q-1} - 1)/\|f\|a^{q-1}$.

Note that $((\lambda + \|f\|^q)/\lambda)^{p-1} > 1$. We now take $a = ((\lambda + \|f\|^q)/\lambda)^{p-1}$. Then

$$h_{p,k,\ell}(a) = \frac{\|f\|^p\left((\lambda + \|f\|^q)/\lambda\right)^{p-1}}{\left(((\lambda + \|f\|^q)/\lambda) - 1\right)^{p-1}} = \left(\lambda + \|f\|^q\right)^{p-1}. \tag{3.2}$$

Hence, in this case, (3.1) becomes (1.3). Also, the equality condition is

$$|1 - f(x)| = \frac{\lambda}{\lambda + \|f\|^q}, \qquad \|x\| = \frac{((\lambda + \|f\|^q)/\lambda) - 1}{\|f\|(\lambda + \|f\|^q)/\lambda} = \frac{\|f\|^{q-1}}{\lambda + \|f\|^q}. \tag{3.3}$$

Here the latter equation yields

$$|1 - f(x)| \geq 1 - |f(x)| \geq 1 - \|f\|\,\|x\| = 1 - \frac{\|f\|^q}{\lambda + \|f\|^q} = \frac{\lambda}{\lambda + \|f\|^q} \tag{3.4}$$

and so the former equation says that the two inequalities above are the equalities. This implies that $0 \leq f(x) \leq 1$ and $|f(x)| = \|f\|\,\|x\|$. Hence $f(x) = \|f\|\,\|x\|$. Thus if the equality holds in (1.3), then $f(x) = \|f\|\,\|x\|$ and $\|x\| = \|f\|^{q-1}/(\lambda + \|f\|^q)$. The converse is easily checked by a simple computation.

Next, we show that

$$S(k,\ell) \subset \overline{\Omega} \quad \text{in the above setting.} \tag{3.5}$$

Pick $(\sigma, \tau) \in S(k,\ell)$ arbitrarily. Then $k\sigma + \ell\tau = 1$, namely, $\sigma + \|f\|\tau = 1$. Noting that $\|f\| = \sup\{|f(e)| : e \in X,\ \|e\| = 1\}$, we can find a sequence $\{e_n\}$ in X such that $\|e_n\| = 1$,

$f(e_n) \geq 0$ and $f(e_n) \to \|f\|$. Put $x_n = \tau e_n$ for $n = 1,2,\ldots$, and consider the sequence $\{(|1 - f(x_n)|, \|x_n\|)\}$ in Ω. Then we have

$$\begin{aligned} \big|\,|1 - f(x_n)| - \sigma\big| &\leq \big|1 - f(x_n) - \sigma\big| \\ &= \big|1 - \tau f(e_n) - (1 - \|f\|\tau)\big| = \tau\big|f(e_n) - \|f\|\big| \longrightarrow 0 \end{aligned} \tag{3.6}$$

as $n \to \infty$. Also, $\|x_n\| = \tau\|e_n\| = \tau$. Hence $(|1 - f(x_n)|, \|x_n\|) \to (\sigma, \tau)$. Thus we conclude that $(\sigma, \tau) \in \overline{\Omega}$, and (3.5) was proved.

Once we have established (3.5), we can apply Theorem 2.1(iii) in the setting of Theorem 1.3. Thus we conclude that the pair of coefficients

$$\left(\left(\frac{\lambda + \|f\|^q}{\lambda}\right)^{p-1}, (\lambda + \|f\|^q)^{p-1}\right) \tag{3.7}$$

is one of the best pairs of nonnegative constants (a, b) such that $a|1 - f(x)|^p + b\|x\|^p \geq 1$ for all $x \in X$. In this sense, we can say that the inequality (1.3) is best possible. Moreover, we know the best possibility of the inequalities (1.1) and (1.2), because Theorem 1.2 is equivalent to Theorem 1.1 and Theorem 1.3 is a special case of Theorem 1.2. □

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