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Research Article

Fixed Points in Functional Inequalities

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Using fixed point methods, we prove the generalized Hyers-Ulam stability of the following functional inequalities $\|f(x)+f(y)+f(z)\| \leq \|f(x+y+z)\|$ and $\|f(x)+f(y)+2f(z)\| \leq \|2f((x+y)/2+z)\|$ in the spirit of Th. M. Rassias stability approach.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach. In 1982, J. M. Rassias [6] followed the innovative approach of the Th. M. Rassias' theorem [4] in which he replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p+q \neq 1$.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [7] for mappings $f: X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [8] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [9] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10–20]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 (see [21, 22]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.2}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1 - L)) d(y, Jy)$ for all $y \in Y$.*

In 1996, Isac and Th. M. Rassias [23] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [24–28]).

In [29], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|, \tag{1.3}$$

then, f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y). \tag{1.4}$$

See also [30]. Gilányi [31] and Fechner [32] proved the generalized Hyers-Ulam stability of the functional inequality (1.3).

In this paper, using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional inequalities

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|, \tag{1.5}$$

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x + y}{2} + z\right)\right\|, \tag{1.6}$$

in Banach spaces.

2. Fixed points and generalized Hyers-Ulam stability of the functional inequalities (1.5) and (1.6)

Throughout this paper, assume that X is a normed vector space with norm $\|\cdot\|$ and that Y is a Banach space with norm $\|\cdot\|$.

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional inequality (1.5) in Banach spaces.

Theorem 2.1. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^3 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,y,z) \leq (1/2)L\varphi(2x,2y,2z)$ for all $x,y,z \in X$, and*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| + \varphi(x,y,z) \tag{2.1}$$

for all $x,y,z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{L}{2-2L}\varphi(x,x,-2x) \tag{2.2}$$

for all $x \in X$.

Proof. Consider the set

$$S := \{g : X \longrightarrow Y\}, \tag{2.3}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq K\varphi(x,x,-2x),\ \forall x \in X\}. \tag{2.4}$$

It is easy to show that (S,d) is complete. (See [33, proof of Theorem 2.5].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right) \tag{2.5}$$

for all $x \in X$.

It follows from [21, proof of Theorem 3.1] that

$$d(Jg,Jh) \leq Ld(g,h) \tag{2.6}$$

for all $g,h \in S$.

Since $f : X \to Y$ is odd, $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in X$. Letting $y = x$ and $z = -2x$ in (2.1), we get

$$\|2f(x) - f(2x)\| = \|2f(x) + f(-2x)\| \leq \varphi(x,x,-2x) \tag{2.7}$$

for all $x \in X$.

It follows from (2.7) that

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}, -x\right) \le \frac{L}{2}\varphi(x, x, -2x) \tag{2.8}$$

for all $x \in X$. Hence $d(f, Jf) \le L/2$.

By Theorem 1.1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{2.9}$$

for all $x \in X$. Then $A : X \to Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}. \tag{2.10}$$

This implies that A is a unique mapping satisfying (2.9) such that there exists a $K \in (0, \infty)$ satisfying

$$\|f(x) - A(x)\| \le K\varphi(x, x, -2x) \tag{2.11}$$

for all $x \in X$;

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = A(x) \tag{2.12}$$

for all $x \in X$;

(3) $d(f, A) \le (1/(1-L))d(f, Jf)$, which implies the inequality

$$d(f, A) \le \frac{L}{2-2L}. \tag{2.13}$$

This implies that the inequality (2.2) holds.

It follows from (2.1) and (2.12) that

$$\|A(x) + A(y) + A(z)\| \le \|A(x + y + z)\| \tag{2.14}$$

for all $x, y, z \in X$. By [34, Proposition 2.2], the mapping $A : X \to Y$ is a Cauchy additive mapping.

Therefore, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying (2.2), as desired. □

Corollary 2.2. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{2.15}$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r + 2}{2^r - 2}\theta\|x\|^r \tag{2.16}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r) \tag{2.17}$$

for all $x, y, z \in X$. Then, we can choose $L= 2^{1-r}$ and we get the desired result. □

Corollary 2.3. *Let $r > 1/3$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\| + \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \tag{2.18}$$

for all $x, y, z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r}{2^{3r} - 2}\theta\|x\|^{3r} \tag{2.19}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y, z) := \theta \cdot \|x\|^r \cdot \|y\|^r \cdot \|z\|^r \tag{2.20}$$

for all $x, y, z \in X$. Then we can choose $L= 2^{1-3r}$ and we get the desired result. □

Remark 2.4. Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^3 \to [0, \infty)$ satisfying (2.1). By a similar method to the proof of Theorem 2.1, one can show that if there exists an $L < 1$ such that $\varphi(x, y, z) \leq 2L\varphi(x/2, y/2, z/2)$ for all $x, y, z \in X$, then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{1}{2 - 2L}\varphi(x, x, -2x) \tag{2.21}$$

for all $x \in X$.

For the case $0 < r < 1$, one can obtain a similar result to Corollary 2.2. Let $0 < r < 1$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.15). Then, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{2+2^r}{2-2^r}\theta\|x\|^r \tag{2.22}$$

for all $x \in X$.

For the case $0 < r < 1/3$, one can obtain a similar result to Corollary 2.3. Let $0 < r < 1/3$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.18). Then, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{2^r}{2-2^{3r}}\theta\|x\|^{3r} \tag{2.23}$$

for all $x \in X$.

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional inequality (1.6) in Banach spaces.

Theorem 2.5. *Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^3 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,y,z) \leq (1/2)L\varphi(2x,2y,2z)$ for all $x,y,z \in X$, and*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + \varphi(x,y,z) \tag{2.24}$$

for all $x,y,z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \frac{L}{2-2L}\varphi(0,2x,-x) \tag{2.25}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 2.6. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : X \to Y$ be an odd mapping such that*

$$\|f(x) + f(y) + 2f(z)\| \leq \left\|2f\left(\frac{x+y}{2} + z\right)\right\| + \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right) \tag{2.26}$$

for all $x,y,z \in X$. Then, there exists a unique Cauchy additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^r+1}{2^r-2}\theta\|x\|^r \tag{2.27}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(x,y,z) := \theta\left(\|x\|^r + \|y\|^r + \|z\|^r\right) \tag{2.28}$$

for all $x, y, z \in X$. Then, we can choose $L = 2^{1-r}$ and we get the desired result. □

Remark 2.7. Let $f : X \to Y$ be an odd mapping for which there exists a function $\varphi : X^3 \to [0,\infty)$ satisfying (2.24). By a similar method to the proof of Theorem 2.5, one can show that if there exists an $L < 1$ such that $\varphi(x,y,z) \le 2L\varphi(x/2, y/2, z/2)$ for all $x, y, z \in X$, then there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\left\|f(x) - A(x)\right\| \le \frac{1}{2-2L}\varphi(0, 2x, -x) \tag{2.29}$$

for all $x \in X$.

For the case $0 < r < 1$, one can obtain a similar result to Corollary 2.6. Let $0 < r < 1$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be an odd mapping satisfying (2.26). Then, there exists a unique Cauchy additive mapping $A : X \to Y$ satisfying

$$\left\|f(x) - A(x)\right\| \le \frac{1+2^r}{2-2^r}\theta\|x\|^r \tag{2.30}$$

for all $x \in X$.

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References

[1] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1960.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, no. 4, pp. 222–224, 1941.
[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[5] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[6] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.
[7] F. Skof, "Proprieta' locali e approssimazione di operatori," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, no. 1, pp. 113–129, 1983.
[8] P. W. Cholewa, "Remarks on the stability of functional equations," *Aequationes Mathematicae*, vol. 27, no. 1-2, pp. 76–86, 1984.
[9] St. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, no. 1, pp. 59–64, 1992.
[10] St. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.
[11] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[12] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and Their Applications, 34, Birkhäuser, Basel, Switzerland, 1998.

[13] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[14] C. Park and J. Cui, "Generalized stability of C^*-ternary quadratic mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 23282, 6 pages, 2007.

[15] C. Park and A. Najati, "Homomorphisms and derivations in C^*-algebras," *Abstract and Applied Analysis*, vol. 2007, Article ID 80630, 12 pages, 2007.

[16] Th. M. Rassias, "The problem of S. M. Ulam for approximately multiplicative mappings," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 352–378, 2000.

[17] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.

[18] Th. M. Rassias, "On the stability of functional equations and a problem of Ulam," *Acta Applicandae Mathematicae*, vol. 62, no. 1, pp. 23–130, 2000.

[19] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[20] Th. M. Rassias and P. Šemrl, "On the Hyers-Ulam stability of linear mappings," *Journal of Mathematical Analysis and Applications*, vol. 173, no. 2, pp. 325–338, 1993.

[21] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, 7 pages, 2003.

[22] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.

[23] G. Isac and Th. M. Rassias, "Stability of ϕ-additive mappings: applications to nonlinear analysis," *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.

[24] L. Cădariu and V. Radu, "Fixed point methods for the generalized stability of functional equations in a single variable," *Fixed Point Theory and Applications*, vol. 2008, Article ID 749392, 15 pages, 2008.

[25] M. Mirzavaziri and M. S. Moslehian, "A fixed point approach to stability of a quadratic equation," *Bulletin of the Brazilian Mathematical Society*, vol. 37, no. 3, pp. 361–376, 2006.

[26] C. Park, "Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras," *Fixed Point Theory and Applications*, vol. 2007, Article ID 50175, 15 pages, 2007.

[27] C. Park, "Generalized Hyers-Ulam stability of quadratic functional equations: a fixed point approach," *Fixed Point Theory and Applications*, vol. 2008, Article ID 493751, 9 pages, 2008.

[28] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.

[29] A. Gilányi, "Eine zur Parallelogrammgleichung äquivalente Ungleichung," *Aequationes Mathematicae*, vol. 62, no. 3, pp. 303–309, 2001.

[30] J. Rätz, "On inequalities associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 66, no. 1-2, pp. 191–200, 2003.

[31] A. Gilányi, "On a problem by K. Nikodem," *Mathematical Inequalities & Applications*, vol. 5, no. 4, pp. 707–710, 2002.

[32] W. Fechner, "Stability of a functional inequality associated with the Jordan-von Neumann functional equation," *Aequationes Mathematicae*, vol. 71, no. 1-2, pp. 149–161, 2006.

[33] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," *Grazer Mathematische Berichte*, vol. 346, pp. 43–52, 2004.

[34] C. Park, Y. S. Cho, and M.-H. Han, "Functional inequalities associated with Jordan-von Neumann-type additive functional equations," *Journal of Inequalities and Applications*, vol. 2007, Article ID 41820, 13 pages, 2007.