Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 70465, 15 pages
doi:10.1155/2007/70465

Research Article

Some Nonlinear Integral Inequalities on Time Scales

Wei Nian Li and Weihong Sheng

Received 28 August 2007; Accepted 6 November 2007

Recommended by Alberto Cabada

The purpose of this paper is to investigate some nonlinear integral inequalities on time scales. Our results unify and extend some continuous inequalities and their corresponding discrete analogues. The theoretical results are illustrated by a simple example at the end of this paper.

1. Introduction

The theory of time scales was introduced by Hilger [1] in his Ph.D. thesis in 1988 in order to unify continuous and discrete analysis. Recently, many authors have extended some fundamental integral inequalities used in the theory of differential and integral equations on time scales. For example, we refer the reader to the literatures [2–8] and the references cited therein.

In this paper, we investigate some nonlinear integral inequalities on time scales, which unify and extend some inequalities established by Pachpatte in [9]. The obtained inequalities can be used as important tools in the study of certain properties of dynamic equations on time scales.

2. Preliminaries on time scales

We first briefly introduce the time scales calculus, which can be found in [4, 5].

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{C}$ denotes the set of complex numbers, and $C(M,S)$ denotes the class of all continuous functions defined on set M with range in the set S.

We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The *forward jump operator* σ on $\mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T} \quad \forall t \in \mathbb{T}. \tag{2.1}$$

In this definition we put $\inf \varnothing = \sup \mathbb{T}$, where $\varnothing$ is the empty set. If $\sigma(t) > t$, then we say that t is *right-scattered*. If $\sigma(t) = t$ and $t < \sup\mathbb{T}$, then we say that t is *right-dense*. The *backward jump operator*, *left-scattered* and *left-dense* points are defined in a similar way. The *graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left–scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Remark 2.1. Clearly, we see that $\sigma(t) = t$ if $\mathbb{T} = \mathbb{R}$ and $\sigma(t) = t + 1$ if $\mathbb{T} = \mathbb{Z}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \geq t_0$, $t \in \mathbb{T}^\kappa$, we define $f^\triangle(t)$ to be the number (provided it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t with

$$\left| [f(\sigma(t)) - f(s)] - f^\triangle(t)[\sigma(t) - s] \right| \leq \varepsilon |\sigma(t) - s| \quad \forall s \in U. \tag{2.2}$$

We call $f^\triangle(t)$ the *delta derivative* of f at t.

Remark 2.2. $f^\triangle$ is the usual derivative f' if $\mathbb{T} = \mathbb{R}$ and the usual forward difference $\boldsymbol{\Delta} f$ (defined by $\boldsymbol{\Delta} f(t) = f(t+1) - f(t)$) if $\mathbb{T} = \mathbb{Z}$.

We say that $f : \mathbb{T} \to \mathbb{R}$ is *rd–continuous* provided f is continuous at each right–dense point of $\mathbb{T}$ and has a finite left–sided limit at each left–dense point of $\mathbb{T}$. As usual, the set of rd–continuous functions is denoted by C_{rd}. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\triangle(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. In this case we define the *Cauchy integral* of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for } a, b \in \mathbb{T}. \tag{2.3}$$

We say that $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. We denote by $\mathcal{R}$ the set of all regressive and rd–continuous functions. We define the set of all positively regressive functions by $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all t} \in \mathbb{T}\}$.

For $h > 0$, we define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh), \tag{2.4}$$

where Log is the principal logarithm function, and

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \qquad \mathbb{Z}_{h=} \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\}. \tag{2.5}$$

For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

If $p \in \mathcal{R}$, then we define the *exponential function* by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right) \quad \text{for } s, t \in \mathbb{T}. \tag{2.6}$$

THEOREM 2.3. *If $p \in \mathcal{R}$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_p(\cdot, t_0)$ is for the unique solution of the initial value problem*

$$x^\Delta = p(t)x, \quad x(t_0) = 1 \quad on\, \mathbb{T}. \tag{2.7}$$

THEOREM 2.4. *If $p \in \mathcal{R}$, then*
(i) $e_p(t,t) \equiv 1$ *and* $e_0(t,s) \equiv 1$;
(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;
(iii) *if $p \in \mathcal{R}^+$, then $e_p(t,t_0) > 0$ for all $t \in \mathbb{T}$.*

Remark 2.5. Clearly, the exponential function is given by

$$e_p(t,s) = e^{\int_s^t p(\tau)d\tau}, \qquad e_\alpha(t,s) = e^{\alpha(t-s)}, \qquad e_\alpha(t,0) = e^{\alpha t} \tag{2.8}$$

for $s,t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function if $\mathbb{T} = \mathbb{R}$, and the exponential function is given by

$$e_p(t,s) = \prod_{\tau=s}^{t-1}[1+p(\tau)], \qquad e_\alpha(t,s) = (1+\alpha)^{t-s}, \qquad e_\alpha(t,0) = (1+\alpha)^t \tag{2.9}$$

for $s,t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$ if $\mathbb{T} = \mathbb{Z}$.

THEOREM 2.6. *If $p \in \mathcal{R}$ and $a,b,c \in \mathbb{T}$, then*

$$\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b). \tag{2.10}$$

THEOREM 2.7. *Let $t_0 \in \mathbb{T}^\kappa$ and $w : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ be continuous at (t,t), $t \in \mathbb{T}^\kappa$ with $t > t_0$. Assume that $w_1^\Delta(t,\cdot)t$ is rd-continuous on $[t_0,\sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood $U$of t, independent of $\tau \in [t_0,\sigma(t)]$, such that*

$$\left| w(\sigma(t),\tau) - w(s,\tau) - w_1^\Delta(t,\tau)(\sigma(t)-s) \right| \le \varepsilon \left|\sigma(t)-s\right| \quad \forall s \in U, \tag{2.11}$$

where w_1^Δdenotes the derivative of w with respect to the first variable, then

$$v(t) := \int_{t_0}^t w(t,\tau)\Delta\tau \tag{2.12}$$

implies

$$v^\Delta(t) = \int_{t_0}^t w_1^\Delta(t,\tau)\Delta\tau + w(\sigma(t),t). \tag{2.13}$$

The following theorem, which can be found in [4, Theorem 6.1, p.253], is a foundational result in dynamic inequalities.

THEOREM 2.8 (Comparison theorem). *Suppose $u,b \in C_{rd}$, $a \in \mathcal{R}^+$. Then*

$$u^\Delta(t) \le a(t)u(t) + b(t), \quad t \ge t_0,\ t \in \mathbb{T}^\kappa \tag{2.14}$$

implies

$$u(t) \le u(t_0)e_a(t,t_0) + \int_{t_0}^{t} e_a(t,\sigma(\tau))b(\tau)\Delta\tau, \quad t \ge t_0,\ t \in \mathbb{T}^{\kappa}. \tag{2.15}$$

3. Main results

In this section, we deal with integral inequalities on time scales. Throughout this section, we always assume that $p \ge q > 0$, p and q are real constants, and $t \ge t_0$, $t_0 \in \mathbb{T}^{\kappa}$.

The following lemma is useful in our main results.

Lemma 3.1. *Let $a \ge 0$. Then*

$$a^{q/p} \le \left(\frac{q}{p}K^{(q-p)/p}a + \frac{p-q}{p}K^{q/p}\right) \quad \text{for any } K > 0. \tag{3.1}$$

Proof. If $a = 0$, then we easily see that the inequality (3.1) holds. Thus we only prove that the inequality (3.1) holds in the case of $a > 0$.

Letting

$$f(K) = \frac{q}{p}K^{(q-p)/p}a + \frac{p-q}{p}K^{q/p}, \quad K > 0, \tag{3.2}$$

we have

$$f'(K) = \frac{q(p-q)}{p^2}K^{(q-2p)/p}(K-a). \tag{3.3}$$

It is easy to see that

$$\begin{aligned} f'(K) &\ge 0, \quad K > a, \\ f'(K) &= 0, \quad K = a, \\ f'(K) &\le 0, \quad 0 < K < a. \end{aligned} \tag{3.4}$$

Therefore,

$$f(K) \ge f(a) = a^{q/p}. \tag{3.5}$$

The proof of Lemma 3.1 is complete. □

Theorem 3.2. *Assume that $u,a,b,g,h \in C_{\text{rd}}$, $u(t)$, $a(t)$, $b(t)$, $g(t)$, and $h(t)$ are nonnegative. Then*

$$u^p(t) \le a(t) + b(t)\int_{t_0}^{t} [g(\tau)u^p(\tau) + h(\tau)u^q(\tau)]\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{3.6}$$

implies

$$\begin{aligned} u(t) \le \Bigg\{ a(t) + b(t)\int_{t_0}^{t} \left[a(\tau)g(\tau) + h(\tau)\left(\frac{K(p-q) + qa(\tau)}{pK^{(p-q)/p}}\right)\right] \\ \times e_F(t,\sigma(\tau))\Delta\tau \Bigg\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \end{aligned} \tag{3.7}$$

where

$$F(t) = b(t)\left(g(t) + \frac{qh(t)}{pK^{(p-q)/p}}\right). \tag{3.8}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} \left[g(\tau)u^p(\tau) + h(\tau)u^q(\tau)\right]\Delta\tau, \quad t \in \mathbb{T}^\kappa. \tag{3.9}$$

Then $z(t_0) = 0$ and (3.6) can be restated as

$$u^p(t) \le a(t) + b(t)z(t), \quad t \in \mathbb{T}^\kappa. \tag{3.10}$$

Using Lemma 3.1, from (3.10), for any $K > 0$, we easily obtain

$$\begin{aligned} u^q(t) &\le \left(a(t) + b(t)z(t)\right)^{q/p} \\ &\le \frac{K(p-q) + qa(t)}{pK^{(p-q)/p}} + \frac{qb(t)z(t)}{pK^{(p-q)/p}}. \end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11), we get

$$\begin{aligned} z^\Delta(t) &= g(t)u^p(t) + h(t)u^q(t) \\ &\le g(t)\left[a(t) + b(t)z(t)\right] + h(t)\left(\frac{K(p-q) + qa(t)}{pK^{(p-q)/p}} + \frac{qb(t)z(t)}{pK^{(p-q)/p}}\right) \\ &= \left[a(t)g(t) + \frac{K(p-q) + qa(t)}{pK^{(p-q)/p}}h(t)\right] + F(t)z(t), \quad t \in \mathbb{T}^\kappa, \end{aligned} \tag{3.12}$$

where $F(t)$ is defined as in (3.8).

It is easy to see that $F(t) \in \mathcal{R}^+$. Therefore, using Theorem 2.8 and noting $z(t_0) = 0$, from (3.12) we obtain

$$z(t) \le \int_{t_0}^{t} \left[a(\tau)g(\tau) + \frac{K(p-q) + qa(\tau)}{pK^{(p-q)/p}}h(\tau)\right] e_F(t, \sigma(\tau))\Delta\tau, \quad t \in \mathbb{T}^\kappa. \tag{3.13}$$

Clearly, the desired inequality (3.7) follows from (3.10) and (3.13). This completes the proof of Theorem 3.2. □

Corollary 3.3. *Let* $\mathbb{T} = \mathbb{R}$ *and assume that* $u(t), a(t), b(t), g(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. *Then the inequality*

$$u^p(t) \le a(t) + b(t)\int_0^t \left[g(s)u^p(s) + h(s)u^q(s)\right]ds, \quad t \in \mathbb{R}_+, \tag{3.14}$$

implies

$$u(t) \leq \left\{a(t)+b(t)\int_0^t \left[a(\tau)g(\tau)+h(\tau)\left(\frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\right] \times \exp\left(\int_\tau^t F(s)ds\right)d\tau\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{R}_+, \tag{3.15}$$

where $F(t)$ *is defined as in Theorem 3.2.*

Corollary 3.4. *Let* $\mathbb{T}=\mathbb{Z}$ *and assume that* $u(t)$, $a(t)$, $b(t)$, $g(t)$, *and* $h(t)$ *are nonnegative functions defined for* $t\in\mathbb{N}_0$. *Then the inequality*

$$u^p(t) \leq a(t)+b(t)\sum_{s=0}^{t-1}\left[g(s)u^p(s)+h(s)u^q(s)\right], \quad t\in\mathbb{N}_0, \tag{3.16}$$

implies

$$u(t) \leq \left\{a(t)+b(t)\sum_{\tau=0}^{t-1}\left[a(\tau)g(\tau)+h(\tau)\left(\frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\right] \times \prod_{s=\tau+1}^{t-1}\left(1+F(s)\right)\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{N}_0, \tag{3.17}$$

where $F(t)$ *is defined as in Theorem 3.2.*

Remark 3.5. Letting $p>1$, $K=q=1$ in Corollaries 3.3 and 3.4, we easily obtain Theorem $1(a_1)$ and Theorem $3(c_1)$ established by Pachpatte [9], respectively.

Corollary 3.6. *Assume that* $u,h\in C_{rd}$, $u(t)$ *and* $h(t)$ *are nonnegative. If* $\beta\geq 0$ *is a real constant, then*

$$u^p(t) \leq \beta+\int_{t_0}^t h(\tau)u^q(\tau)\Delta\tau, \quad t\in\mathbb{T}^\kappa, \tag{3.18}$$

implies

$$u(t) \leq \left\{\frac{1}{q}\left[(K(p-q)+q\beta)e_{\overline{F}}(t,t_0)-K(p-q)\right]\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{T}^\kappa, \tag{3.19}$$

where

$$\overline{F}(t) = \frac{qh(t)}{pK^{(p-q)/p}}. \tag{3.20}$$

Proof. Using Theorem 3.2, it follows from (3.18) that

$$\begin{aligned}
u(t) &\le \left\{\beta + \int_{t_0}^{t} h(\tau)\frac{K(p-q)+q\beta}{pK^{(p-q)/p}} e_{\overline{F}}(t,\sigma(\tau))\Delta\tau\right\}^{1/p} \\
&= \left\{\beta + \left(\frac{K(p-q)}{q}+\beta\right)\int_{t_0}^{t} \overline{F}(\tau) e_{\overline{F}}(t,\sigma(\tau))\Delta\tau\right\}^{1/p} \\
&= \left\{\beta + \left(\frac{K(p-q)}{q}+\beta\right)\left[e_{\overline{F}}(t,t_0) - e_{\overline{F}}(t,t)\right]\right\}^{1/p} \\
&= \left\{\beta + \left(\frac{K(p-q)}{q}+\beta\right)e_{\overline{F}}(t,t_0) - \frac{K(p-q)}{q} - \beta\right\}^{1/p} \\
&= \left\{\frac{1}{q}\left[(K(p-q)+q\beta)e_{\overline{F}}(t,t_0) - K(p-q)\right]\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa},
\end{aligned} \tag{3.21}$$

where the second equation holds because of Theorem 2.6, and the third equation holds because of Theorem 2.4(i). This completes the proof. □

Theorem 3.7. *Assume that $u,a,b,g,h_i \in C_{\mathrm{rd}}$, $u(t)$, $a(t)$, $b(t)$, $g(t)$, and $h_i(t)$ are nonnegative, and $i=1,2,\ldots,n$. If there exists a sequence of positive real numbers $q_1,q_2,\ldots,q_n$ such that $p \ge q_i > 0$, $i=1,2,\ldots,n$, then*

$$u^p(t) \le a(t) + b(t)\int_{t_0}^{t}\left[g(\tau)u^p(\tau)\sum_{i=1}^{n} h_i(\tau)u^{q_i}(\tau)\right]\Delta\tau, \quad t\in\mathbb{T}^{\kappa}, \tag{3.22}$$

implies

$$\begin{aligned}
u(t) \le \Bigg\{ a(t) + b(t)\int_{t_0}^{t}\left[a(\tau)g(\tau) + \sum_{i=1}^{n} h_i(\tau)\left(\frac{K(p-q_i)+q_i a(\tau)}{pK^{(p-q_i)/p}}\right)\right] \\
\times e_{F^*}(t,\sigma(\tau))\Delta\tau\Bigg\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa},
\end{aligned} \tag{3.23}$$

where

$$F^*(t) = b(t)\left(g(t) + \sum_{i=1}^{n}\frac{q_i h_i(t)}{pK^{(p-q_i)/p}}\right). \tag{3.24}$$

Proof. Define $z(t)$ by

$$z(t) = \int_{t_0}^{t}\left[g(\tau)u^p(\tau) + \sum_{i=1}^{n} h_i(\tau)u^{q_i}(\tau)\right]\Delta\tau, \quad t\in\mathbb{T}^{\kappa}. \tag{3.25}$$

Then $z(t_0)=0$, and as in the proof of Theorem 3.2, we have (3.10) and

$$u^{q_i}(t) \le \frac{K(p-q_i)+q_i a(t)}{pK^{(p-q_i)/p}} + \frac{q_i b(t)z(t)}{pK^{(p-q_i)/p}} \quad \text{for any } K>0,\ i=1,2,\dots,n. \tag{3.26}$$

Therefore,

$$\begin{aligned} z^{\Delta}(t) &= g(t)u^p(t) + \sum_{i=1}^{n} h_i(t)u^{q_i}(t) \\ &\le g(t)[a(t)+b(t)z(t)] + \sum_{i=1}^{n} h_i(t)\left(\frac{K(p-q_i)+q_i a(t)}{pK^{(p-q_i)/p}} + \frac{q_i b(t)z(t)}{pK^{(p-q_i)/p}}\right) \\ &= \left[a(t)g(t) + \sum_{i=1}^{n} h_i(t)\left(\frac{K(p-q_i)+q_i a(t)}{pK^{(p-q_i)/p}}\right)\right] + F^*(t)z(t), \quad t\in\mathbb{T}^{\kappa}, \end{aligned} \tag{3.27}$$

where $F^*(t)$ is defined as in (3.24).

The remainder of the proof is similar to that of Theorem 3.2 and we omit it here. □

Theorem 3.8. *Assume that $u,a,b,g,h \in C_{\mathrm{rd}}$, $u(t)$, $a(t)$, $b(t)$, $g(t)$, and $h(t)$ are nonnegative, and $w(t,s)$ is defined as in Theorem 2.7 such that $w(t,s)\ge 0$ and $w_1^{\Delta}(t,s)\ge 0$ for $t,s\in\mathbb{T}$ with $s\le t$. If for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that for all $s\in U$,*

$$\left|\left[w(\sigma(t),\tau)-w(s,\tau)-w_1^{\Delta}(t,\tau)(\sigma(t)-s)\right]\left[g(\tau)u^p(\tau)+h(\tau)u^q(\tau)\right]\right| \le \varepsilon|\sigma(t)-s|, \tag{3.28}$$

then

$$u^p(t) \le a(t) + b(t)\int_{t_0}^{t} w(t,\tau)\left[g(\tau)u^p(\tau)+h(\tau)u^q(\tau)\right]\Delta\tau, \quad t\in\mathbb{T}^{\kappa}, \tag{3.29}$$

implies

$$u(t) \le \left\{a(t)+b(t)\int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa}, \tag{3.30}$$

where

$$\begin{aligned} A(t) &= w(\sigma(t),t)b(t)\left(g(t)+\frac{qh(t)}{pK^{(p-q)/p}}\right) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)b(\tau)\left(g(\tau)+\frac{qh(\tau)}{pK^{(p-q)/p}}\right)\Delta\tau, \\ B(t) &= w(\sigma(t),t)\left[a(t)g(t)+h(t)\left(\frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right)\right] \\ &\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[a(\tau)g(\tau)+h(\tau)\left(\frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\right]\Delta\tau, \quad t\in\mathbb{T}^{\kappa}. \end{aligned} \tag{3.31}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} k(t,\tau)\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{3.32}$$

where

$$k(t,\tau) = w(t,\tau)\big[g(\tau)u^p(\tau) + h(\tau)u^q(\tau)\big], \quad t \in \mathbb{T}^{\kappa}. \tag{3.33}$$

Then $z(t_0) = 0$. As in the proof of Theorem 3.2, we easily obtain (3.10) and (3.11).

It follows from (3.33) that

$$k(\sigma(t),t) = w(\sigma(t),t)\big[g(t)u^p(t) + h(t)u^q(t)\big], \tag{3.34}$$

$$k_1^{\Delta}(t,\tau) = w_1^{\Delta}(t,\tau)\big[g(\tau)u^p(\tau) + h(\tau)u^q(\tau)\big]. \tag{3.35}$$

Therefore, noting the condition (3.28), using Theorem 2.7, and combining (3.32)–(3.35), (3.10), and (3.11), we have

$$\begin{aligned}
z^{\Delta}(t) &= k(\sigma(t),t) + \int_{t_0}^{t} k_1^{\Delta}(t,\tau)\Delta\tau \\
&= w(\sigma(t),t)\big[g(t)u^p(t) + h(t)u^q(t)\big] + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\big[g(\tau)u^p(\tau) + h(\tau)u^q(\tau)\big]\Delta\tau \\
&\le w(\sigma(t),t)\left[a(t)g(t) + h(t)\left(\frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right) + b(t)\left(g(t) + \frac{qh(t)}{pK^{(p-q)/p}}\right)z(t)\right] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[a(\tau)g(\tau) + h(\tau)\left(\frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\right. \\
&\qquad\qquad \left. + b(\tau)\left(g(\tau) + \frac{qh(\tau)}{pK^{(p-q)/p}}\right)z(\tau)\right]\Delta\tau \\
&\le \left[w(\sigma(t),t)b(t)\left(g(t) + \frac{qh(t)}{pK^{(p-q)/p}}\right) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)b(\tau)\left(g(\tau) + \frac{qh(\tau)}{pK^{(p-q)/p}}\right)\Delta\tau\right]z(t) \\
&\quad + w(\sigma(t),t)\left[a(t)g(t) + h(t)\left(\frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right)\right] \\
&\quad + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\left[a(\tau)g(\tau) + h(\tau)\left(\frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\right]\Delta\tau \\
&= A(t)z(t) + B(t), \quad t \in \mathbb{T}^{\kappa}.
\end{aligned} \tag{3.36}$$

Therefore, using Theorem 2.8 and noting $z(t_0) = 0$, we get

$$z(t) \le \int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau, \quad t \in \mathbb{T}^{\kappa}. \tag{3.37}$$

It is easy to see that the desired inequality (3.30) follows from (3.10) and (3.37). The proof of Theorem 3.8 is complete. □

Corollary 3.9. *Let $\mathbb{T} = \mathbb{R}$ and assume that $u(t), a(t), b(t), g(t), h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$. If $w(t,s)$ and its partial derivative $(\partial/\partial t)\ w(t,s)$ are real-valued nonnegative continuous functions for $t,s \in \mathbb{R}_+$ with $s \le t$, then the inequality*

$$u^p(t) \le a(t) + b(t)\int_0^t w(t,\tau)[g(\tau)u^p(\tau) + h(\tau)u^q(\tau)]d\tau, \quad t \in \mathbb{R}_+, \tag{3.38}$$

implies

$$u(t) \le \left\{a(t) + b(t)\int_0^t \exp\left(\int_\tau^t \overline{A}(s)ds\right)\overline{B}(\tau)d\tau\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{R}_+, \tag{3.39}$$

where

$$\begin{aligned}
\overline{A}(t) &= w(t,t)b(t)\left(g(t) + \frac{qh(t)}{pK^{(p-q)/p}}\right) + \int_0^t \frac{\partial w(t,\tau)}{\partial t} b(\tau)\left(g(\tau) + \frac{qh(\tau)}{pK^{(p-q)/p}}\right)d\tau, \\
\overline{B}(t) &= w(t,t)\left[a(t)g(t) + h(t)\left(\frac{K(p-q) + qa(t)}{pK^{(p-q)/p}}\right)\right] \\
&\quad + \int_0^t \frac{\partial w(t,\tau)}{\partial t}\left[a(\tau)g(\tau) + h(\tau)\left(\frac{K(p-q) + qa(\tau)}{pK^{(p-q)/p}}\right)\right]d\tau, \quad t \in \mathbb{R}_+.
\end{aligned} \tag{3.40}$$

Corollary 3.10. *Let $\mathbb{T} = \mathbb{Z}$ and assume that $u(t)$, $a(t)$, $b(t)$, $g(t)$, and $h(t)$ are nonnegative functions defined for $t \in \mathbb{N}_0$. If $w(t,s)$and $\Delta_1 w(t,s)$ are real-valued nonnegative functions for $t,s \in \mathbb{N}_0$ with $s \le t$, then the inequality*

$$u^p(t) \le a(t) + b(t)\sum_{\tau=0}^{t-1} w(t,\tau)[g(\tau)u^p(\tau) + h(\tau)u^q(\tau)], \quad t \in \mathbb{N}_0, \tag{3.41}$$

implies

$$u(t) \le \left\{a(t) + b(t)\sum_{\tau=0}^{t-1} \widetilde{B}(\tau)\prod_{s=\tau+1}^{t-1}(1 + \widetilde{A}(s))\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{N}_0, \tag{3.42}$$

where $\Delta_1 w(t,s) = w(t+1,s) - w(t,s)$ for $t,s \in \mathbb{N}_0$ with $s \le t$,

$$\begin{aligned}
\widetilde{A}(t) &= w(t+1,t)b(t)\left(g(t) + \frac{qh(t)}{pK^{(p-q)/p}}\right) + \sum_{\tau=0}^{t-1}\Delta_1 w(t,\tau)b(\tau)\left(g(\tau) + \frac{qh(\tau)}{pK^{(p-q)/p}}\right), \\
\widetilde{B}(t) &= w(t+1,t)\left[a(t)g(t) + h(t)\left(\frac{K(p-q) + qa(t)}{pK^{(p-q)/p}}\right)\right] \\
&\quad + \sum_{\tau=0}^{t-1}\Delta_1 w(t,\tau)\left[a(\tau)g(\tau) + h(\tau)\left(\frac{K(p-q) + qa(\tau)}{pK^{(p-q)/p}}\right)\right], \quad t \in \mathbb{N}_0.
\end{aligned} \tag{3.43}$$

Remark 3.11. Let $p > 1$, $K = q = 1$. Then the inequality established in Corollary 3.9 reduces to the inequality established by Pachpatte in [9, Theorem 1(a_3)], and the inequality established in Corollary 3.10 reduces to the inequality in [9, Theorem 3(c_3)].

Corollary 3.12. *Suppose that $\alpha \ge 0$ is a constant, $u(t)$ and $w(t,s)$ are defined as in Theorem 3.8. If for any $\varepsilon > 0$, there exists a neighborhood U of t, independent of $\tau \in [t_0, \sigma(t)]$, such that for all $s \in U$,*

$$\left| u^q(\tau)\left[w(\sigma(t),\tau) - w(s,\tau) - w_1^{\Delta}(t,\tau)(\sigma(t)-s)\right]\right| \le \varepsilon \left|\sigma(t)-s\right|, \tag{3.44}$$

then

$$u^p(t) \le \alpha + \int_{t_0}^{t} w(t,\tau)u^q(\tau)\Delta\tau, \quad t \in \mathbb{T}^{\kappa}, \tag{3.45}$$

implies

$$u(t) \le \left\{\frac{1}{q}\left[(K(p-q)+q\alpha)e_{\widehat{A}}(t,t_0) - K(p-q)\right]\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}, \tag{3.46}$$

where

$$\widehat{A}(t) = \frac{q}{pK^{(p-q)/p}}\left(w(\sigma(t),t) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Delta\tau\right), \quad t \in \mathbb{T}^{\kappa}. \tag{3.47}$$

Proof. Letting $b(t) = 1$, $g(t) = 0$ and $h(t) = 1$ in Theorem 3.8, we obtain

$$\begin{aligned} A(t) &= \frac{q}{pK^{(p-q)/p}}\left(w(\sigma(t),t) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Delta\tau\right) := \widehat{A}(t), \quad t \in \mathbb{T}^{\kappa}, \\ B(t) &= \frac{K(p-q)+q\alpha}{pK^{(p-q)/p}}\left\{w(\sigma(t),t) + \int_{t_0}^{t} w_1^{\Delta}(t,\tau)\Delta\tau\right\} \\ &= \frac{K(p-q)+q\alpha}{q}\widehat{A}(t), \quad t \in \mathbb{T}^{\kappa}. \end{aligned} \tag{3.48}$$

Therefore, by Theorem 3.8, noting (3.48), we easily obtain

$$\begin{aligned} u(t) &\le \left\{\alpha + \int_{t_0}^{t} e_A(t,\sigma(\tau))B(\tau)\Delta\tau\right\}^{1/p} \\ &= \left\{\alpha + \int_{t_0}^{t} e_{\widehat{A}}(t,\sigma(\tau))\frac{K(p-q)+q\alpha}{q}\widehat{A}(\tau)\Delta\tau\right\}^{1/p} \\ &= \left\{\alpha + \frac{K(p-q)+q\alpha}{q}\int_{t_0}^{t} e_{\widehat{A}}(t,\sigma(\tau))\widehat{A}(\tau)\Delta\tau\right\}^{1/p} \\ &= \left\{\alpha + \frac{K(p-q)+q\alpha}{q}\left[e_{\widehat{A}}(t,t_0) - e_{\widehat{A}}(t,t)\right]\right\}^{1/p} \\ &= \left\{\frac{K(p-q)+q\alpha}{q}e_{\widehat{A}}(t,t_0) - \frac{K(p-q)}{q}\right\}^{1/p} \quad \text{for any } K > 0,\ t \in \mathbb{T}^{\kappa}. \end{aligned} \tag{3.49}$$

The proof of Corollary 3.12 is complete. □

By investigating the proof procedure of Theorem 3.8 carefully, we easily obtain the following result.

THEOREM 3.13. *Assume that $u,a,b,g,h_i \in C_{\mathrm{rd}}$, $u(t)$, $a(t)$, $b(t)$, $g(t)$, and $h_i(t)$ are nonnegative, $i=1,2,\dots,n$, and there exists a sequence of positive real numbers $q_1,q_2,\dots,q_n$ such that $p \ge q_i > 0$, $i=1,2,\dots,n$. Let $w(t,s)$ be defined as in Theorem 2.7 such that $w(t,s)\ge 0$ and $w_1^{\Delta}(t,s)\ge 0$ for $t,s\in\mathbb{T}$ with $s\le t$. If for any $\varepsilon>0$, there exists a neighborhood U of t, independent of $\tau\in[t_0,\sigma(t)]$, such that for all $s\in U$,*

$$\left|\left[w(\sigma(t),\tau)-w(s,\tau)-w_1^{\Delta}(t,\tau)(\sigma(t)-s)\right]\left[g(\tau)u^p(\tau)+\sum_{i=1}^{n}h_i(\tau)u^{q_i}(\tau)\right]\right| \le \varepsilon\,|\sigma(t)-s|, \tag{3.50}$$

then

$$u^p(t)\le a(t)+b(t)\int_{t_0}^{t}w(t,\tau)\left[g(\tau)u^p(\tau)+\sum_{i=1}^{n}h_i(\tau)u^{q_i}(\tau)\right]\Delta\tau,\quad t\in\mathbb{T}^{\kappa}, \tag{3.51}$$

implies

$$u(t)\le\left\{a(t)+b(t)\int_{t_0}^{t}e_{A^*}(t,\sigma(\tau))B^*(\tau)\Delta\tau\right\}^{1/p}\quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa}, \tag{3.52}$$

where

$$\begin{aligned}
A^*(t)&=w(\sigma(t),t)b(t)\left(g(t)+\sum_{i=1}^{n}\frac{q_ih_i(t)}{pK^{(p-q_i)/p}}\right)\\
&\quad+\int_{t_0}^{t}w_1^{\Delta}(t,\tau)b(\tau)\left(g(\tau)+\sum_{i=1}^{n}\frac{q_ih_i(\tau)}{pK^{(p-q_i)/p}}\right)\Delta\tau,\\
B^*(t)&=w(\sigma(t),t)\left[a(t)g(t)+\sum_{i=1}^{n}h_i(t)\left(\frac{K(p-q_i)+q_ia(t)}{pK^{(p-q_i)/p}}\right)\right]\\
&\quad+\int_{t_0}^{t}w_1^{\Delta}(t,\tau)\left[a(\tau)g(\tau)+\sum_{i=1}^{n}h_i(\tau)\left(\frac{K(p-q_i)+qa(\tau)}{pK^{(p-q_i)/p}}\right)\right]\Delta\tau,\quad t\in\mathbb{T}^{\kappa}.
\end{aligned} \tag{3.53}$$

THEOREM 3.14. *Assume that $u,a,b\in C_{\mathrm{rd}}$, $u(t)$, $a(t)$, and $b(t)$ are nonnegative. Let $f:\mathbb{T}^{\kappa}\times\mathbb{R}\to\mathbb{R}_+$ be a continuous function such that*

$$0\le f(t,x)-f(t,y)\le\phi(t,y)(x-y), \tag{3.54}$$

for $t\in\mathbb{T}^{\kappa}$ and $x\ge y\ge 0$, where $\phi:\mathbb{T}^{\kappa}\times\mathbb{R}\to\mathbb{R}_+$ is a continuous function. Then

$$u^p(t)\le a(t)+b(t)\int_{t_0}^{t}f(\tau,u^q(\tau))\Delta\tau,\quad t\in\mathbb{T}^{\kappa}, \tag{3.55}$$

implies

$$u(t)\le\left\{a(t)+b(t)\int_{t_0}^{t}e_M(t,\sigma(\tau))f\left(\tau,\frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\Delta\tau\right\}^{1/p}\quad\text{for any } K>0,\ t\in\mathbb{T}^{\kappa}, \tag{3.56}$$

where

$$M(t) = \phi\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right) \frac{qb(t)}{pK^{(p-q)/p}}. \tag{3.57}$$

Proof. Define a function $z(t)$ by

$$z(t) = \int_{t_0}^{t} f(\tau, u^q(\tau))\Delta\tau, \quad t \in \mathbb{T}^\kappa. \tag{3.58}$$

Then $z(t_0) = 0$ and (3.55) can be written as (3.10). As in the proof of Theorem 3.2, from (3.10), we easily obtain (3.11). Obviously, it follows from (3.58), (3.11), and (3.54) that

$$\begin{aligned} z^\Delta(t) &= f(t, u^q(t)) \\ &\le f\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}} + \frac{qb(t)}{pK^{(p-q)/p}} z(t)\right) - f\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right) \\ &\quad + f\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right) \\ &\le \phi\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right) \frac{qb(t)}{pK^{(p-q)/p}} z(t) + f\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right) \\ &= M(t)z(t) + f\left(t, \frac{K(p-q)+qa(t)}{pK^{(p-q)/p}}\right), \quad t \in \mathbb{T}^\kappa, \end{aligned} \tag{3.59}$$

where $M(t)$ is defined as in (3.57). Using Theorem 2.8 and noting $z(t_0) = 0$, from (3.59), we get

$$z(t) \le \int_{t_0}^{t} e_M(t, \sigma(\tau)) f\left(\tau, \frac{K(p-q)+qa(\tau)}{pK^{(p-q)/p}}\right)\Delta\tau, \quad t \in \mathbb{T}^\kappa. \tag{3.60}$$

It is easy to see that the desired inequality (3.56) follows from (3.10) and (3.60). The proof of Theorem 3.14 is complete. □

Remark 3.15. Let $p > 1$, $K = q = 1$. We easily see that Theorem 3.14 reduces to in [9, Theorem 2(b_1)] if $\mathbb{T} = \mathbb{R}$, and in [9, Theorem 4(d_1)] if $\mathbb{T} = \mathbb{Z}$.

By Theorem 3.14, we can establish the following more general result.

Theorem 3.16. *Assume that $u, a, b \in C_{rd}$, $u(t)$, $a(t)$, and $b(t)$ are nonnegative, and $f_i : \mathbb{T}^\kappa \times \mathbb{R} \to \mathbb{R}_+$ is a continuous function such that*

$$0 \le f_i(t,x) - f_i(t,y) \le \phi_i(t,y)(x-y), \tag{3.61}$$

for $t \in \mathbb{T}^\kappa$ and $x \ge y \ge 0$, where $\phi_i : \mathbb{T}^\kappa \times \mathbb{R} \to \mathbb{R}_+$ is a continuous function, $i = 1,2,\dots,n$. If there exists a sequence of positive real numbers $q_1, q_2, \dots, q_n$ such that $p \ge q_i > 0, i = 1,2,\dots,n$, then

$$u^p(t) \le a(t) + b(t)\sum_{i=1}^{n} \int_{t_0}^{t} f_i(\tau, u^{q_i}(\tau))\Delta\tau, \quad t \in \mathbb{T}^\kappa, \tag{3.62}$$

implies

$$u(t) \leq \left\{a(t)+b(t)\sum_{i=1}^{n}\int_{t_0}^{t} e_{M^*}(t,\sigma(\tau)) f_i\left(\tau, \frac{K(p-q_i)+q_i a(\tau)}{pK^{(p-q_i)/p}}\right)\Delta\tau\right\}^{1/p} \quad \text{for any } K>0,\ t\in T^{\kappa}, \tag{3.63}$$

where

$$M^*(t)=\sum_{i=1}^{n}\phi_i\left(t, \frac{K(p-q_i)+q_i a(t)}{pK^{(p-q_i)/p}}\right)\frac{q_i b(t)}{pK^{(p-q_i)/p}}. \tag{3.64}$$

Remark 3.17. Using our main results in this paper, we can obtain many dynamic inequalities on some peculiar time scales. Due to limited space, their statements are omitted here.

At the end of this paper, we present an application of Corollary 3.6 to obtain the explicit estimates on the solutions of a dynamic equation on time scales.

Consider the following initial value problem on time scales

$$(u^p(t))^{\Delta}=H(t,u^q(t)), \quad u(t_0)=C, \quad t\in\mathbb{T}^{\kappa}, \tag{3.65}$$

where C, p, and q are constants, $p\geq q>0$, and $H:\mathbb{T}^{\kappa}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Assume that

$$|H(t,u^q(t))|\leq h(t)|u^q(t)|, \quad t\in\mathbb{T}^{\kappa}. \tag{3.66}$$

If $u(t)$ is a solution of IVP (3.65), then

$$|u(t)|\leq\left\{\frac{1}{q}\left[(K(p-q)+q|C|^p)e_{\overline{F}}(t,t_0)-K(p-q)\right]\right\}^{1/p} \quad \text{for any } K>0,\ t\in\mathbb{T}^{\kappa}, \tag{3.67}$$

where $h(t)$ is a nonnegative function, and $\overline{F}(t)$ is defined by (3.20).

In fact, the solution $u(t)$ of IVP (3.65) satisfies the following equivalent equation:

$$u^p(t)=C^p+\int_{t_0}^{t}H(\tau,u^q(\tau))\Delta\tau, \quad t\in\mathbb{T}^{\kappa}. \tag{3.68}$$

Noting the assumption (3.66), we easily obtain

$$|u(t)|^p\leq|C|^p+\int_{t_0}^{t}h(\tau)|u(\tau)|^q\Delta\tau, \quad t\in\mathbb{T}^{\kappa}. \tag{3.69}$$

Now a suitable application of Corollary 3.6 to (3.69) yields (3.67).

Acknowledgments

The authors thank the referees very much for their careful comments and valuable suggestions on this paper. This work is supported by the Natural Science Foundation of Shandong Province (Y2007A05), the National Natural Science Foundation of China

(60674026, 10671127), the Project of Science and Technology of the Education Department of Shandong Province (J06P51), and the Science Foundation of Binzhou University (2006Y01, BZXYLG200708).

References

[1] S. Hilger, "Analysis on measure chains—a unified approach to continuous and discrete calculus," *Results in Mathematics*, vol. 18, no. 1-2, pp. 18–56, 1990.

[2] R. Agarwal, M. Bohner, and A. Peterson, "Inequalities on time scales: a survey," *Mathematical Inequalities & Applications*, vol. 4, no. 4, pp. 535–557, 2001.

[3] E. Akin-Bohner, M. Bohner, and F. Akin, "Pachpatte inequalities on time scales," *Journal of Inequalities in Pure Applied Mathematics*, vol. 6, no. 1, article 6, pp. 23 pages, 2005.

[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[5] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[6] W. N. Li, "Some new dynamic inequalities on time scales," *Journal of Mathematical Analysis and Applications*, vol. 319, no. 2, pp. 802–814, 2006.

[7] D. B. Pachpatte, "Explicit estimates on integral inequalities with time scale," *Journal of Inequalities in Pure Applied Mathematics*, vol. 7, no. 4, article 143, pp. 8 pages, 2006.

[8] F.-H. Wong, C.-C. Yeh, and C.-H. Hong, "Gronwall inequalities on time scales," *Mathematical Inequalities & Applications*, vol. 9, no. 1, pp. 75–86, 2006.

[9] B. G. Pachpatte, "On some new inequalities related to a certain inequality arising in the theory of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 2, pp. 736–751, 2000.

Wei Nian Li: Department of Mathematics, Binzhou University, Shandong 256603, China
Email address: wnli@263.net

Weihong Sheng: Department of Mathematics, Binzhou University, Shandong 256603, China
Email address: wh-sheng@163.com