THE FUGLEDE-PUTNAM THEOREM FOR (p,k)-QUASIHYPONORMAL OPERATORS

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We show that if $T \in \mathcal{B}(\mathcal{H})$ is a (p,k)-quasihyponormal operator and $S^* \in \mathcal{B}(\mathcal{K})$ is a p-hyponormal operator, and if $TX = XS$, where $X : \mathcal{K} \to \mathcal{H}$ is a quasiaffinity (i.e., a one-one map having dense range), then T is a normal and moreover T is unitarily equivalent to S.

Let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. The spectrum of an operator T, denoted by $\sigma(T)$, is the set of all complex numbers λ for which $T - \lambda I$ is not invertible. The numerical range of an operator T, denoted by $W(T)$, is the set defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}. \tag{1}$$

The norm closure of a subspace $\mathcal{M}$ of $\mathcal{H}$ is denoted by $\overline{\mathcal{M}}$. We denote the kernel and the range of an operator T by $\ker(T)$ and $\operatorname{ran}(T)$, respectively.

For p such as $0 < p \le 1$ and positive integer k, an operator $T \in \mathcal{B}(\mathcal{H})$ is called (p,k)-*quasihyponormal* if $T^{*k}(|T|^{2p} - |T^*|^{2p})T^k \ge 0$. A (p,k)-quasihyponormal operator is an extension of p-hyponormal operator (i.e., $(T^*T)^p - (TT^*)^p \ge 0$), k-quasihyponormal operator (i.e., $T^{*k}(|T|^2 - |T^*|^2)T^k \ge 0$) and p-quasihyponormal operator (i.e., $T^*(|T|^{2p} - |T^*|^{2p})T \ge 0$). Aluthge [1], Campbell and Gupta [3], Arora and Arora [5], and the author [8] introduced p-hyponormal, k-quasihyponormal, p-quasihyponormal, and (p,k)-quasihyponormal operators, respectively. It was known that these operators share many interesting properties with hyponormal operators (see [1–8, 11, 12]). In this paper, we consider the extension of results of Sheth [9] and Gupta and Ramanujan [6]. The main result is as follows.

If $T \in \mathcal{B}(\mathcal{H})$ is a (p,k)-quasihyponormal operator and $S^* \in \mathcal{B}(\mathcal{K})$ is a p-hyponormal operator, and if $TX = XS$, where $X : \mathcal{K} \to \mathcal{H}$ is an injective bounded linear operator with dense range, then T is a normal operator unitarily equivalent to S.

In general, the conditions $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ do not imply that T is normal. For example, (see [13]), if $T = SB$, where S is positive and invertible, B is self-adjoint, and

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S and B do not commute, then $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, but T is not normal. Therefore the following question arises naturally.

QUESTION 1. *Which operator T satisfying the condition $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$ is normal?*

In 1966, Sheth [9] showed that if T is a hyponormal operator and $S^{-1}TS = T^*$ for any operator S, where $0 \notin \overline{W(S)}$, then T is self-adjoint. We extend the result of Sheth to the class of p-hyponormal operators as follows.

THEOREM 2. *If T or T^* is p-hyponormal operator and S is an operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

To prove Theorem 2 we need the following lemma.

LEMMA 3 [13, Theorem 1]. *If $T \in \mathcal{B}(\mathcal{H})$ is any operator such that $S^{-1}TS = T^*$, where $0 \notin \overline{W(S)}$, then $\sigma(T) \subseteq \mathbb{R}$.*

Proof of Theorem 2. Suppose that T or T^* is p-hyponormal operator. Since $\sigma(S) \subseteq \overline{W(S)}$, S is invertible and hence $ST = T^*S$ becomes $S^{-1}T^*S = T = (T^*)^*$. Apply Lemma 3 to T^* to get $\sigma(T^*) \subset \mathbb{R}$. Then $\sigma(T) = \overline{\sigma(T^*)} = \sigma(T^*) \subset \mathbb{R}$. Thus $m_2(\sigma(T)) = m_2(\sigma(T^*)) = 0$ for the planer Lebesgue measure m_2. Now apply Putnam's inequality for p-hyponormal operators to T or to T^* (depending upon which is p-hyponormal) to get

$$||(T^*T)^p - (TT^*)^p|| \leq \frac{p}{\pi} \iint_{\sigma(T)} r^{2p-1}\, dr\, d\theta = 0 \tag{2}$$

or

$$||(TT^*)^p - (T^*T)^p|| \leq \frac{p}{\pi} \iint_{\sigma(T^*)} r^{2p-1}\, dr\, d\theta = 0. \tag{3}$$

It follows that T or T^* is normal. Since $\sigma(T) = \sigma(T^*) \subset \mathbb{R}$ here, T must be selfadjoint. □

We can extend the result of Theorem 2 to the class of p-quasihyponormal operators. We use the following lemma.

LEMMA 4 [8, Lemma 1]. *If T is (p,k)-quasihyponormal operator, then T has the following matrix representation:*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \tag{4}$$

where T_1 is p-hyponormal on $\overline{\operatorname{ran}(T^k)}$ and ${T_3}^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

THEOREM 5. *If T is (p,k)-quasihyponormal operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is direct sum of a self-adjoint and nilpotent operator.*

Proof. Since T is (p,k)-quasihyponormal operator, we have the following matrix representation:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k}), \tag{5}$$

where T_1 is p-hyponormal and $T_3^k = 0$. Since $S^{-1}TS = T^*$ and $0 \notin \overline{W(S)}$, we have $\sigma(T) \subseteq \mathbb{R}$ by Lemma 3. Therefore $\sigma(T_1) \subseteq \mathbb{R}$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$ and hence T_1 is self-adjoint by Theorem 2 because T_1 is p-hyponormal operator. Now let P is the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran}(T^k)}$. Since T is (p,k)-quasihyponormal operator we have

$$\begin{aligned} \begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} &= (TPT^*)^p \le P(TT^*)^pP \le P(T^*T)^pP \le (PT^*TP)^p \\ &= \begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{6}$$

by Löwner-Heinz's inequality and Hansen's inequality. By Löwner's inequality, for $0 < q \le p \le 1$, we have

$$\begin{pmatrix} (T_1T_1^*)^q & 0 \\ 0 & 0 \end{pmatrix} \le P(TT^*)^qP \le P(T^*T)^qP \le \begin{pmatrix} (T_1^*T_1)^q & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

Since T_1 is normal, $(TT^*)^q$ has the following matrix representation:

$$(TT^*)^q = \begin{pmatrix} (T_1T_1^*)^q & A \\ A^* & B \end{pmatrix} \quad \text{on } \overline{\operatorname{ran}(T^k)} \oplus \ker(T^{*k}). \tag{8}$$

Put $q = p/2$. Then by straightforward calculation we have

$$\begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = P(TT^*)^pP = P(TT^*)^q(TT^*)^qP = \begin{pmatrix} (T_1T_1^*)^p + AA^* & 0 \\ 0 & 0 \end{pmatrix}, \tag{9}$$

which implies $A = 0$. Thus we have

$$TT^* = \begin{pmatrix} (T_1T_1^*)^q & 0 \\ 0 & B \end{pmatrix}^{1/q} = \begin{pmatrix} T_1T_1^* & 0 \\ 0 & B^{1/q} \end{pmatrix}, \tag{10}$$

and by matrix representation of T we also have

$$TT^* = \begin{pmatrix} T_1T_1^* + T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{pmatrix}. \tag{11}$$

Therefore $T_1T_1^* + T_2T_2^* = T_1T_1^*$ and hence $T_2 = 0$, which implies the proof. □

The following corollary is an extension of the result of Theorem 2 to the class of p-quasihyponormal operators.

Corollary 6. *If T or T^* is p-quasihyponormal operator and S is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then T is self-adjoint.*

Proof. If T is p-quasihyponormal operator, T has the following matrix representation by Lemma 4:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where T_1 is p-hyponormal on $\overline{\operatorname{ran}(T^k)}$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. Since T_1 is self-adjoint and $T_2 = 0$ by Theorem 5, $T = \left(\begin{smallmatrix} T_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is also self-adjoint. On the other hand, if T^* is (p,k)-quasihyponormal operator, then using the arguments of the proof of Theorem 2 we can conclude that T is self-adjoint. □

In 1977, Stampli and Wadhwa [10] showed that if $A^* \in \mathcal{B}(\mathcal{H})$ is hyponormal, $B \in \mathcal{B}(\mathcal{K})$ is dominant, $C \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is injective and has dense range, and if $CA = BC$, then A and B are normal. On the other hand, in 1981, Gupta and Ramanujan [6] showed that if $T \in \mathcal{B}(\mathcal{H})$ is k-quasihyponormal operator and $S \in \mathcal{B}(\mathcal{K})$ is a normal operator for which $TX = XS$ where $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ is one to one operator with dense range, then T is normal operator unitarily equivalent to S. In the following theorem, we extend the result of Gupta and Ramanujan to the class of (p,k)-quasihyponormal operators. We need the following lemma due to Jeon and Duggal [7].

Lemma 7 [7, Corollary 7]. *Let $T \in \mathcal{B}(\mathcal{H})$ be a p-hyponormal operator and let $S^* \in \mathcal{B}(\mathcal{K})$ be a p-hyponormal operator. If $TX = XS$, where $X : \mathcal{K} \to \mathcal{H}$ is an injective bounded linear operator with dense range then T is a normal operator unitarily equivalent to S.*

Theorem 8. *Let $T \in \mathcal{B}(\mathcal{H})$ is a (p,k)-quasihyponormal operator and let $S^* \in \mathcal{B}(\mathcal{K})$ is a p-hyponormal operator. If $TX = XS$, where $X : \mathcal{K} \to \mathcal{H}$ is an injective bounded linear operator with dense range then T is a normal operator unitarily equivalent to S.*

Proof. Let $T_1 := T|_{\overline{\operatorname{ran}(T^k)}}$ and $S_1 := S|_{\overline{\operatorname{ran}(S^k)}}$. Then we have the following matrix representations:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{13}$$

where T_1 is p-hyponormal, ${T_3}^k = 0$ and S_1^* is p-hyponormal. Notice that $T^kX = XS^k$ for all positive integer k. Thus $\overline{X(\overline{\operatorname{ran}(S^k)})} = \overline{\operatorname{ran}(T^k)}$. If we denote the restriction of X to $\overline{\operatorname{ran}(S^k)}$ by X_1 then $X_1 : \overline{\operatorname{ran}(S^k)} \to \overline{\operatorname{ran}(T^k)}$ is one to one and has dense range. Since $X_1S_1x = XSx = TXx = T_1X_1x$ for every $x \in \overline{\operatorname{ran}(S^k)}$, it follows that $X_1S_1 = T_1X_1$. On the other hand, since T_1 and S_1^* are p-hyponormal operators, it follows from Lemma 7 that T_1 is a normal operator unitarily equivalent to S_1. Now let P be the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran}(T^k)}$. Since T is (p,k)-quasihyponormal operator and T_1 is normal operator, from the arguments of the proof of the Theorem 5 we have $T_2 = 0$ and hence $\overline{\operatorname{ran}(T^k)}$ reduces T. Since $X^*(\ker({T^*}^k)) \subseteq \ker({S^*}^k) = \ker(S^*)$, we have that for each $x \in \ker({T^*}^k)$,

$$X^*T_3^*x = X^*T^*x = S^*X^*x = 0. \tag{14}$$

But since X has dense range, X^* is one to one and hence $T_3^* x = 0$ for every $x \in \ker(T^{*k})$. Thus $T_3 = 0$, so that $T = T_1 \oplus 0$. This completes the proof. □

Lemma 9 [11, Lemma 5]. *The restriction $T|_{\mathcal{M}}$ of the (p,k)-quasihyponormal operator T on $\mathcal{H}$ to an invariant subspace $\mathcal{M}$ of T is also (p,k)-quasihyponormal operator.*

Lemma 10. *Let $T \in \mathcal{B}(\mathcal{H})$ be a (p,k)-quasihyponormal operator and $\mathcal{M}$ be an invariant subspace of T for which $T|_{\mathcal{M}}$ is an injective normal operator. Then $\mathcal{M}$ reduces T.*

Proof. Suppose that P is a orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran}(T^k)}$. Then since T is (p,k)-quasihyponormal operator, we have $P\{(T^*T)^p - (TT^*)^p\}P \geq 0$. Put $T_1 = T|_{\mathcal{M}}$ and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp. \tag{15}$$

Since by assumption T_1 is injective normal operator, we have $E \leq P$ for the orthogonal projection E of $\mathcal{H}$ onto $\mathcal{M}$ and $\overline{\operatorname{ran}(T_1^k)} = \mathcal{M}$ because T_1 has dense range. Therefore $\mathcal{M} \subseteq \overline{\operatorname{ran}(T^k)}$ and hence $E\{(T^*T)^p - (TT^*)^p\}E \geq 0$. Since T is (p,k)-quasihyponormal operator, using the Löwner-Heinz inequality and Hansen's inequality we have

$$\begin{aligned} \begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} &= E(TET^*)^pE \leq E(TT^*)^pE \leq E(T^*T)^pE \leq (ET^*TE)^p \\ &= \begin{pmatrix} (T_1^*T_1)^p & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{16}$$

Since T_1 is normal, we have, by Löwner's inequality,

$$(TT^*)^{p/2} = \begin{pmatrix} (T_1T_1^*)^{p/2} & A \\ A^* & B \end{pmatrix}. \tag{17}$$

So

$$\begin{pmatrix} (T_1T_1^*)^p & 0 \\ 0 & 0 \end{pmatrix} = E(TT^*)^pE = \begin{pmatrix} (T_1T_1^*)^p + AA^* & 0 \\ 0 & 0 \end{pmatrix}, \tag{18}$$

and hence $A = 0$ and $TT^* = \begin{pmatrix} T_1T_1^* & 0 \\ 0 & B^{2/p} \end{pmatrix}$. Since

$$TT^* = \begin{pmatrix} T_1T_1^* + T_2T_2^* & T_2T_3^* \\ T_3T_2^* & T_3T_3^* \end{pmatrix}, \tag{19}$$

it follows that $T_2 = 0$ and hence T is reduced by $\mathcal{M}$. □

Theorem 11. *If $T^* \in \mathcal{B}(\mathcal{H})$ is p-hyponormal, $S \in \mathcal{B}(\mathcal{K})$ is injective (p,k)-quasihyponormal, and if $XT = SX$ for $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then $XT^* = S^*X$.*

Proof. Since by assumption $XT = SX$, we can see that $(\ker X)^\perp$ and $\overline{\operatorname{ran} X}$ are invariant subspaces of T^* and S, respectively. Therefore by Lemma 9 we have that $T^*|_{(\ker X)^\perp}$ is p-hyponormal and $S|_{\overline{\operatorname{ran} X}}$ is also (p,k)-quasihyponormal. Now consider the decompositions $\mathcal{H} = (\ker X)^\perp \oplus \ker X$ and $\mathcal{K} = \overline{\operatorname{ran} X} \oplus (\overline{\operatorname{ran} X})^\perp$. Then we have the following matrix representations:

$$T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}, \qquad S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{20}$$

where T_1^* is p-hyponormal, S_1 is injective (p,k)-quasihyponormal and X_1 is injective with dense range. Therefore we have

$$X_1T_1x = XTx = SXx = S_1X_1x \quad \text{for } x \in (\ker X)^\perp. \tag{21}$$

That is, $X_1T_1 = S_1X_1$ and hence T_1 and S_1 are normal by Theorem 8 and $X_1T_1^* = S_1^*X_1$ by the Fuglede-Putnam theorem. Therefore by Lemma 10, $(\ker X)^\perp$ and $\overline{\operatorname{ran} X}$ reduces T^* and S, respectively. Hence we obtain the $XT^* = S^*X$. □

In Lemma 10, we can drop the injective condition if T is p-hyponormal instead of (p,k)-quasihyponormality (see [7, Lemma 2]). Therefore we recapture a generalized Fuglede-Putnam theorem for p-hyponormal operators.

Corollary 12. *Let $T^* \in \mathcal{B}(\mathcal{H})$ is a p-hyponormal operator and let $S \in \mathcal{B}(\mathcal{K})$ is a p-hyponormal operator. If $XT = SX$ for $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then $XT^* = S^*X$.*

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References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations and Operator Theory **13** (1990), no. 3, 307–315.
[2] S. C. Arora and P. Arora, *On p-quasihyponormal operators for* $0 < P < 1$, Yokohama Mathematical Journal **41** (1993), no. 1, 25–29.
[3] S. L. Campbell and B. C. Gupta, *On k-quasihyponormal operators*, Mathematica Japonica **23** (1978/1979), no. 2, 185–189.
[4] M. Chō and M. Itoh, *Putnam's inequality for p-hyponormal operators*, Proceedings of the American Mathematical Society **123** (1995), no. 8, 2435–2440.
[5] B. P. Duggal, *On p-quasihyponormal operators for* $0 < p < 1$, Yokohama Mathematical Journal **41** (1993), 25–29.
[6] B. C. Gupta and P. B. Ramanujan, *On k-quasihyponormal operators II*, The Tohoku Mathematical Journal **20** (1968), 417–424.
[7] I. H. Jeon and B. P. Duggal, *p-hyponormal operators and quasisimilarity*, Integral Equations and Operator Theory **49** (2004), no. 3, 397–403.
[8] I. H. Kim, *On (p,k)-quasihyponormal operators*, Mathematical Inequalities & Applications **7** (2004), no. 4, 629–638.
[9] I. H. Sheth, *On hyponormal operators*, Proceedings of the American Mathematical Society **17** (1966), 998–1000.

[10] J. G. Stampfli and B. L. Wadhwa, *An asymmetric Putnam-Fuglede theorem for dominant operators*, Indiana University Mathematics Journal **25** (1976), no. 4, 359–365.
[11] K. Tanahashi, A. Uchiyama, and M. Chō, *Isolated points of spectrum of* (p,k)*-quasihyponormal operators*, Linear Algebra and its Applications **382** (2004), 221–229.
[12] A. Uchiyama, *An example of a* p*-quasihyponormal operato*, Yokohama Mathematical Journal **46** (1999), no. 2, 179–180.
[13] J. P. Williams, *Operators similar to their adjoints*, Proceedings of the American Mathematical Society **20** (1969), 121–123.

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