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Research Article

Integral Means Inequalities for Fractional Derivatives of a Unified Subclass of Prestarlike Functions with Negative Coefficients

H. Özlem Güney and Shigeyoshi Owa

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Integral means inequalities are obtained for the fractional derivatives of order $p+\lambda(0 \le p \le n,\ 0 \le \lambda < 1)$ of functions belonging to a unified subclass of prestarlike functions. Relevant connections with various known integral means inequalities are also pointed out.

1. Introduction

Let $\mathcal{S}$ denote the class of (*normalized*) functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* and *univalent* in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathcal{T}$ denote the subclass of $\mathcal{S}$ consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0). \tag{1.2}$$

The Hadamard product (or convolution) of two functions f given by (1.1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.3}$$

is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.4}$$

We denote the subclass $\mathcal{R}(\alpha,\beta)$ of $\mathcal{S}$ consisting of α-*prestarlike functions* of order β by

$$\mathcal{R}(\alpha,\beta) = \{f \in \mathcal{S} : (f*s_\alpha)(z) \in \mathcal{S}^*(\beta),\ 0 \le \alpha < 1,\ 0 \le \beta < 1\}, \tag{1.5}$$

where $\mathcal{S}^*(\beta)$ denotes the class of starlike functions of order $\beta(0 \le \beta < 1)$ and s_α is the well-known extremal function for $\mathcal{S}^*(\alpha)$ given by

$$s_\alpha(z) = z(1-z)^{-2(1-\alpha)} \tag{1.6}$$

(cf. [1, 2]). Letting

$$c_n(\alpha) = \frac{\prod_{k=2}^{n}(k-2\alpha)}{(n-1)!} \quad (n = 2,3,\dots), \tag{1.7}$$

s_α can be written in the form

$$s_\alpha(z) = z + \sum_{n=2}^{\infty} c_n(\alpha) z^n. \tag{1.8}$$

The class $\mathcal{R}(\alpha,\beta)$ was investigated by Sheil-Small et al. [3]. We also denote the subclass $\mathcal{C}(\alpha,\beta)$ of $\mathcal{S}$, which was investigated by Owa and Uralegaddi [4], by

$$\mathcal{C}(\alpha,\beta) = \{f \in \mathcal{S} : zf'(z) \in \mathcal{R}(\alpha,\beta)\}. \tag{1.9}$$

In particular, the subclasses

$$\mathcal{R}[\alpha,\beta] = \mathcal{R}(\alpha,\beta) \cap \mathcal{T}, \qquad \mathcal{C}[\alpha,\beta] = \mathcal{C}(\alpha,\beta) \cap \mathcal{T} \tag{1.10}$$

were considered earlier by Srivastava and Aouf [5]. Let us define the unified class $\mathcal{P}(\alpha, \beta,\sigma)$ of the classes $\mathcal{R}[\alpha,\beta]$ and $\mathcal{C}[\alpha,\beta]$ by

$$\mathcal{P}(\alpha,\beta,\sigma) = (1-\sigma)\mathcal{R}[\alpha,\beta] + \sigma\mathcal{C}[\alpha,\beta] \quad (0 \le \sigma \le 1), \tag{1.11}$$

so that

$$\mathcal{P}(\alpha,\beta,0) = \mathcal{R}[\alpha,\beta], \qquad \mathcal{P}(\alpha,\beta,1) = \mathcal{C}[\alpha,\beta]. \tag{1.12}$$

The unified class $\mathcal{P}(\alpha,\beta,\sigma)$ was studied by Raina and Srivastava [6].

We begin by recalling the following useful characterizations of the function class $\mathcal{P}(\alpha,\beta,\sigma)$ due to Raina and Srivastava [6].

Lemma 1.1. *A function f defined by (1.2) belongs to the class $\mathcal{P}(\alpha,\beta,\sigma)$ if and only if*

$$\sum_{n=2}^{\infty}\left\{\frac{(n-\beta)(1-\sigma+\sigma n)}{1-\beta}\right\}c_n(\alpha)a_n \le 1, \tag{1.13}$$

for some $\alpha(0\le\alpha<1)$, $\beta(0\le\beta<1)$, $\sigma(0\le\sigma\le 1)$.

We continue by proving the following lemma.

Lemma 1.2. *Let*

$$f_1(z)=z,\qquad f_k(z)=z-\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}z^k\quad (k=2,3,\dots). \tag{1.14}$$

Then $f\in\mathcal{P}(\alpha,\beta,\sigma)$ if and only if it can be expressed in the form

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z), \tag{1.15}$$

where $\lambda_k\ge 0$ and $\sum_{k=1}^{\infty}\lambda_k=1$.

Proof. Assume that

$$f(z)=\sum_{k=1}^{\infty}\lambda_k f_k(z). \tag{1.16}$$

Then

$$\begin{aligned} f(z)&=\lambda_1 f_1(z)+\sum_{k=2}^{\infty}\lambda_k f_k(z)\\ &=\lambda_1 z+\sum_{k=2}^{\infty}\lambda_k\left(z-\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}z^k\right)\\ &=\left(\sum_{k=1}^{\infty}\lambda_k\right)z-\sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}z^k\\ &=z-\sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}z^k. \end{aligned} \tag{1.17}$$

Thus

$$\begin{aligned} &\sum_{k=2}^{\infty}\lambda_k\left(\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}\right)\left(\frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta}\right)\\ &\quad=\sum_{k=2}^{\infty}\lambda_k=\sum_{k=1}^{\infty}\lambda_k-\lambda_1=1-\lambda_1\le 1. \end{aligned} \tag{1.18}$$

Therefore, we have $f\in\mathcal{P}(\alpha,\beta,\sigma)$. $\square$

Conversely, suppose that $f \in \mathcal{P}(\alpha,\beta,\sigma)$. Since

$$|a_k| \leq \frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)} \quad (k=2,3,\ldots), \tag{1.19}$$

we can set

$$\lambda_k = \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \quad (k=2,3,\ldots), \qquad \lambda_1 = 1-\sum_{k=1}^{\infty}\lambda_k. \tag{1.20}$$

Then

$$\begin{aligned} f(z) &= z-\sum_{k=2}^{\infty}a_k z^k \\ &= z-\sum_{k=2}^{\infty}\lambda_k\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}z^k \\ &= \left(1-\sum_{k=2}^{\infty}\lambda_k\right)z+\sum_{k=2}^{\infty}\lambda_k f_k(z) \\ &= \lambda_1 f_1(z)+\sum_{k=2}^{\infty}\lambda_k f_k(z) \\ &= \sum_{k=1}^{\infty}\lambda_k f_k(z). \end{aligned} \tag{1.21}$$

This completes the assertion of Lemma 1.2.

Lemma 1.2 gives us the following.

Corollary 1.3. *The extreme points of $\mathcal{P}(\alpha,\beta,\sigma)$ are given by*

$$f_1(z) = z, \qquad f_k(z) = z-\frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}z^k. \tag{1.22}$$

We will make use of the following definitions of fractional derivatives by Owa [7] (also by Srivastava and Owa [8]).

Definition 1.4. The fractional derivative of order λ is defined, for a function f, by

$$D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\int_0^z\frac{f(\xi)}{(z-\xi)^{\lambda}}d\xi \quad (0\leq\lambda<1), \tag{1.23}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)>0$.

Definition 1.5. Under the hypothesis of Definition 1.4, the fractional derivative of order $(n+\lambda)$ is defined, for a function f, by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{1.24}$$

where $0 \leq \lambda < 1$ and $n = 0, 1, 2, \ldots$.

It readily follows from (1.23) in Definition 1.4 that

$$D_z^{\lambda} z^k = \frac{\Gamma(k+1)}{\Gamma(k-\lambda+1)} z^{k-\lambda} \quad (0 \leq \lambda < 1). \tag{1.25}$$

We will also need the concept of subordination between analytic functions and a subordination theorem of Littlewood [9] in our investigation.

Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be *subordinate* to g in $\mathbb{U}$ if there exists a function w analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \quad (z \in \mathbb{U}), \tag{1.26}$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \tag{1.27}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1.28}$$

Lemma 1.6. *If the functions f and g are analytic in $\mathbb{U}$ with*

$$g(z) \prec f(z), \tag{1.29}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |g(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^{\mu} d\theta. \tag{1.30}$$

2. The main integral means inequalities

We discuss the integral means inequalities for functions f in $\mathcal{P}(\alpha,\beta,\sigma)$. Our main theorem is contained in the following.

Theorem 2.1. *Let $f \in \mathcal{P}(\alpha,\beta,\sigma)$ and suppose that*

$$\sum_{n=2}^{\infty} (n-p)_{p+1} a_n \leq \frac{(1-\beta)\Gamma(k+1)\Gamma(3-\lambda-p)}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)\Gamma(k+1-\lambda-p)\Gamma(2-p)} \quad (k \geq 2) \tag{2.1}$$

for $0 \leq \lambda < 1$, where $(n-p)_{p+1}$ denotes the Pochhammer symbol defined by

$$(n-p)_{p+1} = (n-p)(n-p+1)\cdots n. \tag{2.2}$$

Also let the function f_k be defined by

$$f_k(z) = z - \frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)} z^k. \tag{2.3}$$

If there exists an analytic function w defined by

$$\{w(z)\}^{k-1} = \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k+1-\lambda-p)}{\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1} \Phi(n) a_n z^{n-1} \tag{2.4}$$

with

$$\Phi(n) = \frac{\Gamma(n-p)}{\Gamma(n+1-\lambda-p)} \quad (0 \le \lambda < 1,\ n = 2,3,\dots), \tag{2.5}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left| D_z^{p+\lambda} f(z) \right|^{\mu} d\theta \le \int_0^{2\pi} \left| D_z^{p+\lambda} f_k(z) \right|^{\mu} d\theta \quad (0 \le \lambda < 1,\ \mu > 0). \tag{2.6}$$

Proof. By virtue of the fractional derivative formula (1.25) and Definition 1.5, we find from (1.1) that

$$\begin{aligned} D_z^{p+\lambda} f(z) &= \frac{z^{1-p-\lambda}}{\Gamma(2-\lambda-p)} \left(1 - \sum_{n=2}^{\infty} \frac{\Gamma(2-\lambda-p)\Gamma(n+1)}{\Gamma(n+1-\lambda-p)} a_n z^{n-1} \right) \\ &= \frac{z^{1-p-\lambda}}{\Gamma(2-\lambda-p)} \left(1 - \sum_{n=2}^{\infty} \Gamma(2-\lambda-p)(n-p)_{p+1} \Phi(n) a_n z^{n-1} \right), \end{aligned} \tag{2.7}$$

where

$$\Phi(n) = \frac{\Gamma(n-p)}{\Gamma(n+1-\lambda-p)} \quad (0 \le \lambda < 1,\ n = 2,3,\dots). \tag{2.8}$$

Since Φ is a decreasing function of n, we have

$$0 < \Phi(n) \le \Phi(2) = \frac{\Gamma(2-p)}{\Gamma(3-\lambda-p)} \quad (0 \le \lambda < 1,\ n = 2,3,\dots). \tag{2.9}$$

Similarly, from (2.3), (1.25), and Definition 1.5, we obtain

$$D_z^{p+\lambda} f_k(z) = \frac{z^{1-p-\lambda}}{\Gamma(2-\lambda-p)} \left(1 - \frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} z^{k-1} \right). \tag{2.10}$$

For $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$, we must show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} \Gamma(2-\lambda-p)(n-p)_{p+1}\Phi(n)a_n z^{n-1} \right|^{\mu} d\theta \\ \leq \int_0^{2\pi} \left| 1 - \frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} z^{k-1} \right|^{\mu} d\theta. \tag{2.11}$$

Thus, by applying Lemma 1.6, it would suffice to show that

$$1 - \sum_{n=2}^{\infty} \Gamma(2-\lambda-p)(n-p)_{p+1}\Phi(n)a_n z^{n-1} \\ \prec 1 - \frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} z^{k-1}. \tag{2.12}$$

If the subordination (2.12) holds true, then we have an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 - \sum_{n=2}^{\infty} \Gamma(2-\lambda-p)(n-p)_{p+1}\Phi(n)a_n z^{n-1} \\ = 1 - \frac{1-\beta}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)} \frac{\Gamma(2-\lambda-p)\Gamma(k+1)}{\Gamma(k+1-\lambda-p)} \{w(z)\}^{k-1}. \tag{2.13}$$

By the condition of the theorem, we define the function w by

$$\{w(z)\}^{k-1} = \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k+1-\lambda-p)}{\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1}\Phi(n)a_n z^{n-1} \tag{2.14}$$

which readily yields $w(0) = 0$. For such a function w, we have

$$|w(z)|^{k-1} \leq \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k+1-\lambda-p)}{\Gamma(k+1)} \sum_{n=2}^{\infty} (n-p)_{p+1}\Phi(n)a_n |z|^{n-1} \\ \leq |z| \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k+1-\lambda-p)}{\Gamma(k+1)} \Phi(2) \sum_{n=2}^{\infty} (n-p)_{p+1}a_n \\ = |z| \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k+1-\lambda-p)}{\Gamma(k+1)} \frac{\Gamma(2-p)}{\Gamma(3-\lambda-p)} \sum_{n=2}^{\infty} (n-p)_{p+1}a_n \\ = |z| < 1, \tag{2.15}$$

by means of the hypothesis of the theorem. □

This means that the subordination (2.12) holds true; therefore the theorem is proved. As special case $p = 0$, Theorem 2.1 readily yields.

Corollary 2.2. *Let $f \in \mathcal{P}(\alpha,\beta,\sigma)$ and suppose that*

$$\sum_{n=2}^{\infty} n\,|a_n| \leq \frac{(1-\beta)\Gamma(k+1)\Gamma(3-\lambda)}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)\Gamma(k+1-\lambda)} \quad (k \geq 2). \tag{2.16}$$

If there exists an analytic function w given by

$$\{w(z)\}^{k-1} = \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k+1-\lambda)}{\Gamma(k+1)} \sum_{n=2}^{\infty} n\Phi(n)a_n z^{n-1} \tag{2.17}$$

with

$$\Phi(n) = \frac{\Gamma(n)}{\Gamma(n+1-\lambda)} \quad (0 \leq \lambda < 1,\ n = 2,3,\dots), \tag{2.18}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |D_z^{\lambda} f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |D_z^{\lambda} f_k(z)|^{\mu} d\theta \quad (0 \leq \lambda < 1,\ \mu > 0). \tag{2.19}$$

Letting $p = 1$ in Theorem 2.1, we have the following.

Corollary 2.3. *Let $f \in \mathcal{P}(\alpha,\beta,\sigma)$ and suppose that*

$$\sum_{n=2}^{\infty} n(n-1)\,|a_n| \leq \frac{(1-\beta)\Gamma(k+1)\Gamma(2-\lambda)}{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)\Gamma(k-\lambda)} \quad (k \geq 2). \tag{2.20}$$

If there exists an analytic function w given by

$$\{w(z)\}^{k-1} = \frac{(k-\beta)(1-\sigma+\sigma k)c_k(\alpha)}{1-\beta} \frac{\Gamma(k-\lambda)}{\Gamma(k+1)} \sum_{n=2}^{\infty} (n-1)_2 \Phi(n)a_n z^{n-1} \tag{2.21}$$

with

$$\Phi(n) = \frac{\Gamma(n-1)}{\Gamma(n-\lambda)} \quad (0 \leq \lambda < 1,\ n = 2,3,\dots), \tag{2.22}$$

then, for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} |D_z^{1+\lambda} f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |D_z^{1+\lambda} f_k(z)|^{\mu} d\theta \quad (0 \leq \lambda < 1,\ \mu > 0). \tag{2.23}$$

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H. Özlem Güney: Department of Mathematics, Faculty of Science and Letters, University of Dicle, 21280 Diyarbakır, Turkey
Email address: ozlemg@dicle.edu.tr

Shigeyoshi Owa: Department of Mathematics, Kinki University, Osaka 577-8502, Higashi-Osaka, Japan
Email address: owa@math.kindai.ac.jp