

# TURAN'S INEQUALITY FOR APPELL POLYNOMIALS

SLAVKO SIMIC

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We give some necessary and sufficient conditions for the class of Appell polynomials to satisfy well-known Turan's inequality. Among the other corollaries, we apply our results to some classes of orthogonal polynomials.

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## 1. Introduction

A real sequence  $\{a_n\}$ ,  $n = 0, 1, 2, \dots$  generates a sequence of Appell polynomials defined as:

$$A_n(x) := \sum_{k=0}^n a_k \binom{n}{k} x^{n-k}, \quad n = 0, 1, 2, \dots \quad (1.1)$$

This class of polynomials is of importance in real and combinatorial analysis [1, 2]. For example, the classical Bernoulli and Laguerre polynomials belong to Appell class.

It is said that the sequence of polynomials  $\{C_n(x)\}$  have  $T$  property if it satisfies Turan's inequality

$$C_n^2(x) - C_{n-1}(x)C_{n+1}(x) \geq 0, \quad x \in [a, b], \quad n \in N. \quad (1.2)$$

We will also consider inverse Turan's inequality and say that  $\{C_n(x)\}$  have  $T^-$  property if it satisfies

$$C_n^2(x) - C_{n-1}(x)C_{n+1}(x) \leq 0, \quad x \in [a, b], \quad n \in N. \quad (1.3)$$

For  $x \in [-1, 1]$ ,  $T$  property was proved by Turán for Legendre's polynomials [5]; in [3] this property proved Szegő for Laguerre polynomials ( $x \in R$ ).

In 1964, Ikeda proved  $T$  property for Bernoulli polynomials and  $x \in [0, 1]$ .

Nowadays, the possibility of  $T$  ( $T^-$ ) property for various classes of polynomials still attracts the attention of mathematicians.

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### 2. Results

Our task in this paper is to investigate  $T$  property of Appell polynomials in two cases. In the first case we suppose that the generating sequence  $\{a_n\}$  consists of positive numbers. In the second case we assume that all polynomials from the sequence  $\{A_n(x)\}$  have real zeros only.

As concerns the first case, we can formulate the next theorem.

**THEOREM 2.1.** *If the generating sequence  $\{a_n\}$  consists of positive numbers only, then*

- (i) *the sequence  $\{A_n(x)\}$  have  $T (T^-)$  property for  $x \in (0, b]$  if and only if the sequence  $\{a_n\}$  have  $T (T^-)$  property;*
- (ii) *the sequence  $\{B_n(x)\}$  defined by*

$$B_n(x) := (A_n(x)/a_n)^{1/n}; \quad B_0(x) := 1, \quad n \in N, \quad (2.1)$$

*is monotone nondecreasing (nonincreasing) for  $x \in (0, b]$  if and only if the sequence  $\{a_n\}$  have  $T (T^-)$  property.*

From the above theorem follow some  $T$  property criteria.

**PROPOSITION 2.2.** *If the sequence  $\{a_n\}$  of positive numbers have not  $T (T^-)$  property then also  $\{A_n(x)\}$  have not this property for  $x \in [a, b]$ ,  $a \leq 0 < b$ .*

**PROPOSITION 2.3.** *If the sequence  $\{A_n(x)/a_n\}$  have  $T^- (T)$  property for  $x \in (0, b]$  then the sequence  $\{A_n(x)\}$  have  $T (T^-)$  property for  $x \in [0, \infty)$ .*

**PROPOSITION 2.4.** *Define*

$$A_n^{(\sigma)}(x) := \sum_{k=0}^n a_k^\sigma \binom{n}{k} x^{n-k}, \quad \sigma \in R; \quad A_n^{(1)}(x) = A_n(x). \quad (2.2)$$

*If  $\{A_n(x)\}$  have  $T$  property for  $x > 0$  then  $\{A_n^\sigma(x)\}$  have  $T$  property for  $\sigma \geq 0$  and  $T^-$  property for  $\sigma < 0$ .*

*Analogous statement takes place if  $\{A_n(x)\}$  have  $T^-$  property.*

$T$  property implies bounds for  $A_n(x)$ . Namely,

**PROPOSITION 2.5.** *If  $\{A_n(x)\}$  ( $A_0(x) = a_0 = 1$ ,  $a_n > 0$ ,  $n \in N$ ) have  $T$  property for  $x \in (0, b]$ , then*

$$\frac{a_n}{a_1^n} \leq \frac{A_n(x)}{A_1^n(x)} \leq 1, \quad x \in [0, b], \quad n \in N. \quad (2.3)$$

*If  $\{A_n(x)\}$  have  $T^-$  property then the reverse inequalities hold.*

In the sequel we will not assume positivity of the sequence  $\{a_n\}$ , but

**THEOREM 2.6.** *If all the zeros of polynomials  $A_n(x)$ ,  $n \in N$  are real, then*

- (i) *the generating sequence  $\{a_n\}$  have  $T$  property;*
- (ii) *the sequence  $\{A_n(x)\}$  have  $T$  property for  $x \in R$ .*

This theorem is particularly useful in the case of sequences of orthogonal polynomials. Because of orthogonality, all their zeros are real and, as a corollary of Theorem 2.6., we get

PROPOSITION 2.7. *The sequence of Hermite polynomials have T property for  $x \in \mathbb{R}$ .*

Denote by  $\{L_n^{(a)}(x)\}$  the sequence of generalized Laguerre polynomials of order  $a > -1$ . Then

PROPOSITION 2.8. *The sequence  $\{L_n^{(a)}(x)/\binom{n+a}{n}\}$  have T property for  $x \in \mathbb{R}$ .*

For  $a = 0$  we obtain Szegő's theorem concerning Laguerre polynomials, mentioned above.

If the sequence of polynomials  $\{Q_n(x)\}$  is not from Appell class, then the following assertion can be useful.

THEOREM 2.9. *If, for fixed  $t \in [a, b]$ , polynomials  $R_n(x)$  defined by*

$$R_n(x) := \sum_{k=0}^n Q_k(t) \binom{n}{k} x^{n-k}, \quad n \in \mathbb{N}, \tag{2.4}$$

*have all its zeros real, then the sequence  $\{Q_n(x)\}$  have T property for  $x \in [a, b]$ .*

As a corollary we have the next

PROPOSITION 2.10. *If  $\{P_n^{(\lambda)}(x)\}$  denotes the sequence of ultraspherical polynomials with parameter  $\lambda > -1/2$ , then the sequence  $\{P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)\}$  have T property for  $x \in [-1, 1]$ .*

For  $\lambda = 1/2$  we obtain Turan's assertion on Legendre's polynomials and for  $\lambda = 1$  we proved T property for the sequence  $\{U_n(x)/(n+1)\}$ ,  $x \in [-1, 1]$ , where  $U_n(x) = P_n^{(1)}(x)$  is a Tchebychef polynomial of second kind.

### 3. Proofs

To prove part (i) of Theorem 2.1, suppose first T ( $T^-$ ) property of the sequence  $\{A_n(x)\}$  for  $x \in (0, b]$ . Then the polynomial  $P(x)$ ,

$$\begin{aligned} P(x) &:= A_n^2(x) - A_{n-1}(x)A_{n+1}(x) \\ &= a_n^2 - a_{n-1}a_{n+1} + (n-1)(a_n a_{n-1} - a_{n+1} a_{n-2})x + \dots + (a_1^2 - a_0 a_2)x^{2n-2} \end{aligned} \tag{3.1}$$

is nonnegative (nonpositive) for  $x \in (0, b]$ .

Using the identity

$$A'_n(x) = nA_{n-1}(x), \tag{3.2}$$

we obtain

$$P'(x) = (n-1)(A_n(x)A_{n-1}(x) - A_{n+1}(x)A_{n-2}(x)). \tag{3.3}$$

Hence, polynomials  $P(x)$  and  $P'(x)$  are of the same sign, that is,  $P(x)$  is either nonnegative and nondecreasing or nonpositive and nonincreasing for  $x \in (0, b]$ .

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Since it is also continuous in  $x$ , it follows that  $P(0) = a_n^2 - a_{n-1}a_{n+1}$  has the same sign as  $P(x)$ ,  $x \in (0, b]$ .

Suppose now that the sequence  $\{a_n\}$  has  $T$  ( $T^-$ ) property.

Putting  $c_n = c_n(x) := a_n x^{-n}$ ,  $x > 0$ ;  $n = 0, 1, 2, \dots$ , we have to prove that  $T$  ( $T^-$ ) property of  $\{c_n\}$  implies the same property for the sequence  $\{C_n\}$ , where

$$C_n := \sum_{k=0}^n \binom{n}{k} c_k; \quad n = 0, 1, 2, \dots \quad (3.4)$$

It is easy to check that  $T$  ( $T^-$ ) property of  $\{c_n\}$  implies this property for the sequence  $\{c_n^{(1)}\}$ , defined by  $c_n^{(1)} := c_n + c_{n-1}$ .

By induction, the same is valid for sequences  $\{c_n^{(m)}\}$ , where

$$c_n^{(m+1)} := c_n^{(m)} + c_{n-1}^{(m)}, \quad m = 1, 2, \dots \quad (3.5)$$

It is only left to note that  $c_n^{(n)} = C_n$ .

*Remark 3.1.* It follows that  $T$  ( $T^-$ ) property of the sequence  $\{a_n\}$  implies the same property of  $\{A_n(x)\}$  for  $x \in [0, \infty)$ .

To prove part (ii) of Theorem 2.1, assume first that  $\{a_n\}$  have  $T$  property. By (i), the sequence  $\{A_n(t)\}$  also have this property for  $t > 0$ , that is,

$$\frac{A_n(t)}{A_{n+1}(t)} \geq \frac{A_{n-1}(t)}{A_n(t)} \quad (3.6)$$

or, by (3.2),

$$\frac{1}{n+1} \frac{A'_{n+1}(t)}{A_{n+1}(t)} \geq \frac{1}{n} \frac{A'_n(t)}{A_n(t)}. \quad (3.7)$$

Integrating (3.7) over  $t \in [0, x]$ , we get

$$(A_{n+1}(x)/a_{n+1})^{1/(n+1)} \geq (A_n(x)/a_n)^{1/n}; \quad n = 1, 2, \dots \quad (3.8)$$

This means that the sequence  $\{B_n(x)\}$  is monotone nondecreasing for each fixed  $x > 0$ .

$T^-$  case can be treated similarly.

Suppose now that  $\{B_n(x)\}$  is monotone and consider the polynomial  $Q(x)$  defined by

$$Q(x) := (A_{n+1}(x)/a_{n+1})^n - (A_n(x)/a_n)^{n+1}. \quad (3.9)$$

By assumption,  $Q(x)$  is nonnegative (nonpositive) for  $x \in (0, b]$ ,  $b > 0$ .

We have  $Q(0) = 0$  and, by (3.2)

$$Q'(0) = n(n+1)(a_n/a_{n+1} - a_{n-1}/a_n); \quad n = 1, 2, \dots \quad (3.10)$$

Therefore,

$$Q(x) = n(n+1)(a_n/a_{n+1} - a_{n-1}/a_n)x + \dots + \left( (a_0/a_{n+1})^n - (a_0/a_n)^{n+1} \right) x^{n(n+1)}. \quad (3.11)$$

Since  $x$  is independent of  $n$ , we see from (3.11) that, for sufficiently small positive  $x$ , the signs of  $Q(x)$  and  $Q'(0)$  have to be the same, that is, the part (ii) is also proved.

*Proof of Proposition 2.2.* This is a consequence of the assertion (i) of Theorem 2.1.  $\square$

Proof of the next proposition needs the following lemma.

LEMMA 3.2. *If the sequence  $\{b_n\}$ ,  $b_0 := 1$  of positive numbers have  $T$  ( $T^-$ ) property, then the sequence  $\{b_n^{1/n}\}$  is nonincreasing (nondecreasing).*

*Proof.*  $T$  property implies  $b_n^2 \geq b_{n-1}b_{n+1}$ ,  $n \in N$ . Hence

$$(b_0b_2)(b_1b_3)^2(b_2b_4)^3 \cdots (b_{n-1}b_{n+1})^n \leq b_1^2b_2^4b_3^6 \cdots b_n^{2n}. \quad (3.12)$$

This gives  $b_{n+1}^n \leq b_n^{n+1}$ , that is,  $\{b_n^{1/n}\}$  is nonincreasing.

Proof of  $T^-$  case goes along the same lines.  $\square$

*Proof of Proposition 2.3.* Assume that  $\{A_n(x)/a_n\}$  have  $T^-$  property for  $x \in (0, b]$ . Then Lemma 3.2 asserts  $\{(A_n(x)/a_n)^{1/n}\}$  nondecreasing for  $x \in (0, b]$ . By part (ii) of Theorem 2.1 this implies  $T$  property for  $\{a_n\}$  which in turn, by part (i) and Remark 3.1, gives  $T$  property of  $\{A_n(x)\}$  for  $x \in [0, \infty)$ .  $\square$

*Proof of Proposition 2.4.* This is a consequence of Theorem 2.1, part (i) and the fact that, if  $\{a_n\}$  have  $T$  property, then  $\{a_n^\sigma\}$  have  $T$  property for  $\sigma \geq 0$  and  $T^-$  property for  $\sigma < 0$ .  $\square$

*Proof of Proposition 2.5.*  $T$  property and Lemma 3.2 imply  $\{(A_n(x))^{1/n}\}$  nonincreasing. Therefore  $A_n(x)^{1/n} \leq A_1(x)$ .

On the other hand, by Theorem 2.1 part (i),  $T$  property of  $\{A_n(x)\}$  implies  $T$  property for  $\{a_n\}$  which in turn, by part (ii), implies that the sequence  $\{(A_n(x)/a_n)^{1/n}\}$  is nondecreasing. Hence

$$(A_n(x)/a_n)^{1/n} \geq A_1(x)/a_1; \quad x > 0, n = 1, 2, \dots \quad (3.13)$$

This is exactly the left-hand side of the inequality from Proposition 2.4.

The other case can be treated similarly.  $\square$

*Proof of Theorem 2.6.* We will prove first the part (ii) of this theorem. For this purpose, let

$$A_{n+1}(x) = a_0 \prod_{i=0}^n (x - x_i), \quad x_i \in R, n = 0, 1, 2, \dots \quad (3.14)$$

Taking logarithmic derivative, by (3.2), we get

$$(n+1) \frac{A_n(x)}{A_{n+1}(x)} = \frac{A'_{n+1}(x)}{A_{n+1}(x)} = \sum_{i=0}^n \frac{1}{x - x_i}. \quad (3.15)$$

Analogously,

$$(n+1) \frac{nA_{n-1}(x)A_{n+1}(x) - (n+1)A_n^2(x)}{A_{n+1}^2(x)} = \left( \frac{A'_{n+1}(x)}{A_{n+1}(x)} \right)' = - \sum_{i=0}^n \frac{1}{(x - x_i)^2}. \quad (3.16)$$

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Therefore, by Cauchy's inequality, we obtain

$$\begin{aligned} \frac{A_n^2(x)}{A_{n+1}^2(x)} &= \left( \frac{1}{n+1} \sum_{i=0}^n \frac{1}{x-x_i} \right)^2 \leq \frac{1}{n+1} \sum_{i=0}^n \frac{1}{(x-x_i)^2} \\ &= \frac{(n+1)A_n^2(x) - nA_{n-1}(x)A_{n+1}(x)}{A_{n+1}^2(x)}, \end{aligned} \quad (3.17)$$

which is equivalent to  $T$  property for the sequence  $\{A_n(x)\}$ ,  $x \in \mathbb{R}$ .  $\square$

Since  $a_n = A_n(0)$ , the assertion (i) follows from (ii) for  $x = 0$ .

*Proof of Proposition 2.7.* The classical Hermite polynomials  $\{H_n(x)\}$  are defined by [4, page 105],

$$\frac{H_n(x)}{n!} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{(-1)^i (2x)^{n-2i}}{i! (n-2i)!}, \quad n = 0, 1, 2, \dots \quad (3.18)$$

To see that the sequence  $\{H_n(x/2)\}$  belongs to the Appell class, write (3.18) in the form

$$H_n(x/2) = \sum_{i=0}^n h_i \binom{n}{i} x^{n-i}, \quad (3.19)$$

where

$$h_{2k} = \frac{(-1)^k (2k)!}{k!}; \quad h_{2k+1} = 0; \quad k = 0, 1, 2, \dots \quad (3.20)$$

Since all their zeros are real [4, page 110], by Theorem 2.6, part (ii),  $T$  property follows for  $x \in \mathbb{R}$ .  $\square$

*Proof of Proposition 2.8.* The class of generalized Laguerre polynomials  $\{L_n^{(a)}(x)\}$  of order  $a > -1$  is defined by [4, page 100]

$$L_n^{(a)}(x) = \sum_{i=0}^n \binom{n+a}{n-i} \frac{(-x)^i}{i!}. \quad (3.21)$$

All its zeros are real and positive [4, page 110]. For  $x \neq 0$ , an elementary transform gives

$$x^n L_n^{(a)}(1/x) \binom{n+a}{n} = \Gamma(a+1) \sum_{i=0}^n \frac{(-1)^i}{\Gamma(i+a+1)} \binom{n}{i} x^{n-i}. \quad (3.22)$$

Now, we can apply Theorem 2.6, part (ii).  $\square$

*Proof of Theorem 2.9.* This assertion is a consequence of Theorem 2.6, part (i).  $\square$

*Proof of Proposition 2.10.* An explicit form of ultraspherical polynomials of order  $\lambda$  is the following [4, page 85]

$$P_n^{(\lambda)}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \frac{\Gamma(n-i+\lambda)}{\Gamma(\lambda)\Gamma(i+1)\Gamma(n-2i+1)} (2x)^{n-2i}. \quad (3.23)$$

All its zeros are real and belong to the interval  $(-1, 1)$ . We also have

$$P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n} = (-1)^n P_n^{(\lambda)}(-1). \quad (3.24)$$

[4, page 81].

For the proof we will use the following identity [4, page 384]:

$$\sum_{k=0}^n \frac{P_k^{(\lambda)}(t)}{P_k^{(\lambda)}(1)} \binom{n}{k} x^{n-k} = (1+2tx+x^2)^{n/2} \frac{P_n^{(\lambda)}[(1+2tx+x^2)^{-1/2}(t+x)]}{P_n^{(\lambda)}(1)}. \quad (3.25)$$

Taking  $t \in (-1, 1)$  we can see that all the zeros  $\{x_i\}$  of the polynomial on the right-hand side of (3.25) are real and given by

$$x_i = t_i \sqrt{\frac{1-t^2}{1-t_i^2}} - t, \quad (3.26)$$

where  $\{t_i\}$  are corresponding zeros of  $P_n^{(\lambda)}(t)$ .

Hence, by (3.24) and Theorem 2.9, we obtain the assertion of Proposition 2.10.  $\square$

## References

- [1] P. A. MacMahon, *Combinatory Analysis, Two Volumes (bound as one)*, Chelsea Publishing, New York, 1960.
- [2] J. Riordan, *An Introduction to Combinatorial Analysis*, Wiley Publications in Mathematical Statistics, John Wiley & Sons, New York; Chapman & Hall, London, 1958.
- [3] G. Szegő, *On an inequality of P. Turán concerning Legendre polynomials*, American Mathematical Society Bulletin **54** (1948), 401–405.
- [4] ———, *Orthogonal Polynomials*, Revised ed., American Mathematical Society Colloquium Publications, vol. 23, American Mathematical Society, Rhode Island, 1959.
- [5] P. Turán, *On the zeros of the polynomials of Legendre*, Československá Akademie Věd. Časopis Pro Pěstování Matematiky **75** (1950), 113–122.

Slavko Simic: Mathematical Institute SANU, Kneza Mihaila 35, 11001 Belgrade, Serbia and Montenegro

*E-mail address:* ssimic@turing.mi.sanu.ac.yu