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Research Article

Strong Convergence to Common Fixed Points of a Finite Family of Nonexpansive Mappings

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We suggest and analyze an iterative algorithm for a finite family of nonexpansive mappings $T_1, T_2, \ldots, T_r$. Further, we prove that the proposed iterative algorithm converges strongly to a common fixed point of $T_1, T_2, \ldots, T_r$.

1. Introduction

Let C be a closed convex subset of a Banach space E. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let $T_1, T_2, \ldots, T_r$ be a finite family of nonexpansive mappings satisfying that the set $F = \bigcap_{i=1}^{r} F(T_i)$ of common fixed points of $T_1, T_2, \ldots, T_r$ is nonempty. The problem of finding a common fixed point has been investigated by many researchers; see, for example, Atsushiba and Takahashi [1], Bauschke [2], Lions [3], Shimizu and Takahashi [4], Takahashi et al. [5], Zeng et al. [6]. To solve this problem, the iterative scheme $x_1 \in C$ and

$$x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) T_n x_n, \quad n \in \mathbb{N}, \tag{1.1}$$

where $T_{n+r} = T_n$ and $0 < \alpha_n$, is used. Wittmann [7] dealt with the iterative scheme for the case $r = 1$; see originally Halpern [8]. Bauschke [2] dealt with the iterative scheme for a finite family of nonexpansive mappings under the restriction that

$$F = F(T_r T_{r-1} \cdots T_1) = F(T_1 T_r \cdots T_2) = \cdots = F(T_{r-1} \cdots T_1 T_r). \tag{1.2}$$

Recently, Kimura et al. [9] dealt with an iteration scheme which is more general than that of Wittmann's result. They proved the following theorems.

Theorem 1.1 (see [9, Theorem 4]). *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let $T_1, T_2, \dots, T_r$ be nonexpansive mappings of C into itself such that the set $F = \bigcap_{i=1}^r F(T_i)$ of common fixed points of $T_1, T_2, \dots, T_r$ is nonempty. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ which satisfy the following control conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^\infty \alpha_n = \infty$;
(C3) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
(C4) $\lim_{n\to\infty} \beta_n^i = \beta^i$ *and* $\sum_{i=1}^r \beta_n^i = 1$, $n \in \mathbb{N}$ *for some* $\beta^i \in (0,1)$;
(C5) $\sum_{n=1}^\infty \sum_{i=1}^r |\beta_{n+1}^i - \beta_n^i| < \infty$.

Let $x \in C$ and define a sequence $\{x_n\}$ by $x_1 \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \sum_{i=1}^r \beta_n^i T_i x_n, \quad n \in \mathbb{N}. \tag{1.3}$$

Then $\{x_n\}$ converges strongly to the point Px, where P is a sunny nonexpansive retraction of C onto F.

Theorem 1.2 (see [9, Theorem 5]). *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let S, T be nonexpansive mappings of C into itself such that the set $F(S) \cap F(T)$ of common fixed points of S and T is nonempty. Let $x \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)(\beta_n S x_n + (1 - \beta_n) T x_n), \quad n \in \mathbb{N}. \tag{1.4}$$

Assume (C1) and (C2) hold and the following conditions are satisfied:

(C3′) $\lim_{n\to\infty}(\alpha_n/\alpha_{n+1}) = 1$;
(C4′) $\lim_{n\to\infty} \beta_n = \beta \in (0,1)$;
(C5′) $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$ converges strongly to the point Px, where P is a sunny nonexpansive retraction of C onto $F(S) \cap F(T)$.

We remark that the control conditions (C3) and (C3′) were introduced initially by Wittmann [7] and Xu [10], respectively. On the other hand, we have to remark that conditions (C1) and (C2) are necessary for the strong convergence of algorithms (1.3) and (1.4) for nonexpansive mappings. It is unclear if they are sufficient.

The objective of this paper is to show another generalization of Mann and Halpern iterative algorithm to a setting of a finite family of nonexpansive mappings. We deal with the iterative scheme $x_0 \in C$ and

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^r \tau_n^i T_i x_n, \quad n \in \mathbb{N}. \tag{1.5}$$

Using this iterative scheme, we can find a common fixed point of a finite family of nonexpansive mappings under some type of control conditions.

2. Preliminaries

Let E be a Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle\cdot,\cdot\rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.1}$$

for $x \in E$.

A Banach space E is said to be strictly convex if $\|(x+y)/2\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|(x_n+y_n)/2\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided that

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.2}$$

exists for each $x, y \in U$. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. The norm of E is said to be uniformly Gâteaux differentiable if for any $y \in U$ the limit exists uniformly for all $x \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then the normalized duality mapping J is norm to weak star uniformly continuous on any bounded subsets of E.

Let C be a closed convex subset of a Banach space E and let D be a subset of C. Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in C$. A mapping $P : C \to D$ is said to be sunny if $P(Px + t(x - Px)) = Px$ whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geq 0$. If $P^2 = P$, then P is called a retraction. We know that a retraction P of C onto D is sunny and nonexpansive if and only if $\langle x - Px, J(y - Px)\rangle \leq 0$ for all $y \in D$. From this inequality, it is easy to show that there exists at most one sunny nonexpansive retraction of C onto D. If there is a sunny nonexpansive retraction of C onto D, then D is said to be a sunny nonexpansive retraction of C.

Now, we introduce several lemmas for our main results in this paper.

Lemma 2.1 (see [11]). *Let C be a nonempty closed convex subset of a strictly convex Banach space. For each $r \in \mathbb{N}$, let T_r be a nonexpansive mapping of C into E. Let $\{\tau_r\}$ be a sequence of positive real numbers such that $\sum_{r=1}^{\infty}\tau_r = 1$. If $\bigcap_{r=1}^{\infty}F(T_r)$ is nonempty, then the mapping $T = \sum_{r=1}^{\infty}\tau_r T_r$ is well-defined and $F(T) = \bigcap_{r=1}^{\infty}F(T_r)$.*

Lemma 2.2 (see [12]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.3 (see [10]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\gamma_n)a_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

First, we consider the following iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n(\tau_n S x_n + (1-\tau_n)Tx_n), \quad n \ge 0, \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n\}$ are sequences in $[0,1]$.

Theorem 3.1. *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let S and T be nonexpansive mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $f : C \to C$ be a fixed contractive mapping. Assume that $\{z_t\}$ converges strongly to a fixed point z of U as $t \to 0$, where z_t is the unique element of C which satisfies $z_t = tf(z_t) + (1-t)Uz_t$, $U = \tau S + (1-\tau)T$, $0 < \tau < 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n\}$ be four real sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Assume $\{\alpha_n\}$ satisfies conditions (C1) and (C2) and assume the following control conditions hold:*

(D3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(D4) $\lim_{n\to\infty} \tau_n = \tau$.

For arbitrary $x_0 \in C$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point of S and T.

Proof. We show first that $\{x_n\}$ is bounded. To end this, by taking a fixed element $p \in F(S) \cap F(T)$ and using (3.1), we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n(\tau_n \|Sx_n - p\| + (1-\tau_n)\|Tx_n - p\|) \\ &\le \alpha_n \alpha \|x_n - p\| + \alpha_n \|f(p) - p\| + (\beta_n + \gamma_n)\|x_n - p\| \\ &= (1 - \alpha_n + \alpha\alpha_n)\|x_n - p\| + \alpha_n \|f(p) - p\| \\ &\le \max\left\{\|x_n - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\right\}. \end{aligned} \tag{3.2}$$

By induction, we get

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{1}{1-\alpha}\|f(p) - p\|\right\} \tag{3.3}$$

for all $n \ge 0$. This shows that $\{x_n\}$ is bounded, so are $\{Tx_n\}$, $\{Sx_n\}$, and $\{f(x_n)\}$.

We show then that $\|x_{n+1} - x_n\| \to 0 (n \to \infty)$.

Define a sequence $\{y_n\}$ which satisfies

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n. \tag{3.4}$$

Observe that

$$
\begin{aligned}
y_{n+1}-y_n &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)\\
&\quad+\frac{\gamma_{n+1}\tau_{n+1}}{1-\beta_{n+1}}\left(Sx_{n+1}-Sx_n\right)+\left(\frac{\gamma_{n+1}\tau_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n\tau_n}{1-\beta_n}\right)Sx_n\\
&\quad+\frac{\gamma_{n+1}(1-\tau_{n+1})}{1-\beta_{n+1}}\left(Tx_{n+1}-Tx_n\right)+\left(\frac{\gamma_{n+1}(1-\tau_{n+1})}{1-\beta_{n+1}}-\frac{\gamma_n(1-\tau_n)}{1-\beta_n}\right)Tx_n\\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(f(x_{n+1})-f(x_n)\right)+\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)f(x_n)\\
&\quad+\frac{\gamma_{n+1}\tau_{n+1}}{1-\beta_{n+1}}\left(Sx_{n+1}-Sx_n\right)+\frac{\gamma_{n+1}(1-\tau_{n+1})}{1-\beta_{n+1}}\left(Tx_{n+1}-Tx_n\right)\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(\tau_{n+1}-\tau_n\right)Sx_n+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)\tau_nSx_n\\
&\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left(\tau_n-\tau_{n+1}\right)Tx_n+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)(1-\tau_n)Tx_n.
\end{aligned}
\tag{3.5}
$$

It follows that

$$
\begin{aligned}
&\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\\
&\quad\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|f(x_n)\|\\
&\qquad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-1\right|\|x_{n+1}-x_n\|+\tau_n\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|Sx_n\|\\
&\qquad+(1-\tau_n)\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|Tx_n\|\\
&\qquad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left|\tau_n-\tau_{n+1}\right|\left(\|Sx_n\|+\|Tx_n\|\right)\\
&\quad\le \frac{(1+\alpha)\alpha_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\left|\tau_n-\tau_{n+1}\right|\left(\|Sx_n\|+\|Tx_n\|\right)\\
&\qquad+\left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\left(\|f(x_n)\|+\tau_n\|Sx_n\|+(1-\tau_n)\|Tx_n\|\right).
\end{aligned}
\tag{3.6}
$$

Since $\{x_n\}$, $\{Tx_n\}$, $\{Sx_n\}$, and $\{f(x_n)\}$ are bounded, we obtain

$$
\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\right)\le 0. \tag{3.7}
$$

Hence, by Lemma 2.2 we know that $\|y_n - x_n\| \to 0$ as $n \to \infty$. Consequently, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|y_n - x_n\| = 0$.

Define $U = \tau S + (1-\tau)T$. Then, by Lemma 2.1, $F(U) = F(S) \cap F(T)$.

Observing that

$$\begin{aligned} \|x_n - Ux_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - Ux_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - Ux_n\| + \beta_n \|x_n - Ux_n\| \\ &\quad + \gamma_n |\tau - \tau_n| (\|Sx_n\| + \|Tx_n\|) \end{aligned} \tag{3.8}$$

and using control conditions (C1), (D3), and (D4) on $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\tau_n\}$, we conclude that $\lim_{n\to\infty} \|Ux_n - x_n\| = 0$.

We next show that

$$\limsup_{n\to\infty} \langle z - f(z), j(z - x_n) \rangle \leq 0. \tag{3.9}$$

Let x_t be the unique fixed point of the contraction mapping U_t given by

$$U_t x = t f(x) + (1-t) U x. \tag{3.10}$$

Then

$$x_t - x_n = t(f(x_t) - x_n) + (1-t)(Ux_t - x_n). \tag{3.11}$$

We compute as follows:

$$\begin{aligned} \|x_t - x_n\|^2 &\leq (1-t)^2 \|Ux_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n) \rangle \\ &\leq (1-t)^2 (\|Ux_t - Ux_n\| + \|Ux_n - x_n\|)^2 \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n) \rangle + 2t\|x_t - x_n\|^2 \\ &\leq (1-t)^2 \|x_t - x_n\|^2 + a_n(t) + 2t\|x_t - x_n\|^2 \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n) \rangle, \end{aligned} \tag{3.12}$$

where $a_n(t) = \|Ux_n - x_n\|(2\|x_t - x_n\| + \|Ux_n - x_n\|) \to 0$ as $n \to \infty$.

The last inequality implies

$$\langle x_t - f(x_t), j(x_t - x_n) \rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t} a_n(t). \tag{3.13}$$

It follows that

$$\limsup_{n\to\infty} \langle x_t - f(x_t), j(x_t - x_n) \rangle \leq \frac{t}{2} M^2. \tag{3.14}$$

Letting $t \to 0$, we obtain

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \le 0. \tag{3.15}$$

Moreover, we have

$$\begin{aligned}
\langle z - f(z), j(z - x_n)\rangle &= \langle z - f(z), j(z - x_n)\rangle - \langle z - f(z), j(x_t - x_n)\rangle \\
&\quad + \langle z - f(z), j(x_t - x_n)\rangle - \langle x_t - f(z), j(x_t - x_n)\rangle \\
&\quad + \langle x_t - f(z), j(x_t - x_n)\rangle - \langle x_t - f(x_t), j(x_t - x_n)\rangle \\
&\quad + \langle x_t - f(x_t), j(x_t - x_n)\rangle \\
&= \langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle \\
&\quad + \langle z - x_t, j(x_t - x_n)\rangle + \langle f(x_t) - f(z), j(x_t - x_n)\rangle \\
&\quad + \langle x_t - f(x_t), j(x_t - x_n)\rangle.
\end{aligned} \tag{3.16}$$

Then, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle \\
&\quad \le \sup_{n\in\mathbb{N}}\langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle + \|z - x_t\|\limsup_{n\to\infty}\|x_t - x_n\| \\
&\qquad + \|f(x_t) - f(z)\|\limsup_{n\to\infty}\|x_t - x_n\| + \limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \\
&\quad \le \sup_{n\in\mathbb{N}}\langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle + (1+\alpha)\|z - x_t\|\limsup_{n\to\infty}\|x_t - x_n\| \\
&\qquad + \limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle.
\end{aligned} \tag{3.17}$$

By hypothesis $x_t \to z \in F(S)\cap F(T)$ as $t \to 0$ and j is norm-to-weak* uniformly continuous on bounded subset of E, we obtain

$$\limsup_{t\to 0}\sup_{n\in\mathbb{N}}\langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle = 0. \tag{3.18}$$

Therefore, we have

$$\begin{aligned}
\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle \\
&\le \limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \le 0.
\end{aligned} \tag{3.19}$$

Finally, we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \|\beta_n(x_n-z)+\gamma_n(\tau_n Sx_n+(1-\tau_n)Tx_n-z)\|^2 \\
&\quad +2\alpha_n\langle f(x_n)-z, j(x_{n+1}-z)\rangle \\
&\le \beta_n^2\|x_n-z\|^2+\gamma_n^2\|\tau_n(Sx_n-z)+(1-\tau_n)(Tx_n-z)\|^2 \\
&\quad +2\beta_n\gamma_n\|x_n-z\|\|\tau_n(Sx_n-z)+(1-\tau_n)(Tx_n-z)\| \\
&\quad +2\alpha_n\langle f(x_n)-f(z), j(x_{n+1}-z)\rangle+2\alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\le \beta_n^2\|x_n-z\|^2+\gamma_n^2\|x_n-z\|^2+2\beta_n\gamma_n\|x_n-z\|\|x_n-z\| \\
&\quad +2\alpha_n\langle f(x_n)-f(z), j(x_{n+1}-z)\rangle+2\alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\le (1-\alpha_n)^2\|x_n-z\|^2+\alpha\alpha_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\
&\quad +2\alpha_n\langle f(z)-z, j(x_{n+1}-z)\rangle.
\end{aligned} \tag{3.20}$$

It follows that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \frac{1-(2-\alpha)\alpha_n}{1-\alpha\alpha_n}\|x_n-z\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(z)-z, j(x_{n+1}-z)\rangle \\
&\quad +\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-z\|^2 \le [1-2(1-\alpha)\alpha_n]\|x_n-z\|^2 \\
&\quad +2(1-\alpha)\alpha_n\left\{\frac{1}{(1-\alpha)(1-\alpha\alpha_n)}\left[\langle f(z)-z, j(x_{n+1}-z)\rangle+\frac{\alpha_n}{2}\|x_n-z\|^2\right]\right\}.
\end{aligned} \tag{3.21}$$

Noting that $\sum_{n=0}^{\infty}[2(1-\alpha)\alpha_n]=\infty$ and

$$\limsup_{n\to\infty}\left\{\frac{1}{(1-\alpha)(1-\alpha\alpha_n)}\left[\langle f(z)-z, j(x_{n+1}-z)\rangle+\frac{\alpha_n}{2}\|x_n-z\|^2\right]\right\}\le 0. \tag{3.22}$$

Apply Lemma 2.3 to (3.21) to conclude that $x_n \to z$ as $n\to\infty$. This completes the proof. □

Remark 3.2. We note that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm. By Xu [13, Theorem 4.1], we know that $\{z_t\}$ converges strongly to a fixed point of U as $t\to 0$, where z_t is the unique element of C which satisfies $z_t = tf(z_t)+(1-t)Uz_t$.

Corollary 3.3. *Let E be a strictly convex and uniformly smooth Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let S and T be nonexpansive mappings of C into itself such that $F(S)\cap F(T)\neq\emptyset$. Let $f: C\to C$ be a fixed contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n\}$ be four real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$. Assume the control conditions (C1), (C2), (D3), and (D4) are satisfied. For arbitrary $x_0\in C$, then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point of S and T.*

We can obtain the following results from Takahashi and Ueda [14] which is related to the existence of sunny nonexpansive retractions.

Corollary 3.4. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let S and T be nonexpansive mappings of C into itself such that $F(S)\cap F(T)\neq\emptyset$. Let $u\in C$ be a given point. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n\}$ be four real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$. Assume the control conditions (C1), (C2), (D3), and (D4) are satisfied. For arbitrary $x_0\in C$, let the sequence $\{x_n\}$ be defined by*

$$x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n\big(\tau_n Sx_n+(1-\tau_n)Tx_n\big),\quad n\geq 0. \tag{3.23}$$

Then $\{x_n\}$ converges strongly to the point Pu, where P is a sunny nonexpansive retraction of C onto $F(S)\cap F(T)$.

We can also obtain the following theorems for a finite family of nonexpansive mappings. The proof is similar to that of Theorem 3.1, the details of the proof, therefore, are omitted.

Theorem 3.5. *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable and let C be a closed convex subset of E. Let $T_1,T_2,\dots,T_r$ be a finite family of nonexpansive mappings of C into itself such that the set $F=\bigcap_{i=1}^r F(T_i)$ of common fixed points of $T_1,T_2,\dots,T_r$ is nonempty. Let $f:C\to C$ be a fixed contractive mapping. Assume that $\{z_t\}$ converges strongly to a fixed point z of U as $t\to 0$, where z_t is the unique element of C which satisfies $z_t=tf(z_t)+(1-t)Uz_t$, $U=\sum_{i=1}^r\tau^iT_i$, $0<\tau^i<1$, and $\sum_{i=1}^r\tau_n^i=1$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n^i\}$ be real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$. Assume the control conditions (C1), (C2), and (D3) hold. Assume $\{\tau_n^i\}$ satisfies the condition (D4′):*

$$\lim_{n\to\infty}\tau_n^i=\tau^i,\quad i=1,2,\dots,r,\quad \sum_{i=1}^r\tau_n^i=1. \tag{3.24}$$

For arbitrary $x_0\in C$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+\gamma_n\sum_{i=1}^r\tau_n^iT_ix_n,\quad n\geq 0. \tag{3.25}$$

Then $\{x_n\}$ converges strongly to a common fixed point of $T_1,T_2,\dots,T_r$.

Theorem 3.6. *Let E be a strictly convex and uniformly smooth Banach space and let C be a closed convex subset of E. Let $T_1,T_2,\dots,T_r$ be a finite family of nonexpansive mappings of C into itself such that the set $F=\bigcap_{i=1}^r F(T_i)$ of common fixed points of $T_1,T_2,\dots,T_r$ is nonempty. Let $f:C\to C$ be a fixed contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\tau_n^i\}$ be real sequences in $[0,1]$ such that $\alpha_n+\beta_n+\gamma_n=1$. Assume the control conditions (C1), (C2), (D3), and (D4′) are satisfied. For arbitrary $x_0\in C$, then the sequence $\{x_n\}$ defined by (3.25) converges strongly to a common fixed point of $T_1,T_2,\dots,T_r$.*

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