BOUNDARY BEHAVIOUR OF ANALYTIC FUNCTIONS IN SPACES OF DIRICHLET TYPE

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For $0 < p < \infty$ and $\alpha > -1$, we let $\mathcal{D}^p_\alpha$ be the space of all analytic functions f in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that f' belongs to the weighted Bergman space A^p_α. We obtain a number of sharp results concerning the existence of tangential limits for functions in the spaces $\mathcal{D}^p_\alpha$. We also study the size of the exceptional set $E(f) = \{e^{i\theta} \in \partial\mathbb{D} : V(f,\theta) = \infty\}$, where $V(f,\theta)$ denotes the radial variation of f along the radius $[0,e^{i\theta})$, for functions $f \in \mathcal{D}^p_\alpha$.

1. Introduction and main results

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. If $0 < r < 1$ and f is an analytic function in $\mathbb{D}$ (abbreviated $f \in \mathcal{H}ol(\mathbb{D})$), we set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{\mathrm{it}})|^p\,dt\right)^{1/p}, \quad I_p(r,f) = M_p^p(r,f), \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \sup_{0\le t\le 2\pi} |f(re^{\mathrm{it}})|. \tag{1.1}$$

For $0 < p \le \infty$, the Hardy space H^p consists of those functions $f \in \mathcal{H}ol(\mathbb{D})$ for which $\|f\|_{H^p} \overset{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty$. We refer to [10] for the theory of Hardy spaces.

The weighted Bergman space A^p_α $(0 < p < \infty, \alpha > -1)$ is the space of all functions $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \overset{\text{def}}{=} \left(\int_{\mathbb{D}} (1-|z|)^\alpha |f(z)|^p\,dA(z)\right)^{1/p} < \infty, \tag{1.2}$$

where $dA(z) = (1/\pi)\,dx\,dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We mention [11, 16] as general references for the theory of Bergman spaces.

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We will write $\mathcal{D}_\alpha^p$ $(0 < p < \infty, \alpha > -1)$ for the space of all functions $f \in \mathcal{H}ol(\mathbb{D})$ such that $\int_{\mathbb{D}}(1-|z|)^\alpha |f'(z)|^p\, dA(z) < \infty$. In other words,

$$f \in \mathcal{D}_\alpha^p \Longleftrightarrow f' \in A_\alpha^p. \tag{1.3}$$

If $p < \alpha + 1$, it is well known that $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$ with equivalence of norms (see [12, Theorem 6]). If $p > 1$ and $\alpha = p - 2$, we are considering the Besov spaces $\mathcal{B}^p$ which have been extensively studied in [3, 9, 29]. Specially relevant is the space $\mathcal{B}^2 = \mathcal{D}_0^2$, which coincides with the classical Dirichlet space $\mathcal{D}$.

The space $\mathcal{D}_\alpha^p$ is said to be a Dirichlet space if $p \geq \alpha + 1$. Specially interesting are the spaces in the "limit case" $p = \alpha + 1$, that is, the spaces $\mathcal{D}_{p-1}^p$, $0 < p < \infty$. These spaces are closely related to Hardy spaces. Indeed, a direct calculation with Taylor coefficients gives that $H^2 = \mathcal{D}_1^2$. Furthermore, we have

$$H^p \subset \mathcal{D}_{p-1}^p, \quad 2 \leq p < \infty, \tag{1.4}$$

$$\mathcal{D}_{p-1}^p \subset H^p, \quad 0 < p \leq 2. \tag{1.5}$$

The relation (1.4) is a classical result of Littlewood and Paley [21], and (1.5) can be found in [28]. A good number of results on the spaces $\mathcal{D}_{p-1}^p$ have been recently obtained in [4, 13–15, 28]. We remark that the spaces $\mathcal{D}_{p-1}^p$ are not nested. Actually, it is easy to see that if $p \neq q$, then there is no relation of inclusion between $\mathcal{D}_{p-1}^p$ and $\mathcal{D}_{q-1}^q$.

Fatou's theorem asserts that if $0 < p \leq \infty$ and $f \in H^p$, then f has a finite nontangential limit $f(e^{i\theta})$ for a.e. $e^{i\theta} \in \partial\mathbb{D}$. Bearing in mind (1.5), we see that this is true if $f \in \mathcal{D}_{p-1}^p$ and $0 < p \leq 2$. In view of (1.4), it is natural to ask whether or not Fatou's theorem remains true for the spaces $\mathcal{D}_{p-1}^p$, $2 < p < \infty$. The answer to this question is negative. Indeed, [15, Theorem 3.5] asserts that if $2 < p < \infty$, then there exists a function $f \in \mathcal{D}_{p-1}^p$ such that

$$\lim_{r \to 1^-} \frac{|f(re^{\mathrm{it}})|}{\left(\log 1/(1-r)\right)^{1/2-1/p}\left(\log\log 1/(1-r)\right)^{-1}} = \infty, \quad \text{for a.e. } e^{\mathrm{it}} \in \partial\mathbb{D}. \tag{1.6}$$

This function has a nontangential limit almost nowhere in ∂D.

Fatou's theorem is best possible for Hardy spaces in the sense that it cannot be extended further to give the existence of "tangential limits." Indeed, Lohwater and Piranian [22] (see also [8, page 43], [20, 31], and [32, Volume I, page 280] for some related results) proved that if γ_0 is a Jordan curve, internally tangent to $\partial\mathbb{D}$ at $z = 1$, and having no other point in common with $\partial\mathbb{D}$, and γ_θ $(\theta \in \mathbb{R})$ denotes the rotation of γ_0 through an angle θ around the origin, then there exists a function $f \in H^\infty$ such that, for every $\theta \in \mathbb{R}$, f does not approach a limit as $z \to e^{i\theta}$ along γ_θ.

In spite of this, a number of "tangential Fatou's theorems" have been proved for certain spaces of Dirichlet type.

For $A>0$, $\gamma\geq 1$, and $\xi\in\partial\mathbb{D}$, we define

$$R(A,\gamma,\xi)=\{z\in\mathbb{D}:|1-\overline{\xi}z|^{\gamma}\leq A(1-|z|)\}. \tag{1.7}$$

When $\gamma=1$ and $A>1$, the region $R(A,\gamma,\xi)$ is basically a Stolz angle. When $\gamma>1$, $R(A,\gamma,\xi)$ is a region contained in $\mathbb{D}$ which touches $\partial\mathbb{D}$ at ξ tangentially. As γ increases, the degree of tangency increases.

We define also, for $A>1$ and $\beta>0$,

$$\begin{aligned} R_{\exp}(A,\beta,\xi)&=\left\{z\in\mathbb{D}:\exp\left(-|1-\overline{\xi}z|^{-\beta}\right)\leq\frac{(1-|z|)}{A}\right\},\\ R_{\log}(A,\beta,\xi)&=\left\{z\in\mathbb{D}:|1-\overline{\xi}z|\leq A(1-|z|)\left(\log\frac{2}{1-|z|}\right)^{\beta}\right\}. \end{aligned} \tag{1.8}$$

As β increases, the degree of tangency increases in both types of tangential regions.

If $f\in\mathcal{H}ol(\mathbb{D})$, we say that f has the γ-limit L at $e^{i\theta}$, if $f(z)\to L$ as $z\to e^{i\theta}$ within $R(A,\gamma,\xi)$ for every A. Notice that saying that f has the 1-limit L at $e^{i\theta}$ is the same as saying that f has the nontangential limit L at $e^{i\theta}$. Substituting the regions $R(A,\gamma,\xi)$ with the regions $R_{\exp}(A,\beta,\xi)$ and $R_{\log}(A,\beta,\xi)$, we have the notions of $\beta_{\exp}$-limits and $\beta_{\log}$-limits. We observe that these definitions of tangential limits are equivalent to those considered in [2, 7, 23, 26].

Among other results, Kinney [19] and Nagel, Rudin, and Shapiro [23] (see also [26]) proved the following.

(i) If $0<\alpha<1$ and $f\in D^2_{\alpha}$, then f has a finite α^{-1}-limit at a.e. $e^{i\theta}\in\partial\mathbb{D}$.

(ii) If $f\in D^2_0=\mathcal{D}$, then f has a finite $1_{\exp}$-limit almost everywhere.

In view of these results, it is natural to ask whether results of this kind can be proved for the spaces $\mathcal{D}^p_{\alpha}$ for other choices of p and α. We start with a negative result.

Theorem 1.1. (a) *Suppose that $A>1$ and $\beta>1$. Then there exists a function $f\in\bigcap_{1\leq p<\infty}\mathcal{D}^p_{p-1}$ such that for almost every $e^{i\theta}\in\partial\mathbb{D}$, f does not approach a limit as $z\to e^{i\theta}$ inside $R_{\log}(A,\beta,e^{i\theta})$.*

(b) *Suppose that $A>0$ and $\gamma>1$. Then there exists a function $f\in\bigcap_{0<p<\infty}\mathcal{D}^p_{p-1}$ such that for almost every $e^{i\theta}\in\partial\mathbb{D}$, f does not approach a limit as $z\to e^{i\theta}$ inside $R(A,\gamma,e^{i\theta})$.*

Next we turn our attention to the spaces $\mathcal{D}^p_{\alpha}$ with $1\leq p\leq 2$ and $-1<\alpha\leq p-1$. We will prove the following theorem.

Theorem 1.2. (a) *Suppose that $1\leq p\leq 2$, $p-2<\alpha\leq p-1$, and $f\in\mathcal{D}^p_{\alpha}$. Then f has an $(\alpha-p+2)^{-1}$-limit at a.e. $e^{i\theta}\in\partial\mathbb{D}$.*

(b) *Suppose that $1<p\leq 2$ and $f\in\mathcal{D}^p_{p-2}=\mathcal{B}^p$. Then f has a $(p'-1)_{\exp}$-limit at a.e. $e^{i\theta}\in\partial\mathbb{D}$.*

Here and throughout the paper, if $p>1$, we write p' for the exponent conjugate of p, $1/p+1/p'=1$.

We will prove that part (a) of Theorem 1.2 is sharp in the sense that the degree of potential tangency $(\alpha - p + 2)^{-1}$ cannot be substituted by any larger one.

Theorem 1.3. *Suppose that* $1 \le p \le 2$, $p-2 < \alpha \le p-1$, $A > 0$, *and* $\gamma > (\alpha - p + 2)^{-1}$. *Then there exists a function* $f \in \mathcal{D}_\alpha^p$ *such that for almost every* $e^{i\theta} \in \partial\mathbb{D}$, f *does not approach a limit as* $z \to e^{i\theta}$ *inside* $R(A,\gamma,e^{i\theta})$.

Now we turn to questions related to radial variation of analytic functions. If $f \in \mathcal{H}ol(\mathbb{D})$ and $\theta \in [-\pi,\pi)$, we define

$$V(f,\theta) \stackrel{\text{def}}{=} \int_0^1 |f'(re^{i\theta})|\,dr. \tag{1.9}$$

Then $V(f,\theta)$ denotes the radial variation of f along the radius $[0,e^{i\theta})$, that is, the length of the image of this radius under the mapping f. We define the exceptional set $E(f)$ associated to f as

$$E(f) = \{e^{i\theta} \in \partial\mathbb{D} : V(f,\theta) = \infty\}. \tag{1.10}$$

It is clear that if f has finite radial variation at $e^{i\theta}$, then f has a finite radial limit at $e^{i\theta}$. Even though every H^p-function, $0 < p \le \infty$, has finite radial limits a.e., if we take $f \in \mathcal{H}ol(\mathbb{D})$ given by a power series with Hadamard gaps

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k} \quad \text{with } n_{k+1} \ge \lambda n_k,\ \forall k\ (\lambda > 1), \tag{1.11}$$

such that

$$\sum_{k=1}^{\infty} |a_k|^2 < \infty \quad \text{but} \sum_{k=1}^{\infty} |a_k| = \infty, \tag{1.12}$$

then $f \in \bigcap_{0<p<\infty} H^p$, but a result of Zygmund (see [30, Theorem 1, page 194]) shows that $V(f,\theta) = \infty$ for every $\theta \in [-\pi,\pi)$.

We will prove a positive result for $\mathcal{D}_{p-1}^p$-functions, $0 < p \le 1$.

Theorem 1.4. *If* $0 < p \le 1$ *and* $f \in \mathcal{D}_{p-1}^p$, *then* $E(f)$ *has measure* 0.

We note that this result cannot be extended to $p > 1$. Indeed, if we take f given by a power series with Hadamard gaps as in (1.11) with $\sum_{k=1}^{\infty} |a_k|^p < \infty$ and $\sum_{k=1}^{\infty} |a_k| = \infty$, we have that $f \in \mathcal{D}_{p-1}^p$ (see [15, Proposition A]) and so $V(f,\theta) = \infty$ for every $\theta \in [-\pi,\pi)$.

On the other hand, we have the following well-known result of Beurling [5] for functions in $\mathcal{D}_\alpha^2$.

Theorem 1.5. *Let* f *be an analytic function in* $\mathbb{D}$.

(a) *If* $f \in \mathcal{D}$, *then* $E(f)$ *has logarithmic capacity* 0.

(b) *If* $0 < \alpha < 1$ *and* $f \in \mathcal{D}_\alpha^2$, *then* $E(f)$ *has* α*-capacity* 0.

See [17] for the definitions of logarithmic capacity and α-capacity and [27] for an extension of Theorem 1.5.

We will prove the following result for other values of p.

THEOREM 1.6. *Suppose that* $f \in \mathcal{D}^p_\alpha$.

(a) *If* $0 < p \leq 1$ *and* $-1 < \alpha < p-1$, *then* $E(f)$ *has Lebesgue measure* 0.

(b) *If* $1 < p < 2$ *and* $p-2 < \alpha < p-1$, *then* $E(f)$ *has Lebesgue measure* 0.

(c) *If* $1 < p \leq 2$ *and* $\alpha = p-2$, *then* $E(f)$ *has logarithmic capacity* 0.

(d) *If* $2 < p < \infty$ *and* $p-1 > \alpha \geq p/2-1$, *then* $E(f)$ *has* β*-capacity* 0 *for all* $\beta > 2/p(1+\alpha)-1$.

(e) *If* $2 < p < \infty$ *and* $\alpha < p/2-1$, *then* $E(f)$ *has logarithmic capacity* 0.

2. On the membership of Blaschke products in spaces of Dirichlet type

We remark that $H^\infty \not\subset \mathcal{D}^p_\alpha$, if $0 < p < \infty$ and $-1 < \alpha < p-1$ (see, e.g., [13, Section 3] for explicit examples). Clearly, (1.4) gives that $H^\infty \subset \mathcal{D}^p_{p-1}$, if $2 \leq p < \infty$. However, this does not remain true for $0 < p < 2$. Indeed, Vinogradov [28, pages 3822-3823] has shown that there exist Blaschke products B which do not belong to $\bigcup_{0<p<2} \mathcal{D}^p_{p-1}$. In this section, we will find a number of sufficient conditions for the membership of a Blaschke product in some of the spaces $\mathcal{D}^p_\alpha$. These results will be basic in the proofs of Theorems 1.1 and 1.3.

We recall that if a sequence of points $\{a_n\}$ in $\mathbb{D}$ satisfies the *Blaschke condition* $\sum_{n=1}^\infty (1-|a_n|) < \infty$, the corresponding Blaschke product B is defined as

$$B(z) = \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}. \tag{2.1}$$

Such a product is analytic in $\mathbb{D}$, bounded by one, and with nontangential limits of modulus one almost everywhere on the unit circle. We start obtaining sufficient conditions for the membership of a Blaschke product in the spaces $\mathcal{D}^p_{p-1}$, improving the first part of [28, Lemma 2.11].

LEMMA 2.1. *Let B be a Blaschke product with sequence of zeros* $\{a_n\}$.

(a) *If* $\{a_n\}$ *satisfies*

$$\sum_{n=1}^{\infty} (1-|a_n|) \log\left(\frac{1}{1-|a_n|}\right) < \infty, \tag{2.2}$$

then $B \in \bigcap_{1 \leq p < \infty} \mathcal{D}^p_{p-1}$.

(b) *If there exists* $q \in (0,1)$ *such that*

$$\sum_{n=1}^{\infty} (1-|a_n|)^q < \infty, \tag{2.3}$$

then $B \in \bigcap_{0 < p < \infty} \mathcal{D}^p_{p-1}$.

Proof. A result of Rudin's (see [25, Theorem I]) shows that (2.2) implies that $B \in \mathcal{D}_0^1$. Then (a) follows from the Cauchy estimate $|B'(z)| \leq 1/(1-|z|)$.

We turn now to part (b). Suppose that $\{a_n\}$ satisfies (2.3) for a certain $q \in (0,1)$. Assume for now that $p \in (0,1]$. Using [18, Theorem 3.1], we see that $B' \in A^{2-q}$. Using this, Hölder's inequality with exponents $(2-q)/p$ and $(2-q)/(2-q-p)$, and the fact that $(2-q)(1-p)/(2-q-p) < 1$, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |B'(z)|^p (1-|z|^2)^{p-1}\, dA(z) \\
&\quad \leq \left(\int_{\mathbb{D}} |B'(z)|^{2-q}\, dA(z)\right)^{p/(2-q)} \left(\int_{\mathbb{D}} (1-|z|^2)^{(2-q)(p-1)/(2-q-p)}\, dA(z)\right)^{(2-q-p)/(2-q)} \\
&\quad < \infty.
\end{aligned} \tag{2.4}$$

Hence, we have shown that $B \in \mathcal{D}_{p-1}^p$, for all $p \in (0,1]$. Using the Cauchy estimate again, we obtain that $B \in \mathcal{D}_{p-1}^p$ for all $p \in (0,\infty)$, as desired. □

We next give a simplified proof of a result that essentially is Theorem 3.1(i) for $\beta = 1$ and $p \geq 1$ in [18].

Lemma 2.2. *Let p and α be such that $p \geq 1$ and $p-2 < \alpha < p-1$. If B is a Blaschke product whose sequence of zeros $\{a_n\}$ satisfies*

$$\sum_{n=1}^{\infty} (1-|a_n|)^{\alpha+2-p} < \infty, \tag{2.5}$$

then $B \in \mathcal{D}_\alpha^p$.

Proof. We will use the notation and terminology of [1, pages 332-333].

Let p, α, and B be as in the statement. Notice that $0 < \alpha + 2 - p < 1$, and then, using [24, Theorem 1], we deduce that $B' \in \mathrm{B}^{1/(\alpha-p+3)}$ or, equivalently, $B \in \mathcal{D}_{\alpha-p+1}^1$. Then as in the proof of Lemma 2.1, the Cauchy estimate implies $B \in \mathcal{D}_\alpha^p$ since $p - 1 \geq 0$. □

3. Tangential limits for $\mathcal{D}_\alpha^p$-functions

Proof of Theorem 1.1(a). We are going to use an argument which is similar to the one used in the proof of [32, Volume I, Chapter VII, Theorem 7.44].

Take M with $1 < M < A$ and let C_θ be the boundary of $R_{\log}(M,\beta,e^{i\theta})$ ($\theta \in [0,2\pi)$). For all sufficiently large n, let l_n denote the length of the arc of the circle $|z| = 1 - 1/n$ which lies in $R_{\log}(M,\beta,1)$ and let $m_n = E[2\pi/l_n] + 1$, where, for $x \in \mathbb{R}$, $E[x]$ denotes the greatest integer that is smaller than or equal to x. Let $S_n = \{z_{n,1}, z_{n,2}, \ldots, z_{n,m_n}\}$ be any collection of m_n points equally spaced on $|z| = 1 - 1/n$. Since the circular distance between any two consecutive points of S_n is smaller than l_n, for every θ the set $R_{\log}(M,\beta,e^{i\theta})$ contains a point of S_n.

We define

$$\sigma_n = \sum_{k=1}^{m_n} (1 - |z_{n,k}|) \log\left(\frac{1}{1 - |z_{n,k}|}\right) = \frac{m_n \log(n)}{n}. \tag{3.1}$$

Notice that $l_n \asymp (1/n)\log^{\beta} n$. Then it is easy to see that there exists a positive constant C (which does not depend on n) such that

$$\sigma_n = \frac{m_n \log(n)}{n} \leq \frac{(1 + 2\pi/l_n)\log(n)}{n} \leq C\frac{\log(n)}{n l_n} \leq C\frac{1}{\log^{\beta-1} n} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.2}$$

Let us take then an increasing sequence n_k satisfying that $\sum_{k=1}^{\infty} \sigma_{n_k} < \infty$ and let B be the Blaschke product with zeros at the points of $\bigcup_{k=1}^{\infty} S_{n_k}$. By part (a) of Lemma 2.1, $B \in \bigcap_{1 \leq p < \infty} \mathcal{D}_{p-1}^{p}$. Notice that for each $\theta \in \mathbb{R}$, B has infinitely many zeros in the set $R_{\log}(M,\beta,e^{i\theta})$. Thus for every θ, the limit of $B(z)$ as $z \to e^{i\theta}$ inside of $R_{\log}(M,\beta,e^{i\theta})$ must be zero if it exists at all. Since the radial limit of B has absolute value 1 a.e., it follows that for almost every $e^{i\theta} \in \partial\mathbb{D}$, the limit of $B(z)$ as $z \to e^{i\theta}$ inside of $R_{\log}(M,\beta,e^{i\theta})$ does not exist. □

Part(b) of Theorem 1.1 can be proved in a similar way using part (b) of Lemma 2.1. We omit the details.

Next we will obtain a representation formula for functions f in the space $\mathcal{D}_{\alpha}^{p}$, $-1 < \alpha$, $1 \leq p \leq 2$, which will play a basic role in the proof of Theorem 1.2.

Theorem 3.1. *Suppose that either $1 \leq p \leq 2$ and $-1 < \alpha < p - 1$ or $1 < p \leq 2$ and $\alpha = p - 2$, and that $f \in \mathcal{D}_{\alpha}^{p}$. Then there exists a function $h(e^{i\theta}) \in L^p(\partial\mathbb{D})$ such that*

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1 - e^{-i\theta}z)^{(\alpha+1)/p}}\, d\theta, \quad z \in \mathbb{D}. \tag{3.3}$$

Proof. Let p and α be as in the statement and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}_{\alpha}^{p}$. Then $zf'(z) = \sum_{n=0}^{\infty} n a_n z^n \in A_{\alpha}^{p}$. Since $\mathcal{D}_{\alpha}^{p} \subset A_{\alpha}^{p}$, we also have that $f \in A_{\alpha}^{p}$. Then it follows that

$$zf'(z) + \frac{\alpha+1}{p} f(z) = \sum_{n=0}^{\infty}\left(n + \frac{\alpha+1}{p}\right) a_n z^n \in A_{\alpha}^{p}. \tag{3.4}$$

So using [6, Lemma 1.1] (see also [12, part (iii) of Theorem 5]) and [6, Corollary 3.5], we deduce that the fractional integral

$$h(z) \overset{\text{def}}{=} \widetilde{I}^{(\alpha+1)/p}\left(zf'(z) + \frac{\alpha+1}{p} f(z)\right) = \sum_{n=0}^{\infty}\left(n + \frac{\alpha+1}{p}\right) B\left(n+1, \frac{\alpha+1}{p}\right) a_n z^n \tag{3.5}$$

belongs to H^p since $p \leq 2$. Here $B(\cdot,\cdot)$ is the classical beta function. Note that

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{3.6}$$

and recall that $\Gamma(s+1) = s\Gamma(s)$, for all $s \neq 0, -1, \ldots$. Then it is easy to see that

$$h(z) = \sum_{n=0}^{\infty} \frac{n!\Gamma((\alpha+1)/p)}{\Gamma(n+(\alpha+1)/p)} a_n z^n. \tag{3.7}$$

Then,

$$h(e^{i\theta}) = \sum_{n=0}^{\infty} \frac{n!\Gamma((\alpha+1)/p)}{\Gamma(n+(\alpha+1)/p)} a_n e^{in\theta} \in L^p(\partial\mathbb{D}). \tag{3.8}$$

By the binomial theorem,

$$\frac{1}{(1-e^{-i\theta}z)^{(\alpha+1)/p}} = \sum_{k=0}^{\infty} \frac{\Gamma(k+(\alpha+1)/p)}{k!\Gamma((\alpha+1)/p)} e^{-ik\theta} z^k. \tag{3.9}$$

Thus,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{(\alpha+1)/p}}\, dt \\
&\quad = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} \frac{n!\Gamma(\alpha+1/p)}{\Gamma(n+(\alpha+1)/p)} a_n e^{in\theta}\right)\left(\sum_{k=0}^{\infty} \frac{\Gamma(k+(\alpha+1)/p)}{k!\Gamma(\alpha+1/p)} e^{-ik\theta} z^k\right) d\theta \\
&\quad = \sum_{n=0}^{\infty} a_n z^n = f(z).
\end{aligned} \tag{3.10}$$

This finishes the proof. □

Proof of Theorem 1.2. We need to consider three cases.

Case 1. $1 \le p \le 2$ and $\alpha = p-1$. Then $\mathcal{D}_\alpha^p = \mathcal{D}_{p-1}^p \subset H^p$ and the result in this case follows from Fatou's theorem for H^p.

Case 2. $1 \le p \le 2$ and $p-2 < \alpha < p-1$. If $f \in \mathcal{D}_\alpha^p$, then, using Theorem 3.1, we have that there exists $h(e^{i\theta}) \in L^p(\partial\mathbb{D})$ such that

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{(\alpha+1)/p}}\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1-e^{-i\theta}z)^{1-(p-\alpha-1)/p}}\, dt. \tag{3.11}$$

Notice that $p((p-\alpha-1)/p) < 1$, so by [23, part (a) of Theorem A] we have that f has $(\alpha-p+2)^{-1}$-limit at a.e. $e^{i\theta} \in \partial\mathbb{D}$.

Case 3. $1 < p \leq 2$ and $\alpha = p - 2$. Using again Theorem 3.1, we have that if $f \in \mathcal{D}^p_\alpha$, then there exists $h(e^{i\theta}) \in Ł^p(\partial\mathbb{D})$ such that

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{h(e^{i\theta})}{(1 - e^{-i\theta}z)^{1-1/p}}\, dt. \tag{3.12}$$

Using [23, part (b) of Theorem A], we deduce that f has $(p' - 1)_{\exp}$-limit at a.e. $e^{i\theta} \in \partial\mathbb{D}$. □

Theorem 1.3 can be proved arguing as in the proof of part (a) of Theorem 1.1, using Lemma 2.2 instead of Lemma 2.1. Again, we will omit the details.

4. Radial variation of functions in the spaces $\mathcal{D}^p_\alpha$

Proof of Theorem 1.4. Let $0 < p < 1$ and $f \in \mathcal{D}^p_{p-1}$. Set

$$F_f = \{\theta \in [-\pi, \pi] : f \text{ has a finite nontangential limit at } e^{i\theta}\}. \tag{4.1}$$

By (1.5) and Fatou's theorem, $[-\pi, \pi] \setminus F_f$ has Lebesgue measure 0. On the other hand, Zygmund proved in [30, page 81] that

$$(1 - r)\,|\,f'(re^{i\theta})\,| \longrightarrow 0, \quad \text{as } r \longrightarrow 1^-, \tag{4.2}$$

for all $\theta \in F_f$. Consequently the set

$$F_f^* = \{\theta \in [-\pi, \pi] : (1 - r)\,|\,f'(re^{i\theta})\,| \longrightarrow 0\} \tag{4.3}$$

is such that $[-\pi, \pi] \setminus F_f^*$ has Lebesgue measure 0. Since $f \in \mathcal{D}^p_{p-1}$, we deduce that the set

$$T_f = \left\{\theta \in [-\pi, \pi] : \int_0^1 (1 - r)^{p-1}\,|\,f'(re^{i\theta})\,|^p\, dr < \infty\right\} \tag{4.4}$$

is such that $[-\pi, \pi] \setminus T_f$ has Lebesgue measure 0. Thus, $[-\pi, \pi] \setminus \left(F_f^* \cap T_f\right)$ has Lebesgue measure 0. Furthermore, if $\theta \in F_f^* \cap T_f$, there exists a positive constant C_θ such that

$$V(f, \theta) = \int_0^1 |\,f'(re^{i\theta})\,|^p\,|\,f'(re^{i\theta})\,|^{1-p}\, dr \leq C_\theta \int_0^1 (1 - r)^{p-1}\,|\,f'(re^{i\theta})\,|^p\, dr < \infty. \tag{4.5}$$

□

Proof of Theorem 1.6. Since

$$\mathcal{D}^p_\alpha \subset \mathcal{D}^p_\beta, \quad -1 < \alpha \leq \beta,\ 0 < p < \infty, \tag{4.6}$$

(a) follows from Theorem 1.4.

Suppose now that $1 < p < 2$, $p - 2 < \alpha < p - 1$, and $f \in \mathcal{D}^p_\alpha$. Then the set

$$T_f^\alpha = \left\{\theta \in [-\pi, \pi] : \int_0^1 (1 - r)^\alpha\,|\,f'(re^{i\theta})\,|^p\, dr < \infty\right\} \tag{4.7}$$

is such that $[-\pi,\pi] \setminus T_f^\alpha$ has Lebesgue measure 0. Now, using Hölder's inequality, we see that there exists a positive constant $C_{\alpha,p}$ such that

$$\begin{aligned} V(f,\theta) &= \int_0^1 (1-r)^{\alpha/p} \left| f'(re^{i\theta}) \right| (1-r)^{-\alpha/p}\, dr \\ &\le \left(\int_0^1 (1-r)^{\alpha} \left| f'(re^{i\theta}) \right|^p dr \right)^{1/p} \left(\int_0^1 (1-r)^{-p'\alpha/p}\, dr \right)^{1/p'} \\ &\le C_{\alpha,p} \left(\int_0^1 (1-r)^{\alpha} \left| f'(re^{i\theta}) \right|^p dr \right)^{1/p} < \infty, \end{aligned} \tag{4.8}$$

for all $\theta \in T_f^\alpha$. (We have used that $-p'\alpha/p > -1$ since $\alpha < p-1$.) Thus, (b) is proved.

Part (c) follows from the well-known inclusion

$$\mathcal{D}_{p-2}^p = \mathcal{B}^p \subset \mathcal{B}^q = \mathcal{D}_{q-2}^q, \quad 1 < p < q < \infty, \tag{4.9}$$

(see, e.g., [3, page 112]), Theorem 1.5, and the fact that $\mathcal{B}^2 = \mathcal{D}$.

Finally, suppose that $2 < p < \infty$ and $f \in \mathcal{D}_\alpha^p$. Using Hölder's inequality with exponents $p/(p-2)$ and $p/2$, we have that

$$\begin{aligned} \int_{\mathbb{D}} (1-|z|)^{\beta} \left| f'(z) \right|^2 dA(z) &= \int_{\mathbb{D}} (1-|z|)^{\beta - 2\alpha/p} \left| f'(z) \right|^2 (1-|z|)^{2\alpha/p}\, dA(z) \\ &\le \left(\int_{\mathbb{D}} (1-|z|)^{(p\beta - 2\alpha)/(p-2)}\, dA(z) \right)^{(p-2)/p} \\ &\quad \times \left(\int_{\mathbb{D}} (1-|z|)^{\alpha} \left| f'(z) \right|^p dA(z) \right)^{2/p}. \end{aligned} \tag{4.10}$$

Letting $\beta = 0$, we see that the condition $\alpha < p/2 - 1$ implies that $f \in \mathcal{D}$. Hence, (e) follows from part (a) of Theorem 1.5. On the other hand, if $p - 1 > \alpha \ge p/2 - 1$, then β can be chosen so that $\beta > (2/p)(1+\alpha) - 1$ and $0 < \beta < 1$. Then (4.10) implies that $f \in \mathcal{D}_\beta^2$, and (d) follows from part (b) of Theorem 1.5. □

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