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Research Article

On the Strengthened Jordan's Inequality

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The main purpose of this paper is to present two methods of sharpening Jordan's inequality. The first method shows that one can obtain new strengthened Jordan's inequalities from old ones. The other method shows that one can sharpen Jordan's inequality by choosing proper functions in the monotone form of L'Hopital's rule. Finally, we improve a related inequality proposed early by Redheffer.

1. Introduction

The well-known Jordan's inequality states that $\sin x/x \geq 2/\pi$ $(x \in (0,\pi/2])$ holds with equality if and only if $x = \pi/2$ (see [1]). It plays an important role in many areas of pure and applied mathematics. This inequality was first extended to the following:

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2), \quad x \in \left(0, \frac{\pi}{2}\right], \tag{1.1}$$

and then, it was further extended to the following:

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2), \quad x \in \left(0, \frac{\pi}{2}\right], \tag{1.2}$$

which holds with equality if and only if $x = \pi/2$ (see [2–4]). Inequality (1.2) is slightly stronger than inequality (1.1) and is sharp in the sense that $1/\pi^3$ cannot be replaced by a larger constant. More recently, the monotone form of L'Hopital's rule (see [5, Lemma 5.1]) has been successfully used by Zhu [6, 7] and Wu and Debnath [8, 9] to sharpen

Jordan's inequality. For example, it has been shown in [6] that if $0 < x \le \pi/2$, then

$$\frac{2}{\pi}+\frac{1}{\pi^3}\left(\pi^2-4x^2\right) \le \frac{\sin x}{x} \le \frac{2}{\pi}+\frac{\pi-2}{\pi^3}\left(\pi^2-4x^2\right) \tag{1.3}$$

holds with equality if and only if $x = \pi/2$. Furthermore, the constants $1/\pi^3$ and $(\pi-2)/\pi^3$ in (1.3) are the best. Also, in the process of sharpening Jordan's inequality, one can use the same method as in [8] to introduce a parameter θ $(0 < \theta \le \pi)$ to replace the value $\pi/2$. In a recent paper [10], the first author established an identity which states that the function $\sin x/x$ is a power series of $(\pi^2 - 4x^2)$ with positive coefficients for all $x \neq 0$. This enables us to obtain a much better inequality than (1.3) if $0 < x \le \pi/2$.

Motivated by the previous research on Jordan's inequality, in this paper, we present two methods of sharpening Jordan's inequality. The first method shows that one can obtain new strengthened Jordan's inequalities from old ones. The other method shows that one can sharpen Jordan's inequality by choosing proper functions in the monotone form of L'Hopital's rule. Finally, we improve a related inequality proposed early by Redheffer.

2. New inequalities from old ones

The first method related to Jordan's inequality is implied in the following.

Theorem 2.1. *Let $g : [0,\pi/2] \to [0,1]$ be a continuous function. If*

$$\frac{\sin x}{x} \ge g(x), \quad x \in \left(0, \frac{\pi}{2}\right], \tag{2.1}$$

then

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + h(x) - h\left(\frac{\pi}{2}\right), \quad x \in \left(0, \frac{\pi}{2}\right], \tag{2.2}$$

holds with equality if and only if $x = \pi/2$, where

$$h(x) = -\int_0^x \left(\frac{1}{u^2}\int_0^u v^2 g(v)\,dv\right) du \quad \left(x \in \left[0, \frac{\pi}{2}\right]\right). \tag{2.3}$$

Proof. For the given function $g(x)$, we define the function $h(x)$ by (2.3) on $[0,\pi/2]$ with $h(0) = h'(0) = 0$, where the integrand function

$$\frac{1}{u^2}\int_0^u v^2 g(v)\,dv \quad \left(\text{with } \lim_{u\to 0^+}\frac{1}{u^2}\int_0^u v^2 g(v)\,dv = 0\right) \tag{2.4}$$

in (2.3) is bounded at zero. Then,

$$2h'(x) + xh''(x) = -xg(x) \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right). \tag{2.5}$$

Consider the following function:

$$f(x) = \frac{\sin x}{x} - h(x) \quad \left(x \in \left[0, \frac{\pi}{2}\right]\right) \text{ with } f(0) = 1. \tag{2.6}$$

Clearly,

$$f'(x) = \frac{1}{x^2}(x \cos x - \sin x - x^2 h'(x)) := \frac{1}{x^2}\phi(x) \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right), \tag{2.7}$$

where

$$\phi(x) = x \cos x - \sin x - x^2 h'(x). \tag{2.8}$$

Since

$$\phi'(x) = -x(\sin x + 2h'(x) + xh''(x)) = x(-\sin x + xg(x)), \tag{2.9}$$

it follows from (2.1) that $\phi'(x) \le 0$ on $(0,\pi/2)$. Hence, $\phi(x) \le \phi(0) = 0$ on $[0,\pi/2]$, and by (2.7), we obtain that $f(x)$ is a monotone decreasing function. Therefore,

$$\min_{x\in(0,\pi/2]} f(x) = f\left(\frac{\pi}{2}\right) = \frac{2}{\pi} - h\left(\frac{\pi}{2}\right). \tag{2.10}$$

The desired inequality (2.2) follows from (2.6) and (2.10). This completes the proof of Theorem 2.1. □

Theorem 2.1 shows how to get a new estimate (2.2) from inequality (2.1). There are many functions $g(x)$ satisfying (2.1) (i.e., there are many inequalities such as (2.1)). For each $g(x)$, we can obtain a new function $h(x)$ by (2.3), and thus, we get a strengthened Jordan's inequality (2.2). Note that from (2.3) and nonnegativity of $g(x)$, we see that

$$h'(x) = -\frac{1}{x^2}\int_0^x v^2 g(v)\,dv \le 0 \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right), \tag{2.11}$$

and hence, $h(x) - h(\pi/2) \ge 0$ on $[0,\pi/2]$. So inequality (2.2) improves the well-known Jordan's inequality. After obtaining inequality (2.2), one can get another new strengthened Jordan's inequality by applying Theorem 2.1 repeatedly.

From Theorem 2.1 and the above established inequalities (1.1) and (1.2), we have several strengthened Jordan's inequalities. For example, Jordan's inequality can be obtained from (2.1) by taking $g(x) = 0$ simply. Taking $g(x) = 2/\pi$ in (2.1), we see that inequality (2.2) is just (1.1). That is, we can get inequality (1.1) from inequality $\sin x/x \ge 2/\pi$ $(x \in (0,\pi/2])$. If we take

$$g(x) = \frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2) \tag{2.12}$$

in (2.1), we obtain, from Theorem 2.1, that inequality (1.1) yields that

$$\frac{\sin x}{x} \ge \frac{2}{\pi} + \frac{60+\pi^2}{720\pi}(\pi^2 - 4x^2) + \frac{1}{960\pi}(\pi^2 - 4x^2)^2, \quad x \in \left(0, \frac{\pi}{2}\right], \tag{2.13}$$

holds with equality if and only if $x = \pi/2$. However, if we take

$$g(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) \tag{2.14}$$

in (2.1), we obtain, from Theorem 2.1, that inequality (1.2) yields that

$$\frac{\sin x}{x} \geq \frac{2}{\pi} + \frac{1}{60\pi}(\pi^2 - 4x^2) + \frac{1}{80\pi^3}(\pi^2 - 4x^2)^2, \quad x \in \left(0, \frac{\pi}{2}\right], \tag{2.15}$$

holds with equality if and only if $x = \pi/2$. Inequality (2.13) is slightly stronger than inequality (2.15). In the case when $x \in (0, x_0]$, inequality (2.13) is slightly stronger than inequality (1.2), where

$$x_0 = \left(\frac{7\pi^4 + 240\pi^2 - 2880}{12\pi^2}\right)^{1/2} \approx 1.2. \tag{2.16}$$

Applying Theorem 2.1 repeatedly, one can see that $\sin x/x$ is not always less than a polynomial of $(\pi^2 - 4x^2)$ with positive coefficients for all $0 < x \leq \pi/2$.

3. Sharpening Jordan's inequality

The following monotone form of L'Hopital's rule (see [5, Lemma 5.1]) plays an important role in the process of sharpening Jordan's inequality as noticed in [6].

Lemma 3.1. *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{3.1}$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

By choosing proper functions in Lemma 3.1, we sharpen Jordan's inequality as follows. First, define the functions $f_1(x)$ and $f_3(x)$ by

$$f_1(x) = \frac{\sin x}{x}\ (f_1(0) = 1), \quad f_3(x) = \sin x - x\ \cos x \quad \left(x \in \left[0, \frac{\pi}{2}\right]\right). \tag{3.2}$$

Suppose that $f_2(x) \in \mathbf{C}^2[0, \pi/2]$ with

$$f_2'(x) \neq 0, \quad (x^2 f_2'(x))' \neq 0 \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right). \tag{3.3}$$

We define the function $f_4(x)$ by $f_4(x) = -x^2 f_2'(x)$ on $[0, \pi/2]$ with $f_4(0) = 0$. Then,

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin x - x\ \cos x}{-x^2 f_2'(x)} = \frac{f_3(x)}{f_4(x)} \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right), \tag{3.4}$$

$$\frac{f_3'(x)}{f_4'(x)} = \frac{x \sin x}{-(x^2 f_2'(x))'} := H(x) \quad \left(x \in \left(0, \frac{\pi}{2}\right)\right). \tag{3.5}$$

If, in addition, we can choose the function $f_2(x)$ such that

$$H(x) := \frac{x \sin x}{-\left(x^2 f_2'(x)\right)'} \quad \text{is decreasing (resp., increasing) on } \left(0, \frac{\pi}{2}\right), \tag{3.6}$$

then $f_3'(x)/f_4'(x)$ is decreasing (resp., increasing) on $(0, \pi/2)$, which shows that

$$\frac{f_3(x)}{f_4(x)} = \frac{f_3(x) - f_3(0)}{f_4(x) - f_4(0)} \tag{3.7}$$

is decreasing (resp., increasing) on $(0, \pi/2)$ by Lemma 3.1. So $f_1'(x)/f_2'(x)$ is decreasing (resp., increasing) on $(0, \pi/2)$ by (3.4). This shows that the function

$$\varphi(x) := \frac{f_1(x) - f_1(\pi/2)}{f_2(x) - f_2(\pi/2)} = \frac{\sin x/x - 2/\pi}{f_2(x) - f_2(\pi/2)} \tag{3.8}$$

is decreasing (resp., increasing) on $(0, \pi/2)$ by Lemma 3.1.

Therefore, in the case of decreasing, we have

$$\lim_{x \to \pi^-/2} \varphi(x) = \inf_{x \in (0, \pi/2]} \varphi(x) \le \varphi(x) \le \sup_{x \in (0, \pi/2]} \varphi(x) = \lim_{x \to 0^+} \varphi(x), \quad x \in \left(0, \frac{\pi}{2}\right]; \tag{3.9}$$

while in the case of increasing, we have

$$\lim_{x \to 0^+} \varphi(x) = \inf_{x \in (0, \pi/2]} \varphi(x) \le \varphi(x) \le \sup_{x \in (0, \pi/2]} \varphi(x) = \lim_{x \to \pi^-/2} \varphi(x), \quad x \in \left(0, \frac{\pi}{2}\right]. \tag{3.10}$$

Denote $M(f_2) := \lim_{x \to \pi^-/2} \varphi(x)$ and $m(f_2) := \lim_{x \to 0^+} \varphi(x)$. The condition $f_2'(x) \ne 0$ in (3.3) implies, by the Darboux property (intermediate value property) of the derivative, that either $f_2'(x) > 0$ or $f_2'(x) < 0$ on $(0, \pi/2)$ (otherwise, if there were values x_1, x_2 with $f_2'(x_1) > 0$, $f_2'(x_2) < 0$, then by the mentioned property, there was an x_0 between them with $f_2'(x_0) = 0$, which is a contradiction). If we further replace the condition $f_2'(x) \ne 0$ in (3.3) by $f_2'(x) > 0$ or $f_2'(x) < 0$ on $(0, \pi/2)$ in order to get $f_2(x) - f_2(\pi/2) < 0$ or $f_2(x) - f_2(\pi/2) > 0$ on $(0, \pi/2)$, respectively, we obtain from (3.8) and (3.9) that

$$\frac{2}{\pi} + m(f_2)\left(f_2(x) - f_2\left(\frac{\pi}{2}\right)\right) \le \frac{\sin x}{x} \le \frac{2}{\pi} + M(f_2)\left(f_2(x) - f_2\left(\frac{\pi}{2}\right)\right), \quad x \in \left(0, \frac{\pi}{2}\right], \tag{3.11}$$

or

$$\frac{2}{\pi} + M(f_2)\left(f_2(x) - f_2\left(\frac{\pi}{2}\right)\right) \le \frac{\sin x}{x} \le \frac{2}{\pi} + m(f_2)\left(f_2(x) - f_2\left(\frac{\pi}{2}\right)\right), \quad x \in \left(0, \frac{\pi}{2}\right], \tag{3.12}$$

holds, respectively. The similar inequality can be obtained from (3.8) and (3.10). Note that in each case $m(f_2)(f_2(x) - f_2(\pi/2)) \ge 0$ and $M(f_2)(f_2(x) - f_2(\pi/2)) \ge 0$ on $(0, \pi/2]$; also $m(f_2)$ and $M(f_2)$ are the best constants in inequality (3.11) or (3.12).

Finally, the main point of the above method concentrates upon the choice of function $f_2(x) \in \mathbf{C}^2[0, \pi/2]$ satisfying (3.3) and (3.6) with $f_2'(x) \ne 0$ in (3.3) replaced by $f_2'(x) > 0$

or $f_2'(x) < 0$ on $(0, \pi/2)$. One can check that there are many such functions; for example, $f_2(x) = x^n$ $(n \in \mathbb{N})$ and $f_2(x) = e^{-x}$ satisfy the above requirements. Hence, the corresponding inequality (3.11) or (3.12) holds. The following theorem is one of such results.

THEOREM 3.2. *If $0 < x \le \pi/2$, then*

$$\frac{2}{\pi} + \frac{\pi-2}{\pi^2}(\pi - 2x) \le \frac{\sin x}{x} \le \frac{2}{\pi} + \frac{2}{\pi^2}(\pi - 2x), \tag{3.13}$$

$$\frac{2}{\pi} + \frac{2}{n\pi^{n+1}}\left(\pi^n - (2x)^n\right) \le \frac{\sin x}{x} \le \frac{2}{\pi} + \frac{\pi-2}{\pi^{n+1}}\left(\pi^n - (2x)^n\right) \quad (n \ge 2) \tag{3.14}$$

hold with equality if and only if $x = \pi/2$. Furthermore, the constants $(\pi-2)/\pi^2$ and $2/\pi^2$ in (3.13), as well as the constants $2/(n\pi^{n+1})$ and $(\pi-2)/\pi^{n+1}$ in (3.14), are the best.

Note that in the case when $n = 2$, inequality (3.14) reduces to (1.3). By applying Theorem 2.1, one can also strengthen Jordan's inequality from Theorem 3.2 and compare the obtained inequalities.

4. A related inequality

Redheffer et al. [11] proposed the following inequality:

$$\frac{\sin \pi x}{\pi x} \ge \frac{1-x^2}{1+x^2} \tag{4.1}$$

for real $x \in \mathbb{R}$ (one can consider only $x > 0$); see [1]. Williams [12] gave a proof of (4.1). In the case when $x \ge 1$, Williams generalizes the result (4.1) in [12] by proving the following inequality:

$$\frac{\sin \pi x}{\pi x} \ge \frac{1-x^2}{1+x^2} + \frac{(1-x)^2}{x(1+x^2)} \quad (x \ge 1). \tag{4.2}$$

In this section, we extend inequality (4.1) in the case when $0 < x < 1$. In fact, we provide two identities related to inequality (4.1). The first identity comes from the evaluation of an Erdős-Turán-type series established by the first author [13], which states that

$$\frac{\pi x}{\sin \pi x} = 1 + \sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^{n+1} x^2}{n^2 + nx} = 1 + 2x^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2 - x^2} \tag{4.3}$$

for $0 < x < 1$. The second identity comes from harmonic analysis, which states that

$$\sum_{n \in \mathbb{Z}} \left| \frac{\sin(\pi(n+x))}{\pi(n+x)} \right|^2 = 1 \tag{4.4}$$

for any $x \in \mathbb{R}$ (see [14, page 10] and [15, page 212]). From (4.4), we have

$$\frac{\pi x}{\sin \pi x} = \left(1 + 2x^2 \sum_{n=1}^{\infty} \frac{x^2 + n^2}{(x^2 - n^2)^2}\right)^{1/2} \tag{4.5}$$

for $0 < x < 1$.

It follows from the alternating series (4.3) that

$$\frac{\pi x}{\sin \pi x} \leq 1 + 2x^2 \left\{ \frac{1}{1-x^2} - \frac{1}{4-x^2} + \frac{1}{9-x^2} \right\} \quad (0 < x < 1) \tag{4.6}$$

which yields

$$\frac{\sin \pi x}{\pi x} \geq \frac{(1-x^2)(4-x^2)(9-x^2)}{x^6 - 2x^4 + 13x^2 + 36} \quad (0 < x < 1). \tag{4.7}$$

Inequality (4.7) is much better than (4.1) in the case when $0 < x < 1$. Also, one can add more positive terms to the right-hand side of inequality (4.7) to get higher accuracy.

It follows from (4.5) that

$$\frac{\pi x}{\sin \pi x} \geq \left(1 + 2x^2 \frac{1+x^2}{(1-x^2)^2} \right)^{1/2} \quad (0 < x < 1) \tag{4.8}$$

which yields

$$\frac{\sin \pi x}{\pi x} \leq \frac{1-x^2}{\sqrt{1+3x^4}} \quad (0 < x < 1). \tag{4.9}$$

Therefore, we have the following inequality.

Theorem 4.1. *If* $0 < x < 1$, *then*

$$\frac{(1-x^2)(4-x^2)(9-x^2)}{x^6 - 2x^4 + 13x^2 + 36} \leq \frac{\sin \pi x}{\pi x} \leq \frac{1-x^2}{\sqrt{1+3x^4}}. \tag{4.10}$$

Also, one can add more positive terms to the left-hand side of inequality (4.10) and add more negative terms to the right-hand side of inequality (4.10) to get higher accuracy.

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