IMPLICIT PREDICTOR-CORRECTOR ITERATION PROCESS FOR FINITELY MANY ASYMPTOTICALLY (QUASI-)NONEXPANSIVE MAPPINGS

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We study an implicit predictor-corrector iteration process for finitely many asymptotically quasi-nonexpansive self-mappings on a nonempty closed convex subset of a Banach space E. We derive a necessary and sufficient condition for the strong convergence of this iteration process to a common fixed point of these mappings. In the case E is a uniformly convex Banach space and the mappings are asymptotically nonexpansive, we verify the weak (resp., strong) convergence of this iteration process to a common fixed point of these mappings if Opial's condition is satisfied (resp., one of these mappings is semicompact). Our results improve and extend earlier and recent ones in the literature.

1. Introduction and preliminaries

Let E be a real Banach space equipped with norm $\|\cdot\|$, let C be a nonempty subset of E, and let $T : C \to C$. The set $F(T) = \{x \in C : Tx = x\}$ consists of all fixed points of T.

Definition 1.1. T is said to be

(1) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C; \tag{1.1}$$

(2) *asymptotically nonexpansive* [3] if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - T^n y\right\| \leq k_n \|x - y\|, \quad \forall x, y \in C,\ n \geq 1; \tag{1.2}$$

(3) *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$, and there exists a sequence $\{k_n\}_{n=1}^{\infty} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\|T^n x - p\right\| \leq k_n \|x - p\|, \quad \forall x \in C,\ p \in F(T),\ n \geq 1; \tag{1.3}$$

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(4) *semicompact* [9] if for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a strongly convergent subsequence of $\{x_n\}$.

The class of asymptotically nonexpansive mappings, as a natural extension of that of nonexpansive mappings, was introduced by Goebel and Kirk [3]. They proved that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T on C has a fixed point. Furthermore, the study of iterative construction for fixed points of asymptotically nonexpansive mappings began in 1978. Bose [1] first proved that if the uniformly convex Banach space E satisfies Opial's condition [5], then $\{T^n x\}$ converges weakly to a fixed point of T, provided T is asymptotically regular at x, that is, $\lim_{n\to\infty} \|T^n x - T^{n+1}x\| = 0$. A Banach space E is said to satisfy *Opial's condition* [5] if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, one has

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E,\ y \neq x. \tag{1.4}$$

It is well known that every Hilbert space satisfies Opial's condition (see, e.g., [5]).

Xu and Ori [8] first introduced an implicit iteration process for N nonexpansive mappings in a Hilbert space and proved the following weak convergence theorem.

Theorem 1.2 (see [8]). *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings on C such that $F = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $x_0 \in C$ and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $\{x_n\}$ defined implicity by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{n(\bmod N)} x_n, \quad n \geq 1, \tag{1.5}$$

converges weakly to a common fixed point of mappings $\{T_j\}_{j=1}^N$.

Later, Sun [7] introduced and studied another implicit iteration process

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_{n(\bmod N)}^{l_n+1} x_n, \quad n \geq 1, \tag{1.6}$$

for N asymptotically quasi-nonexpansive self-mappings $\{T_j\}_{j=1}^N$ on a nonempty bounded closed convex subset C of a Banach space E, where $\{\alpha_n\}$ is a sequence in $(0,1)$, x_0 is an initial point in C, and $n = l_n N + n\ (\bmod N)$. Moreover, he proved that the sequence $\{x_n\}$ defined by his iteration process converges strongly to a common fixed point of $\{T_j\}_{j=1}^N$ under suitable conditions.

At the same time, in [10], Zhou and Chang introduced and studied the following implicit iteration process:

$$x_n = \alpha_n x_{n-1} + \beta_n T_{n(\bmod N)}^n x_n + \gamma_n u_n, \quad n \geq 1, \tag{1.7}$$

for N asymptotically nonexpansive self-mappings $\{T_j\}_{j=1}^N$ on a nonempty closed convex subset C of a Banach space E, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are three sequences in $[0,1]$, x_0 is an initial point in C, and $\{u_n\}$ is a bounded sequence in C. Moreover, they proved that the sequence $\{x_n\}$ defined by their iteration process converges weakly to a common fixed point of $\{T_j\}_{j=1}^N$ under suitable conditions.

As indicated in [10], if $T_1, T_2, \dots, T_N : C \to C$ are N asymptotically nonexpansive mappings, then there exists a sequence, called *common Lipschitz constants*, $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for each $i = 1,2,\dots,N$,

$$||T_i^n x - T_i^n y|| \le k_n \|x - y\|, \quad \forall x, y \in C,\ n \ge 1. \tag{1.8}$$

A similar situation occurs when $T_1, T_2, \dots, T_N$ are asymptotically quasi-nonexpansive. By convention, we write $T_n := T_{n(\mathrm{mod}N)}$, for integer $n \ge 1$, with the mod function taking values in the set $\{1,2,\dots,N\}$. In other words, if $n = l_n N + q$ for some unique integers $l_n \ge 0$ and $1 \le q \le N$, then we set $T_n = T_q$.

In this paper, we introduce the following implicit predictor-corrector iteration process with an auxiliary finite family of asymptotically quasi-nonexpansive self-mappings on C.

Definition 1.3 (basic setup). Let C be a nonempty closed convex subset of a Banach space E, and let $\{T_1, T_2, \dots, T_N\}$ and $\{\widehat{T}_1, \widehat{T}_2, \dots, \widehat{T}_{\widehat{N}}\}$ be two families of asymptotically quasi-nonexpansive mappings from C into C with common Lipschitz constants $\{k_n\}$ and $\{\widehat{k}_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$ and $\sum_{n=1}^{\infty}(\widehat{k}_n - 1) < +\infty$, respectively. Let $\{x_n\}$ be an iterative sequence in C generated from an arbitrary $x_0 \in C$ by the following three steps.

Auxiliary step. With $x_{n-1}(n \ge 1)$ established, y_n is computed implicitly by

$$y_n = \widehat{\alpha}_n x_{n-1} + \widehat{\beta}_n \widehat{T}_n^{\widehat{l}_n} y_n + \widehat{\gamma}_n \widehat{u}_n. \tag{1.9a}$$

Predictor step. With y_n obtained in the auxiliary step, z_n is computed implicitly by

$$z_n = \overline{\alpha}_n y_n + \overline{\beta}_n T_n^{l_n} z_n + \overline{\gamma}_n \overline{u}_n. \tag{1.9b}$$

Corrector step. With z_n obtained in the predictor step, x_n is computed explicitly by

$$x_n = \alpha_n y_n + \beta_n T_n^{l_n} z_n + \gamma_n u_n. \tag{1.9c}$$

Here, $T_n := T_{n(\mathrm{mod}N)}$ and $\widehat{T}_n := \widehat{T}_{n(\mathrm{mod}\widehat{N})}$ for $n = 1,2,\dots$. On the other hand, $\{u_n\}_{n=1}^{\infty}$, $\{\widehat{u}_n\}_{n=1}^{\infty}$, $\{\overline{u}_n\}_{n=1}^{\infty}$ are three bounded sequences in C; and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\widehat{\alpha}_n\}_{n=1}^{\infty}$, $\{\overline{\alpha}_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\widehat{\beta}_n\}_{n=1}^{\infty}$, $\{\overline{\beta}_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\widehat{\gamma}_n\}_{n=1}^{\infty}$, $\{\overline{\gamma}_n\}_{n=1}^{\infty}$ are nine real sequences in $[0,1]$ such that

$$\begin{gathered}
\alpha_n + \beta_n + \gamma_n = 1 \quad (\forall n \ge 1), \quad \sum_{n=1}^{\infty} \gamma_n < +\infty, \\
\widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1 \quad (\forall n \ge 1), \quad \sum_{n=1}^{\infty} \widehat{\gamma}_n < +\infty, \\
\overline{\alpha}_n + \overline{\beta}_n + \overline{\gamma}_n = 1 \quad (\forall n \ge 1), \quad \sum_{n=1}^{\infty} \overline{\gamma}_n < +\infty, \\
0 < \widehat{\beta}_n, \overline{\beta}_n \le c < K^{-1} \quad (\forall n \ge 1), \quad K = \max\left\{\sup_{n\ge1} k_n, \sup_{n\ge1} \widehat{k}_n\right\} \ge 1.
\end{gathered} \tag{1.10}$$

Remark 1.4. Since $0<\widehat{\beta}_n,\overline{\beta}_n\leq c<K^{-1}$, it is clear that the mappings $y\mapsto\widehat{\alpha}_n x_{n-1}+\widehat{\beta}_n\widehat{T}_n^{\widehat{l}_n}y+\widehat{\gamma}_n\widehat{u}_n$ and $z\mapsto\overline{\alpha}_n y_n+\overline{\beta}_n T_n^{l_n}z+\overline{\gamma}_n\overline{u}_n$ are two contractions from the nonempty closed convex set C into itself. Thus, by the Banach contraction principle, there exist the unique points $y_n,z_n\in C$ such that (1.9a) and (1.9b) hold, respectively. Therefore, the sequence $\{x_n\}$ is well defined.

Our aim is to consider and study the strong and weak convergences of the above implicit predictor-corrector iteration process. To this end, we need the following lemmas.

Lemma 1.5. *Let $\{b_n\}$, $\{\overline{b}_n\}$, $\{\widehat{b}_n\}$ be three nonnegative real sequences with finite sums. Then $\sum_{n=1}^{\infty}\lambda_n<+\infty$, where $\lambda_n=(1+b_n)(1+\overline{b}_n)(1+\widehat{b}_n)-1$ for each ≥ 1.*

Lemma 1.6 (see [10]). *Let $\{a_n\}$, $\{\lambda_n\}$, $\{\mu_n\}$ be three nonnegative real sequences such that $\sum_{n=1}^{\infty}\lambda_n<+\infty$, $\sum_{n=1}^{\infty}\mu_n<+\infty$, and*

$$a_{n+1}\leq(1+\lambda_n)a_n+\mu_n,\quad\forall n\geq 1.\tag{1.11}$$

Then $\lim_{n\to\infty}a_n$ exists.

Lemma 1.7 (see [6]). *Let E be a uniformly convex Banach space, $\{t_n\}\subset[b,c]\subset(0,1)$, and $\{x_n\},\{y_n\}\subset E$. If $\lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\|=d<+\infty$, $\limsup_{n\to\infty}\|x_n\|\leq d$, and $\limsup_{n\to\infty}\|y_n\|\leq d$, then $\lim_{n\to\infty}\|x_n-y_n\|=0$.*

Lemma 1.8 (demiclosed principle [2]). *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $T:C\to C$ be an asymptotically nonexpansive mapping with $F(T)\neq\emptyset$. Then $I-T$ is demiclosed at zero, that is, for any sequence $\{x_n\}\subset C$,*

$$\begin{array}{r}x_n\longrightarrow q\in C\text{ weakly}\\(I-T)x_n\longrightarrow 0\text{ strongly}\end{array}\Longrightarrow(I-T)q=0.\tag{1.12}$$

2. Main results

Lemma 2.1. *Let C be a nonempty closed convex subset of a Banach space E, and let $\{T_i\}_{i=1}^{N}$ and $\{\widehat{T}_j\}_{j=1}^{\widehat{N}}$ be two finite families of asymptotically quasi-nonexpansive self-mappings on C such that $\bigcap_{i=1}^{N}F(T_i)\cap\bigcap_{j=1}^{\widehat{N}}F(\widehat{T}_j)\neq\emptyset$. If $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are the iterative sequences defined by (1.9a), (1.9b), and (1.9c), then for each $p\in\bigcap_{i=1}^{N}F(T_i)\cap\bigcap_{j=1}^{\widehat{N}}F(\widehat{T}_j)$, there hold*

$$\lim_{n\to\infty}||x_n-p||=d,\qquad\limsup_{n\to\infty}||y_n-p||\leq d,\qquad\limsup_{n\to\infty}||z_n-p||\leq d.\tag{2.1}$$

Proof. Since $\{u_n\}_{n=1}^{\infty}$, $\{\widehat{u}_n\}_{n=1}^{\infty}$, $\{\overline{u}_n\}_{n=1}^{\infty}$ are three bounded sequences in C, for any given $p\in\bigcap_{i=1}^{N}F(T_i)\cap\bigcap_{j=1}^{\widehat{N}}F(\widehat{T}_j)$, we have

$$M:=\max\left\{\sup_{n\geq1}||u_n-p||,\sup_{n\geq1}||\widehat{u}_n-p||,\sup_{n\geq1}||\overline{u}_n-p||\right\}<+\infty.\tag{2.2}$$

Note that $1-\overline{\beta}_n k_{l_n} \geq 1-cK>0$ and $1-\widehat{\beta}_n\widehat{k}_{\widehat{l}_n} \geq 1-cK>0$. Put

$$L=\frac{1}{1-cK}, \qquad b_n=\beta_n(k_{l_n}-1), \qquad \overline{b}_n=\frac{1-\overline{\beta}_n}{1-\overline{\beta}_n k_{l_n}}-1, \qquad \widehat{b}_n=\frac{1-\widehat{\beta}_n}{1-\widehat{\beta}_n\widehat{k}_{\widehat{l}_n}}-1. \tag{2.3}$$

Then we have

$$\begin{aligned}
&0 \leq b_n=\beta_n(k_{l_n}-1) \leq k_{l_n}-1, \qquad 1+b_n \leq K,\\
&0 \leq \overline{b}_n=\frac{\overline{\beta}_n(k_{l_n}-1)}{1-\overline{\beta}_n k_{l_n}} \leq L(k_{l_n}-1), \qquad 1+\overline{b}_n \leq L,\\
&0 \leq \widehat{b}_n=\frac{\widehat{\beta}_n(\widehat{k}_{\widehat{l}_n}-1)}{1-\widehat{\beta}_n\widehat{k}_{\widehat{l}_n}} \leq L(\widehat{k}_{\widehat{l}_n}-1), \qquad 1+\widehat{b}_n \leq L.
\end{aligned} \tag{2.4}$$

Observe that

$$\begin{aligned}
||y_n-p|| &= ||\widehat{\alpha}_n(x_{n-1}-p)+\widehat{\beta}_n(\widehat{T}_n^{\widehat{l}_n}y_n-p)+\widehat{\gamma}_n(\widehat{u}_n-p)||\\
&\leq \widehat{\alpha}_n||x_{n-1}-p||+\widehat{\beta}_n\widehat{k}_{\widehat{l}_n}||y_n-p||+\widehat{\gamma}_n||\widehat{u}_n-p||.
\end{aligned} \tag{2.5}$$

It follows

$$\begin{aligned}
||y_n-p|| &\leq \frac{\widehat{\alpha}_n}{1-\widehat{\beta}_n\widehat{k}_{\widehat{l}_n}}||x_{n-1}-p||+\frac{\widehat{\gamma}_n}{1-\widehat{\beta}_n\widehat{k}_{\widehat{l}_n}}||\widehat{u}_n-p||\\
&\leq \frac{1-\widehat{\beta}_n}{1-\widehat{\beta}_n\widehat{k}_{\widehat{l}_n}}||x_{n-1}-p||+LM\widehat{\gamma}_n\\
&= (1+\widehat{b}_n)||x_{n-1}-p||+LM\widehat{\gamma}_n.
\end{aligned} \tag{2.6}$$

Similarly,

$$\begin{aligned}
||z_n-p|| &= ||\overline{\alpha}_n(y_n-p)+\overline{\beta}_n(T_n^{l_n}z_n-p)+\overline{\gamma}_n(\overline{u}_n-p)||\\
&\leq \overline{\alpha}_n||y_n-p||+\overline{\beta}_n k_{l_n}||z_n-p||+\overline{\gamma}_n||\overline{u}_n-p||
\end{aligned} \tag{2.7}$$

Consequently,

$$\begin{aligned}
||z_n-p|| &\leq \frac{\overline{\alpha}_n}{1-\overline{\beta}_n k_{l_n}}||y_n-p||+\frac{\overline{\gamma}_n}{1-\overline{\beta}_n k_{l_n}}||\overline{u}_n-p||\\
&\leq \frac{1-\overline{\beta}_n}{1-\overline{\beta}_n k_{l_n}}||y_n-p||+LM\overline{\gamma}_n\\
&= (1+\overline{b}_n)||y_n-p||+LM\overline{\gamma}_n.
\end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned}
\|x_n - p\| &= \|\alpha_n(y_n - p) + \beta_n(T_n^{l_n}z_n - p) + \gamma_n(u_n - p)\| \\
&\le \alpha_n\|y_n - p\| + \beta_n k_{l_n}\|z_n - p\| + \gamma_n\|u_n - p\| \\
&\le (1-\beta_n)\|y_n - p\| + \beta_n k_{l_n}\left[(1+\overline{b}_n)\|y_n - p\| + LM\overline{\gamma}_n\right] + \gamma_n M \\
&\le (1+\beta_n(k_{l_n}-1))(1+\overline{b}_n)\|y_n - p\| + M\left[KL\overline{\gamma}_n + \gamma_n\right] \\
&\le (1+b_n)(1+\overline{b}_n)\|y_n - p\| + KLM\left[\overline{\gamma}_n + \gamma_n\right] \\
&\le (1+b_n)(1+\overline{b}_n)\left[(1+\widehat{b}_n)\|x_{n-1} - p\| + LM\widehat{\gamma}_n\right] + KLM\left[\overline{\gamma}_n + \gamma_n\right] \\
&\le (1+b_n)(1+\overline{b}_n)(1+\widehat{b}_n)\|x_{n-1} - p\| + KL^2M\widehat{\gamma}_n + KLM\left[\overline{\gamma}_n + \gamma_n\right] \\
&\le (1+b_n)(1+\overline{b}_n)(1+\widehat{b}_n)\|x_{n-1} - p\| + KL^2M\left[\gamma_n + \overline{\gamma}_n + \widehat{\gamma}_n\right] \\
&= (1+\lambda_n)\|x_{n-1} - p\| + \mu_n,
\end{aligned} \tag{2.9}$$

where $\lambda_n = (1+b_n)(1+\overline{b}_n)(1+\widehat{b}_n) - 1$, and $\mu_n = KL^2M[\gamma_n + \overline{\gamma}_n + \widehat{\gamma}_n]$.

Since $\sum_{n=1}^{\infty}(k_{l_n} - 1) < +\infty$ and $\sum_{n=1}^{\infty}(\widehat{k}_{\widehat{l}_n} - 1) < +\infty$, it follows from (2.4) that $\sum_{n=1}^{\infty} b_n < +\infty$, $\sum_{n=1}^{\infty}\overline{b}_n < +\infty$, and $\sum_{n=1}^{\infty}\widehat{b}_n < +\infty$. Hence, we derive $\sum_{n=1}^{\infty}\lambda_n < +\infty$ by Lemma 1.5. Note that $\sum_{n=1}^{\infty}\gamma_n < +\infty$, $\sum_{n=1}^{\infty}\overline{\gamma}_n < +\infty$, and $\sum_{n=1}^{\infty}\widehat{\gamma}_n < +\infty$. This provides $\sum_{n=1}^{\infty}\mu_n < +\infty$. By Lemma 1.6, $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = d$.

Since $\lim_{n\to\infty}\widehat{b}_n = \lim_{n\to\infty}\widehat{\gamma}_n = 0$, from (2.6), we obtain

$$\limsup_{n\to\infty}\|y_n - p\| \le \limsup_{n\to\infty}(1+\widehat{b}_n)\|x_{n-1} - p\| + LM\limsup_{n\to\infty}\widehat{\gamma}_n \le d. \tag{2.10}$$

Further, since $\lim_{n\to\infty}\overline{b}_n = \lim_{n\to\infty}\overline{\gamma}_n = 0$, from (2.8), we obtain

$$\limsup_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}(1+\overline{b}_n)\|y_n - p\| + LM\limsup_{n\to\infty}\overline{\gamma}_n \le d. \tag{2.11}$$

□

Theorem 2.2. *Let C be a nonempty closed convex subset of a Banach space E. Let $\{T_i\}_{i=1}^{N}$ and $\{\widehat{T}_j\}_{j=1}^{\widehat{N}}$ be two finite families of asymptotically quasi-nonexpansive self-mappings on C such that $F := \bigcap_{i=1}^{N}F(T_i) \cap \bigcap_{j=1}^{\widehat{N}}F(\widehat{T}_j) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence defined by (1.9a), (1.9b), and (1.9c). Then $\{x_n\}$ converges strongly to an element of F if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{2.12}$$

Proof. The necessity is obvious. For the sufficiency, we assume $\liminf_{n\to\infty} d(x_n, F) = 0$. Let p be any given element in F. Then from (2.9), we obtain

$$\|x_n - p\| \le (1+\lambda_n)\|x_{n-1} - p\| + \mu_n, \tag{2.13}$$

where $\sum_{n=1}^{\infty}\lambda_n < +\infty$ and $\sum_{n=1}^{\infty}\mu_n < +\infty$. Taking the infimum over all $p \in F$, we get

$$d(x_n,F) \leq (1+\lambda_n)d(x_{n-1},F)+\mu_n. \tag{2.14}$$

Hence, $\lim_{n\to\infty} d(x_n,F)$ exists. Furthermore, we have $\lim_{n\to\infty} d(x_n,F) = 0$.

By Lemma 2.1, we know that $\lim_{n\to\infty}\|x_n - p\|$ exists. Hence $\{x_n\}$ is bounded. Put $\delta_n = \lambda_n\|x_{n-1}-p\|+\mu_n$. Then $\sum_{n=1}^{\infty}\delta_n < +\infty$, and (2.13) can be rewritten as

$$||x_n - p|| \leq ||x_{n-1}-p||+\delta_n. \tag{2.15}$$

For arbitrary $\varepsilon > 0$, choose N_0 such that $d(x_{N_0},F) < \varepsilon/4$ and $\sum_{j=N_0}^{\infty}\delta_j < \varepsilon/4$. Consequently, for all $n,m \geq N_0$, we have

$$\begin{aligned} ||x_n - x_m|| &\leq ||x_n - p|| + ||x_m - p|| \\ &\leq ||x_{N_0} - p|| + \sum_{j=N_0+1}^{n}\delta_j + ||x_{N_0}-p|| + \sum_{j=N_0+1}^{m}\delta_j \\ &\leq 2||x_{N_0}-p|| + 2\sum_{j=N_0}^{\infty}\delta_j. \end{aligned} \tag{2.16}$$

Taking the infimum over all $p \in F$, we obtain

$$||x_n - x_m|| \leq 2d(x_{N_0},F) + 2\sum_{j=N_0}^{\infty}\delta_j \leq \frac{2\varepsilon}{4}+\frac{2\varepsilon}{4} = \varepsilon. \tag{2.17}$$

This shows that $\{x_n\}_{n=1}^{\infty}$ is Cauchy. Let $\lim_{n\to\infty} x_n = u$. It is easy to verify that F is closed. Since $\lim_{n\to\infty} d(x_n,F) = 0$, we must have that $u \in F$. □

As a consequence of Lemma 2.1, the iterated sequence $\{x_n\}$ is bounded. If the underlying space E is reflexive, then we can expect that its weak cluster points provide common fixed points of $T_1,T_2,\dots,T_N$. This leads to the following theorem.

Theorem 2.3. *Let E be a uniformly convex Banach space, let C be a nonempty closed convex subset of E, and let $\{T_i\}_{i=1}^{N}$ (resp., $\{\widehat{T}_j\}_{j=1}^{\widehat{N}}$) be a finite family of asymptotically nonexpansive (resp., asymptotically quasi-nonexpansive) self-mappings on C such that $\bigcap_{j=1}^{\widehat{N}} F(\widehat{T}_j) \cap \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose $\lim_{n\to\infty}\widehat{\beta}_n = 0$ and $\{\beta_n\}_{n=1}^{\infty} \subset [b,c] \subset (0,K^{-1})$, where K is as in (1.10). Then every weak cluster point of the bounded iterative sequence $\{x_n\}$ defined by (1.9a), (1.9b), and (1.9c) belongs to $\bigcap_{i=1}^{N} F(T_i)$.*

Proof. Let $p \in \bigcap_{j=1}^{\widehat{N}} F(\widehat{T}_j) \cap \bigcap_{i=1}^{N} F(T_i)$. By Lemma 2.1, we have

$$\lim_{n\to\infty}||x_n - p|| = d, \qquad \limsup_{n\to\infty}||y_n - p|| \leq d, \qquad \limsup_{n\to\infty}||z_n - p|| \leq d. \tag{2.18}$$

Obviously, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded sequences in C.

Observe that

$$\|x_n - p\| = \|(1-\beta_n)[y_n - p + \gamma_n(u_n - y_n)] + \beta_n[T_n^{l_n} z_n - p + \gamma_n(u_n - y_n)]\| \longrightarrow d, \tag{2.19}$$

as $n \to \infty$. Since $\lim_{n\to\infty} \gamma_n = 0$ and $\{u_n\}$ is bounded, we have

$$\begin{aligned} \limsup_{n\to\infty} \|y_n - p + \gamma_n(u_n - y_n)\| &\leq \limsup_{n\to\infty} [\|y_n - p\| + \gamma_n\|u_n - y_n\|] \leq d, \\ \limsup_{n\to\infty} \|T_n^{l_n} z_n - p + \gamma_n(u_n - y_n)\| &\leq \limsup_{n\to\infty} [k_{l_n}\|z_n - p\| + \gamma_n\|u_n - y_n\|] \leq d. \end{aligned} \tag{2.20}$$

It follows from Lemma 1.7 that

$$\lim_{n\to\infty} \|T_n^{l_n} z_n - y_n\| = 0. \tag{2.21}$$

Thus,

$$\begin{aligned} \lim_{n\to\infty} \|z_n - y_n\| &= \lim_{n\to\infty} \|\overline{\alpha}_n y_n + \overline{\beta}_n T_n^{l_n} z_n + \overline{\gamma}_n \overline{u}_n - y_n\| \\ &= \lim_{n\to\infty} \|\overline{\beta}_n (T_n^{l_n} z_n - y_n) + \overline{\gamma}_n(\overline{u}_n - y_n)\| = 0. \end{aligned} \tag{2.22}$$

Similarly,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - y_n\| &= \lim_{n\to\infty} \|\alpha_n y_n + \beta_n T_n^{l_n} z_n + \gamma_n u_n - y_n\| \\ &= \lim_{n\to\infty} \|\beta_n (T_n^{l_n} z_n - y_n) + \gamma_n(u_n - y_n)\| = 0. \end{aligned} \tag{2.23}$$

Moreover,

$$\begin{aligned} \|y_n - x_{n-1}\| &= \|\widehat{\alpha}_n x_{n-1} + \widehat{\beta}_n \widehat{T}_n^{\widehat{l}_n} y_n + \widehat{\gamma}_n \widehat{u}_n - x_{n-1}\| \\ &= \|\widehat{\beta}_n (\widehat{T}_n^{\widehat{l}_n} y_n - x_{n-1}) + \widehat{\gamma}_n (\widehat{u}_n - x_{n-1})\| \\ &\leq \widehat{\beta}_n \|\widehat{T}_n^{\widehat{l}_n} y_n - x_{n-1}\| + \widehat{\gamma}_n \|\widehat{u}_n - x_{n-1}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{2.24}$$

since $\lim_{n\to\infty} \widehat{\beta}_n = \lim_{n\to\infty} \widehat{\gamma}_n = 0$. As a result, we have

$$\|x_n - x_{n-1}\| \leq \|x_n - y_n\| + \|y_n - x_{n-1}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{2.25}$$

It forces

$$\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0, \quad \text{for each } i = 1, 2, \ldots, N. \tag{2.26}$$

On the other hand, we have

$$\begin{aligned} \|x_n - T_n^{l_n} x_n\| &\leq \|x_n - y_n\| + \|y_n - T_n^{l_n} z_n\| + \|T_n^{l_n} z_n - T_n^{l_n} x_n\| \\ &\leq \|x_n - y_n\| + \|y_n - T_n^{l_n} z_n\| + k_{l_n}\|z_n - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{2.27}$$

As $n = l_nN + n \pmod N$ for $n > N$, we get

$$n - N = (l_n - 1)N + n \pmod N, \tag{2.28}$$

and hence $l_{n-N} = l_n - 1$. Thus, we have

$$T_n^{l_n - 1} = T_{n-N}^{l_{n-N}}. \tag{2.29}$$

Consequently, we derive

$$\begin{aligned}
||x_n - T_nx_n|| &\le ||x_n - T_n^{l_n}x_n|| + ||T_n^{l_n}x_n - T_nx_n|| \\
&\le ||x_n - T_n^{l_n}x_n|| + K||T_n^{l_n-1}x_n - x_n|| \\
&= ||x_n - T_n^{l_n}x_n|| + K||T_{n-N}^{l_{n-N}}x_n - x_n|| \\
&\le ||x_n - T_n^{l_n}x_n|| + K\big[||T_{n-N}^{l_{n-N}}x_n - T_{n-N}^{l_{n-N}}x_{n-N}|| \\
&\qquad + ||T_{n-N}^{l_{n-N}}x_{n-N} - x_{n-N}|| + ||x_{n-N} - x_n||\big] \\
&\le ||x_n - T_n^{l_n}x_n|| + K\big[(1+K)||x_{n-N} - x_n|| \\
&\qquad + ||T_{n-N}^{l_{n-N}}x_{n-N} - x_{n-N}||\big] \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.30}$$

This implies that for each $j = 1,2,\dots,N$,

$$\begin{aligned}
||x_n - T_{n+j}x_n|| &\le ||x_n - x_{n+j}|| + ||x_{n+j} - T_{n+j}x_{n+j}|| + ||T_{n+j}x_{n+j} - T_{n+j}x_n|| \\
&\le (1+K)||x_n - x_{n+j}|| + ||x_{n+j} - T_{n+j}x_{n+j}|| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{2.31}$$

Note that the closedness and convexity of C imply the weak closedness of C. Let $\widetilde{x} \in C$ be any weak cluster point of the bounded sequence $\{x_n\}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \to \widetilde{x}$ weakly (see, e.g., [4, page 313]). Since the pool of mappings $\{T_i : 1 \le i \le N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that for some integer $l \in \{1,2,\dots,N\}$, $T_{n_i} = T_l$ for all $i \ge 1$. Then it follows from (2.31) that for each $j = 1,2,\dots,N$,

$$x_{n_i} - T_{l+j}x_{n_i} \longrightarrow 0, \quad \text{as } i \longrightarrow \infty, \tag{2.32}$$

that is, for each $j = 1,2,\dots,N$,

$$x_{n_i} - T_jx_{n_i} \longrightarrow 0, \quad \text{as } i \longrightarrow \infty. \tag{2.33}$$

By Lemma 1.8, we can conclude that $\widetilde{x} \in \bigcap_{j=1}^N F(T_j)$. □

Theorem 2.4. *In addition to the conditions in Theorem 2.3, assume further that* $\emptyset \ne \bigcap_{i=1}^N F(T_i) \subseteq \bigcap_{j=1}^{\hat{N}} F(\hat{T}_j)$.

(a) *If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to an element of* $\bigcap_{i=1}^N F(T_i)$.

(b) *If one of $\{T_i\}_{i=1}^N$ is semicompact, then $\{x_n\}$ converges strongly to an element of* $\bigcap_{i=1}^N F(T_i)$.

Proof. We continue the argument in the proof of Theorem 2.3.

For (a), we claim that $\{x_n\}$ is weakly convergent. Were this false, there existed another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to \overline{x} \in C$ weakly and $\overline{x} \neq \widetilde{x}$. Utilizing the same argument as in Theorem 2.3, we can prove that $\overline{x} \in \bigcap_{j=1}^{N} F(T_j)$. Note that by Lemma 2.1, both $\lim_{n\to\infty} \|x_n - \widetilde{x}\|$ and $\lim_{n\to\infty} \|x_n - \overline{x}\|$ exist. It follows from the Opial condition of E that

$$\begin{aligned}
\lim_{n\to\infty} ||x_n - \widetilde{x}|| &= \liminf_{i\to\infty} ||x_{n_i} - \widetilde{x}|| \\
&< \liminf_{i\to\infty} ||x_{n_i} - \overline{x}|| = \lim_{n\to\infty} ||x_n - \overline{x}|| = \liminf_{j\to\infty} ||x_{n_j} - \overline{x}|| \\
&< \liminf_{j\to\infty} ||x_{n_j} - \widetilde{x}|| = \lim_{n\to\infty} ||x_n - \widetilde{x}||.
\end{aligned} \tag{2.34}$$

This contradiction indicates that $\overline{x} = \widetilde{x}$, and so $\{x_n\}$ converges weakly to $\widetilde{x}$.

For (b), by (2.33), we can assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ exists such that $x_{n_i} \to \hat{x} \in \bigcap_{i=1}^{N} F(T_i)$ in norm. It then follows from Lemma 2.1 that

$$\lim_{n\to\infty} ||x_n - \hat{x}|| = \lim_{i\to\infty} ||x_{n_i} - \hat{x}|| = 0. \tag{2.35}$$

This completes the proof. □

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