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Research Article

On General Summability Factor Theorems

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The goal of this paper is to obtain sufficient and (different) necessary conditions for a series $\sum a_n$, which is absolutely summable of order k by a triangular matrix method A, $1 < k \le s < \infty$, to be such that $\sum a_n\lambda_n$ is absolutely summable of order s by a triangular matrix method B. As corollaries, we obtain two inclusion theorems.

In the recent papers [1, 2], the author obtained necessary and sufficient conditions for a series $\sum a_n$ which is absolutely summable of order k by a weighted mean method, $1 < k \le s < \infty$, to be such that $\sum a_n\lambda_n$ is absolutely summable of order s by a triangular matrix method. In this paper, we obtain sufficient and (different) necessary conditions for a series $\sum a_n$ which is absolutely summable $|A|_k$ to imply the series $\sum a_n\lambda_n$ which is absolutely summable $|B|_s$.

Let T be a lower triangular matrix, $\{s_n\}$ a sequence.

Then

$$T_n := \sum_{\nu=0}^{n} t_{n\nu}s_\nu. \tag{1}$$

A series $\sum a_n$ is said to be summable $|T|_k$, $k \ge 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}\left|T_n - T_{n-1}\right|^k < \infty. \tag{2}$$

We may associate with T two lower triangular matrices $\overline{T}$ and $\widehat{T}$ as follows:

$$\begin{aligned}\bar{t}_{n\nu} &= \sum_{r=\nu}^{n} t_{nr}, \quad n,\nu = 0,1,2,\ldots, \\ \hat{t}_{n\nu} &= \bar{t}_{n\nu} - \bar{t}_{n-1,\nu}, \quad n = 1,2,3,\ldots.\end{aligned} \tag{3}$$

With $s_n := \sum_{i=0}^{n} a_i\lambda_i$,

$$\begin{aligned}y_n &:= \sum_{i=0}^{n} t_{ni}s_i = \sum_{i=0}^{n} t_{ni}\sum_{\nu=0}^{i} a_\nu\lambda_\nu = \sum_{\nu=0}^{n} a_\nu\lambda_\nu\sum_{i=\nu}^{n} t_{ni} = \sum_{\nu=0}^{n} \bar{t}_{n\nu}a_\nu\lambda_\nu, \\ Y_n &:= y_n - y_{n-1} = \sum_{\nu=0}^{n}(\bar{t}_{n\nu} - \bar{t}_{n-1})\lambda_\nu a_\nu = \sum_{\nu=0}^{n}\hat{t}_{n\nu}\lambda_\nu a_\nu.\end{aligned} \tag{4}$$

We will call T as a triangle if T is lower triangular and $t_{nn} \neq 0$ for each n. The notation $\Delta_\nu\hat{a}_{n\nu}$ means $\hat{a}_{n\nu} - \hat{a}_{n,\nu+1}$. The notation $\lambda \in (|A|_k, |B|_s)$ will be used to represent the statement that if $\sum a_n$ is summable $|A|_k$, then $\sum a_n\lambda_n$ is summable $|B|_s$.

Theorem 1. *Let $1 < k \leq s < \infty$. Let $\{\lambda_n\}$ be a sequence of constants, A and B triangles satisfying*

(i) $|b_{nn}\lambda_n|/|a_{nn}| = O(\nu^{1/s-1/k})$,
(ii) $(n|X_n|)^{s-k} = O(1)$,
(iii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|)$,
(iv) $\sum_{\nu=0}^{n-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)| = O(|b_{nn}\lambda_n|)$,
(v) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}\lambda_n|)^{s-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)| = O(\nu^{s-1}|b_{\nu\nu}\lambda_\nu|^s)$,
(vi) $\sum_{\nu=0}^{n-1}|b_{\nu\nu}\lambda_{\nu+1}||\hat{b}_{n,\nu+1}\lambda_{\nu+1}| = O(|b_{nn}\lambda_{n+1}|)$,
(vii) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}\lambda_{n+1}|)^{s-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}| = O((\nu|b_{\nu\nu}\lambda_{\nu+1}|)^{s-1})$,
(viii) $\sum_{n=1}^{\infty} n^{s-1}|\sum_{\nu=2}^{n}\hat{b}_{n\nu}\lambda_\nu\sum_{i=0}^{\nu-2}\hat{a}'_{\nu i}X_i|^s = O(1)$,

then $\lambda \in (|A|_k, |B|_s)$.

Proof. If y_n denotes the nth term of the B-transform of a sequence $\{s_n\}$, then

$$\begin{aligned}y_n &= \sum_{i=0}^{n} b_{ni}s_i = \sum_{i=0}^{n} b_{ni}\sum_{\nu=0}^{i}\lambda_\nu a_\nu = \sum_{\nu=0}^{n}\lambda_\nu a_\nu\sum_{i=\nu}^{n} b_{ni} = \sum_{\nu=0}^{n}\bar{b}_{n\nu}\lambda_\nu a_\nu, \\ y_{n-1} &= \sum_{\nu=0}^{n-1}\bar{b}_{n-1,\nu}\lambda_\nu a_\nu,\end{aligned} \tag{5}$$

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n}\hat{b}_{n\nu}\lambda_\nu a_\nu, \tag{6}$$

where $s_n = \sum_{i=0}^{n}\lambda_i a_i$.

Let x_n denote the nth term of the A-transform of a series $\sum a_n$, then as in (6),

$$X_n := x_n - x_{n-1} = \sum_{\nu=0}^{n}\hat{a}_{n\nu}a_\nu. \tag{7}$$

Since $\widehat{A}$ is a triangle, it has a unique two-sided inverse, which we will denote by A'. Thus we may solve (7) for a_n to obtain

$$a_n = \sum_{\nu=0}^{n} \widehat{a}'_{n\nu} X_\nu. \tag{8}$$

Substituting (8) into (6) yields

$$\begin{aligned}
Y_n &= \sum_{\nu=0}^{n} \widehat{b}_{n\nu}\lambda_\nu a_\nu = \sum_{\nu=0}^{n} \widehat{b}_{n\nu}\lambda_\nu \left(\sum_{i=0}^{\nu} \widehat{a}'_{\nu i} X_i\right) = \sum_{\nu=0}^{n} \widehat{b}_{n\nu}\lambda_\nu \left(\sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i + \widehat{a}'_{\nu,\nu-1} X_{\nu-1} + \widehat{a}'_{\nu\nu} X_\nu\right) \\
&= \sum_{\nu=0}^{n} \widehat{b}_{n\nu}\lambda_\nu \widehat{a}'_{\nu\nu} X_\nu + \sum_{\nu=1}^{n} \widehat{b}_{n\nu}\lambda_\nu \widehat{a}'_{\nu,\nu-1} X_{\nu-1} + \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i \\
&= \widehat{b}_{nn}\lambda_n \widehat{a}'_{nn} X_n + \sum_{\nu=0}^{n-1} \widehat{b}_{n\nu}\lambda_\nu \widehat{a}'_{\nu\nu} X_\nu + \sum_{\nu=0}^{n-1} \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \widehat{a}'_{\nu+1,\nu} X_\nu + \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \left(\widehat{b}_{n\nu}\lambda_\nu a'_{\nu\nu} + \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \widehat{a}'_{\nu+1,\nu}\right) X_\nu + \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \left(\widehat{b}_{n\nu}\lambda_\nu a'_{\nu\nu} + \widehat{b}_{n,\nu+1}\lambda_{\nu+1} a'_{\nu\nu} - \widehat{b}_{n,\nu+1}\lambda_{\nu+1} a'_{\nu\nu} + \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \widehat{a}'_{\nu+1,\nu}\right) X_\nu \\
&\quad + \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \left(a'_{\nu\nu} + \widehat{a}_{\nu+1,\nu}\right) X_\nu + \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i.
\end{aligned} \tag{9}$$

Using the fact that

$$a'_{\nu\nu} + \widehat{a}'_{\nu+1,\nu} = \frac{1}{a_{\nu\nu}}\left(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu+1,\nu+1}}\right), \tag{10}$$

and substituting (10) into (9), we have the following:

$$\begin{aligned}
Y_n &= \frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \left(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}}\right) X_\nu + \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}, \text{ say.}
\end{aligned} \tag{11}$$

By Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{s-1} \left| T_{ni} \right|^{s} < \infty, \quad i = 1,2,3,4. \tag{12}$$

Using (i),

$$\begin{aligned} J_1 := \sum_{n=1}^{\infty} n^{s-1} \left| T_{n1} \right|^{s} &= \sum_{n=1}^{\infty} n^{s-1} \left| \frac{b_{nn}\lambda_n}{a_{nn}} X_n \right|^{s} \\ &= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(n^{1/s-1/k}\right)^{s} \left| X_n \right|^{s} \\ &= O(1) \sum_{n=1}^{\infty} n^{k-1} \left| X_n \right|^{k} \left(n^{s-s/k-k+1} \left| X_n \right|^{s-k}\right). \end{aligned} \tag{13}$$

But $n^{s-s/k-k+1}|X_n|^{s-k} = (n^{1-1/k}|X_n|)^{s-k} = O((n|X_n|)^{s-k}) = O(1)$, from (ii) of Theorem 1. Since $\sum a_n$ is summable $|A|_k$, $J_1 = O(1)$.

Using (i), (iv), (v), (ii), and Hölder's inequality,

$$\begin{aligned} J_2 &:= \sum_{n=1}^{\infty} n^{s-1} \left| T_{n2} \right|^{s} = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} X_\nu \right|^{s} \\ &\leq \sum_{n=1}^{\infty} n^{s-1} \left\{ \sum_{\nu=0}^{n-1} \nu^{1/s-1/k} \left| b_{\nu\nu}\lambda_\nu \right|^{-1} \left| \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu) \right| \left| X_\nu \right| \right\}^{s} \\ &= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \nu^{1-s/k} \left| b_{\nu\nu}\lambda_\nu \right|^{-s} \left| \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu) \right| \left| X_\nu \right|^{s} \right) \times \left(\sum_{\nu=0}^{n-1} \left| \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu) \right| \right)^{s-1} \\ &= O(1) \sum_{n=1}^{\infty} \left(n \left| b_{nn}\lambda_n \right|\right)^{s-1} \sum_{\nu=0}^{n-1} \nu^{1-s/k} \left| b_{\nu\nu}\lambda_\nu \right|^{-s} \left| \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu) \right| \left| X_\nu \right|^{s} \\ &= O(1) \sum_{\nu=1}^{\infty} \nu^{1-s/k} \left| b_{\nu\nu}\lambda_\nu \right|^{-s} \left| X_\nu \right|^{s} \sum_{n=\nu+1}^{\infty} \left(n \left| b_{nn}\lambda_n \right|\right)^{s-1} \left| \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu) \right| \\ &= O(1) \sum_{\nu=1}^{\infty} \nu^{1-s/k} \left| b_{\nu\nu}\lambda_\nu \right|^{-s} \left| X_\nu \right|^{s} \nu^{s-1} \left| b_{\nu\nu}\lambda_\nu \right|^{s} = O(1) \sum_{\nu=1}^{\infty} \nu^{s-s/k} \left| X_\nu \right|^{s} \\ &= O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} \left| X_\nu \right|^{k} \left(\nu^{s-s/k-k+1} \left| X_\nu \right|^{s-k}\right) = O(1) \sum_{\nu=1}^{\infty} \nu^{k-1} \left| X_\nu \right|^{k} = O(1). \end{aligned} \tag{14}$$

Using (iii), (vi), (vii), (ii), and Hölder's inequality,

$$\begin{aligned}
J_3 &:= \sum_{n=1}^{\infty} n^{s-1} \left| T_{n3} \right|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=0}^{n-1} \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \left(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}} \right) X_\nu \right|^s \\
&\leq \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \left| \frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}} \right| \left| X_\nu \right| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \left| X_\nu \right| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \left(\frac{\left| b_{\nu\nu}\lambda_{\nu+1} \right|}{\left| b_{\nu\nu}\lambda_{\nu+1} \right|} \right) \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \left| X_\nu \right| \right)^s \\
&= O(1) \sum_{n=1}^{\infty} n^{s-1} \left(\sum_{\nu=0}^{n-1} \left| b_{\nu\nu}\lambda_{\nu+1} \right|^{1-s} \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \left| X_\nu \right|^s \right) \times \left(\sum_{\nu=0}^{n-1} \left| b_{\nu\nu}\lambda_{\nu+1} \right| \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \right)^{s-1} \\
&= O(1) \sum_{n=1}^{\infty} \left(n \left| b_{nn}\lambda_{n+1} \right| \right)^{s-1} \sum_{\nu=0}^{n-1} \left| b_{\nu\nu}\lambda_{\nu+1} \right|^{1-s} \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \left| X_\nu \right|^s \\
&= O(1) \sum_{\nu=0}^{\infty} \left| b_{\nu\nu}\lambda_{\nu+1} \right|^{1-s} \left| X_\nu \right|^s \sum_{n=\nu+1}^{\infty} \left(n \left| b_{nn}\lambda_{n+1} \right| \right)^{s-1} \left| \widehat{b}_{n,\nu+1}\lambda_{\nu+1} \right| \\
&= O(1) \sum_{\nu=0}^{\infty} \left| b_{\nu\nu}\lambda_{n+1} \right|^{1-s} \left| X_\nu \right|^s \nu^{s-1} \left| b_{\nu\nu}\lambda_{\nu+1} \right|^{s-1} = O(1) \sum_{\nu=0}^{\infty} \nu^{s-1} \left| X_\nu \right|^s \\
&= O(1) \sum_{\nu=0}^{\infty} \nu^{k-1} \left| X_\nu \right|^k \left(\nu \left| X_\nu \right| \right)^{s-k} = O(1) \sum_{\nu=0}^{\infty} \nu^{k-1} \left| X_\nu \right|^k = O(1).
\end{aligned} \tag{15}$$

From (viii),

$$\sum_{n=1}^{\infty} n^{s-1} \left| T_{n4} \right|^s = \sum_{n=1}^{\infty} n^{s-1} \left| \sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i \right|^s = O(1). \tag{16}$$

□

We now state sufficient conditions, when $k = s$.

Corollary 1. *Let $\{\lambda_n\}$ be a sequence of constants, A and B triangles satisfying*

(i) $|b_{nn}|/|a_{nn}| = O(1/|\lambda_n|)$,

(ii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|)$,

(iii) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)| = O(|b_{nn}\lambda_n|)$,

(iv) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}\lambda_n|)^{k-1} |\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)| = O(\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k)$,

(v) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}\lambda_{\nu+1}||\widehat{b}_{n,\nu+1}\lambda_{\nu+1}| = O(|b_{nn}\lambda_{n+1}|)$,

(vi) $\sum_{n=\nu+1}^{\infty} (n|b_{nn}\lambda_{n+1}|)^{k-1} |\widehat{b}_{n,\nu+1}\lambda_{n+1}| = O((\nu|b_{\nu\nu}\lambda_{\nu+1}|)^{k-1})$,

(vii) $\sum_{n=1}^{\infty} n^{k-1} |\sum_{\nu=2}^{n} \widehat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i} X_i|^k = O(1)$,

then $\lambda \in (|A|_k, |B|_k)$.

A weighted mean matrix is a lower triangular matrix with entries p_k/P_n, $0 \leq k \leq n$, where $P_n = \sum_{k=0}^{n} p_k$.

Corollary 2. *Let $1 < k \leq s < \infty$. Let $\{\lambda_n\}$ be a sequence of constants, let B be a triangle such that B, and let $\{p_n\}$ satisfy*

(i) $b_{\nu\nu}\lambda_\nu = O((p_\nu/P_\nu)\nu^{1/s-1/k})$,

(ii) $(n|X_n|)^{s-k} = O(1)$,

(iii) $\sum_{\nu=1}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)| = O(|b_{nn}\lambda_n|)$,

(iv) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}\lambda_n|)^{s-1}|\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)| = O(\nu^{s-1}|b_{\nu\nu}\lambda_\nu|^s)$,

(v) $\sum_{\nu=1}^{n-1} |b_{\nu\nu}\lambda_\nu||\widehat{b}_{n,\nu}\lambda_\nu| = O(|b_{nn}\lambda_n|)$,

(vi) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}\lambda_n|)^{s-1}|\widehat{b}_{n,\nu}\lambda_\nu| = O((\nu|b_{\nu,\nu}\lambda_\nu|)^{s-1})$,

then $\lambda \in (|\overline{N}, p_n|_k, |B|_s)$.

Proof. Conditions (i), (ii), (iii)–(vii) of Theorem 1 reduce to conditions (i)–(vi), respectively, of Corollary 1.

With $A = (\overline{N}, p_n)$,

$$a_{nn} - a_{n+1,n} = \frac{p_n}{P_n} - \frac{p_n}{P_{n+1}} = \frac{p_n p_{n+1}}{P_n P_{n+1}} = a_{nn}a_{n+1,n+1}, \tag{17}$$

and condition (ii) of Theorem 1 is automatically satisfied.

A matrix A is said to be factorable if $a_{nk} = b_n c_k$ for each n and k.

Since A is a weighted mean matrix, $\widehat{A}$ is a factorable triangle and it is easy to show that its inverse is bidiagonal. Therefore condition (viii) of Theorem 1 is trivially satisfied. □

We now turn our attention to obtaining necessary conditions.

Theorem 2. *Let $1 < k \leq s < \infty$, and let A and B be two lower triangular matrices with A satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{a}_{n\nu}|^k = O(|a_{\nu\nu}|^k). \tag{18}$$

Then necessary conditions for $\lambda \in (|A|_k, |B|_s)$ are

(i) $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}|\nu^{1/s-1/k})$,

(ii) $\sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu \widehat{b}_{n\nu}\lambda_\nu|^s) = O(|a_{\nu\nu}|^s \nu^{s-s/k})$,

(iii) $\sum_{n=\nu+1}^{\infty} n^{s-1}|\widehat{b}_{n,\nu+1}\lambda_{\nu+1}|^s = O(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k)^{s/k}$.

Proof. Define

$$A^* = \Big\{\{a_i\} : \sum a_i \text{ is summable } |A|_k\Big\},$$
$$B^* = \Big\{\{b_i\} : \sum b_i\lambda_i \text{ is summable } |B|_s\Big\}. \tag{19}$$

With Y_n and X_n as defined by (6) and (7), the spaces A^* and B^* are BK-spaces, with norms given by

$$\|a\|_1 = \left\{ |X_0|^k + \sum_{n=1}^{\infty} n^{k-1} |X_n|^k \right\}^{1/k},$$
$$\|a\|_2 = \left\{ |Y_0|^s + \sum_{n=1}^{\infty} n^{s-1} |Y_n|^s \right\}^{1/s}, \tag{20}$$

respectively.

From the hypothesis of the theorem, $\|a\|_1 < \infty$ implies that $\|a\|_2 < \infty$. The inclusion map $i : A^* \to B^*$ defined by $i(x) = x$ is continuous, since A^* and B^* are BK-spaces. Applying the Banach-Steinhaus theorem, there exists a constant $K > 0$ such that

$$\|a\|_2 \le K \|a\|_1. \tag{21}$$

Let e_n denote the nth coordinate vector. From (6) and (7), with $\{a_n\}$ defined by $a_n = e_n - e_{n+1}$, $n = \nu$, $a_n = 0$ otherwise, we have

$$X_n = \begin{cases} 0, & n < \nu, \\ \hat{a}_{n\nu}, & n = \nu, \\ \Delta_\nu \hat{a}_{n\nu}, & n > \nu, \end{cases}$$
$$Y_n = \begin{cases} 0, & n < \nu, \\ \hat{b}_{n\nu}\lambda_\nu, & n = \nu, \\ \Delta_\nu (\hat{b}_{n\nu}\lambda_\nu), & n > \nu. \end{cases} \tag{22}$$

From (20),

$$\|a\|_1 = \left\{ \nu^{k-1} |a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{a}_{n\nu}|^k \right\}^{1/k},$$
$$\|a\|_2 = \left\{ \nu^{s-1} |b_{\nu\nu}\lambda_\nu|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu (\hat{b}_{n\nu}\lambda_\nu)|^s \right\}^{1/s}, \tag{23}$$

recalling that $\hat{b}_{\nu\nu} = \bar{b}_{\nu\nu} = b_{\nu\nu}$.

From (21), using (18), we obtain

$$\begin{aligned} &\nu^{s-1} |b_{\nu\nu}\lambda_\nu|^s + \sum_{n=\nu+1}^{\infty} n^{s-1} |\Delta_\nu (\hat{b}_{n\nu}\lambda_\nu)|^s \\ &\quad \le K^s \left(\nu^{k-1} |a_{\nu\nu}|^k + \sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu \hat{a}_{n\nu}|^k \right)^{s/k} \le K^s \left(\nu^{k-1} |a_{\nu\nu}|^k + O(1) |a_{\nu\nu}|^k \right)^{s/k} \\ &\quad = O\left(|a_{\nu\nu}|^k (\nu^{k-1} + 1) \right)^{s/k} = O\left(\nu^{k-1} |a_{\nu\nu}|^k \right)^{s/k}. \end{aligned} \tag{24}$$

The above inequality will be true if and only if each term on the left-hand side is $O(\nu^{k-1}|a_{\nu\nu}|^k)^{s/k}$. Using the first term,

$$\nu^{s-1}|b_{\nu\nu}\lambda_\nu|^s = O(\nu^{k-1}|a_{\nu\nu}|^k)^{s/k}, \tag{25}$$

which implies that $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}|\nu^{1/s-1/k})$, and (i) is necessary.

Using the second term, we obtain

$$\sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^s = O(\nu^{k-1}|a_{\nu\nu}|^k)^{s/k} = O(\nu^{s-s/k}|a_{\nu\nu}|^s), \tag{26}$$

which is condition (ii).

If we now define $a_n = e_{n+1}$ for $n = \nu$, $a_n = 0$ otherwise, then from (6) and (7), we obtain

$$X_n = \begin{cases} 0, & n \le \nu, \\ \hat{a}_{n,\nu+1}, & n > \nu, \end{cases}$$
$$Y_n = \begin{cases} 0, & n \le \nu, \\ \hat{b}_{n,\nu+1}\lambda_{\nu+1}, & n > \nu. \end{cases} \tag{27}$$

The corresponding norms are

$$\|a\|_1 = \left\{\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k\right\}^{1/k},$$
$$\|a\|_2 = \left\{\sum_{n=\nu+1}^{\infty} n^{s-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^s\right\}^{1/s}. \tag{28}$$

Applying (21),

$$\sum_{n=\nu+1}^{\infty} n^{s-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^s \le K^s\left\{\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k\right\}^{s/k}, \tag{29}$$

which implies condition (iii). □

Corollary 3. *Let A and B be two lower triangular matrices with A satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu\hat{a}_{n\nu}|^k = O(|a_{\nu\nu}|^k). \tag{30}$$

Then necessary conditions for $\lambda \in (|A|_k, |B|_k)$ are

(i) $|b_{\nu\nu}\lambda_\nu| = O(|a_{\nu\nu}|)$,

(ii) $(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu\hat{b}_{n\nu}\lambda_\nu|^k)^{1/k} = O(|a_{\nu\nu}|\nu^{1-1/k})$,

(iii) $\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k = O(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{a}_{n,\nu+1}|^k)$.

Corollary 4. *Let $1 < k \leq s < \infty$. Let B be a lower triangular matrix, $\{p_n\}$ is a sequence satisfying*

$$\sum_{n=\nu+1}^{\infty} n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k = O\left(\frac{1}{P_\nu^k}\right). \tag{31}$$

Then necessary conditions for $\lambda \in (|\overline{N}, p_n|_k, |B|_s)$ are

(i) $P_\nu|b_{\nu\nu}\lambda_\nu|/p_\nu = O(\nu^{1/s-1/k})$,

(ii) $\sum_{n=\nu+1}^{\infty} n^{s-1}|\Delta_\nu(\widehat{b}_{n\nu}\lambda_\nu)|^s = O(\nu^{s-s/k}(p_\nu/P_\nu)^s)$,

(iii) $\sum_{n=\nu+1}^{\infty} n^{s-1}|\widehat{b}_{n,\nu+1}\lambda_{\nu+1}|^s = O(1)$.

Proof. With $A = (\overline{N}, p_n)$, (18) becomes (31), and conditions (i)–(iii) of Theorem 2 become conditions (i)–(iii) of Corollary 4, respectively. □

Every summability factor theorem becomes an inclusion theorem by setting each $\lambda_n = 1$.

Corollary 5 (see [3]). *Let $1 < k \leq s < \infty$. Let A and B be triangles satisfying*

(i) $|b_{nn}|/|a_{nn}| = O(\nu^{1/s-1/k})$,

(ii) $(n|X_n|)^{s-k} = O(1)$,

(iii) $|a_{nn} - a_{n+1,n}| = O(|a_{nn}a_{n+1,n+1}|)$,

(iv) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu})| = O(|b_{nn}|)$,

(v) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}|)^{s-1}|\Delta_\nu(\widehat{b}_{n\nu})| = O(\nu^{s-1}|b_{\nu\nu}|^s)$,

(vi) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}||\widehat{b}_{n,\nu+1}| = O(|b_{nn}|)$,

(vii) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}|)^{s-1}|\widehat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{s-1})$,

(viii) $\sum_{n=1}^{\infty} n^{s-1}|\sum_{\nu=2}^{n} \widehat{b}_{n\nu} \sum_{i=0}^{\nu-2} \widehat{a}'_{\nu i}X_i|^s = O(1)$.

Then if $\sum a_n$ is summable $|A|_k$, it is summable $|B|_s$.

Corollary 6 (see [4]). *Let be $\{p_n\}$ a sequence of positive constants, B is a triangle satisfying*

(i) $P_n|b_{nn}| = O((p_n)\nu^{1/s-1/k})$,

(ii) $(n|X_n|)^{s-k} = O(1)$,

(iii) $\sum_{\nu=0}^{n-1} |\Delta_\nu(\widehat{b}_{n\nu})| = O(|b_{nn}|)$,

(iv) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}|)^{s-1}|\Delta_\nu(\widehat{b}_{n\nu})| = O(\nu^{s-1}|b_{\nu\nu}|^s)$,

(v) $\sum_{\nu=0}^{n-1} |b_{\nu\nu}\widehat{b}_{n,\nu+1}| = O(|b_{nn}|)$,

(vi) $\sum_{n=\nu+1}^{\infty}(n|b_{nn}|)^{s-1}|\widehat{b}_{n,\nu+1}| = O((\nu|b_{\nu\nu}|)^{s-1})$.

Then if $\sum a_n$ is summable $|\overline{N}, p_n|_k$, it is summable $|B|_s$.

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