Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 374756, 8 pages
doi:10.1155/2008/374756

Research Article

Sufficient Conditions for Subordination of Multivalent Functions

Öznur Özkan Kılıç

Department of Statistics and Computer Sciences, Başkent University, Bağlıca, Ankara 06530, Turkey

Correspondence should be addressed to Öznur Özkan Kılıç, oznur@baskent.edu.tr

Received 13 January 2008; Accepted 28 March 2008

Recommended by Ram Mohapatra

The authors investigate various subordination results for some subclasses of analytic functions in the unit disc. We obtain some sufficient conditions for multivalent close-to-starlikeness.

1. Introduction and definitions

Let $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$, and let $\mathscr{H}(\mathbb{U})$ be the set of all functions *analytic in* $\mathbb{U}$, and let

$$\mathscr{A}_p = \{f \in \mathscr{H}(\mathbb{U}) : f(z) = z^p + a_{p+1}z^{p+1} + \cdots\} \tag{1.1}$$

for all $z \in \mathbb{U}$ and $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ with $\mathscr{A}_1 = \mathscr{A}$.

For $p \in \mathbb{N}$, let

$$\mathscr{H}_p = \{f \in \mathscr{H}(\mathbb{U}) : f(z) = p + b_p z^p + \cdots\} \tag{1.2}$$

with $\mathscr{H}_1 = \mathscr{H}$.

A function $f(z)$ in $\mathscr{A}_p$ is said to be *p-valently starlike of order* α $(0 \le \alpha < p)$ in $\mathbb{U}$, that is, $f \in \mathcal{S}^*(\alpha)$, if and only if

$$\frac{f(z)}{z} \neq 0, \qquad \Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{1.3}$$

for $z \in \mathbb{U},\ 0 \le \alpha < p,\ p \in \mathbb{N}$.

Similarly, a function $f(z)$ in $\mathcal{A}_p$ is said to be *p-valently convex of order* α $(0 \leq \alpha < p)$ in $\mathbb{U}$, that is, $f \in \mathcal{K}(\alpha)$, if and only if

$$f'(z) \neq 0, \qquad \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{1.4}$$

for $z \in \mathbb{U}$, $0 \leq \alpha < p$, $p \in \mathbb{N}$.

We denote by $\mathcal{C}(\alpha)$ to be the family of functions $f(z)$ in $\mathcal{A}_p$ such that

$$\Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha \tag{1.5}$$

for $z \in \mathbb{U} \setminus \{0\}$, $0 \leq \alpha < p$, $p \in \mathbb{N}$.

Similarly, we denote by $\mathcal{CS}^*(\alpha)$ to be the family of functions $f(z)$ in $\mathcal{A}_p$ such that

$$\Re\left\{\frac{f(z)}{z^p}\right\} > \alpha \tag{1.6}$$

for $z \in \mathbb{U} \setminus \{0\}$, $0 \leq \alpha < p$, $p \in \mathbb{N}$.

We note that the classes $\mathcal{C}(\alpha)$ and $\mathcal{CS}^*(\alpha)$ are special classes of the class of *p-valently close-to-convex of order* α $(0 \leq \alpha < p)$, the class of *p-valently close-to-starlike of order* α $(0 \leq \alpha < p)$ in $\mathbb{U}$, respectively.

In particular, the classes $\mathcal{S}$, $\mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{K}(0) = \mathcal{K}$, $\mathcal{C}(0) = \mathcal{C}$, $\mathcal{CS}^*(0) = \mathcal{CS}^*$ are the familiar classes of univalent, starlike, convex, close-to-convex, and close-to-starlike functions in $\mathbb{U}$, respectively. Also, we note that

(i) $f \in \mathcal{K}(\alpha) \Leftrightarrow zf' \in \mathcal{S}^*(\alpha)$;

(ii) $\mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$.

Let

$$J(\lambda, f; z) \equiv (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right), \quad (z \in \mathbb{U}) \tag{1.7}$$

for λ real number and $f \in \mathcal{A}_p$.

The class of *λ-convex* functions are defined by

$$\mathcal{M}_\lambda = \left\{f \in \mathcal{A}_p : \Re\ J(\lambda, f; z) > 0\right\}. \tag{1.8}$$

We note that $\mathcal{M}_\lambda \subset \mathcal{M}_\beta \subset \mathcal{M}_0 = \mathcal{S}^*$ for $0 \leq \lambda/\beta \leq 1$ and $\mathcal{M}_\lambda \subset \mathcal{M}_1 \subset \mathcal{K}$ for $\lambda \geq 1$.

Let

$$I_p(\mu, f; z) = (1-\mu)\frac{f(z)}{z^p} + \mu\frac{f'(z)}{z^{p-1}}, \quad \left(z \in \mathbb{U} \setminus \{0\}\right) \tag{1.9}$$

for μ real number and $f \in \mathcal{A}_p$. We note that $I_1(\mu, f; z) = I(\mu, f; z)$.

The class of functions is defined by $I_p(\mu, f; z)$ as above:

$$\mathcal{T}_\mu := \left\{f \in \mathcal{A}_p : \Re\ I_p(\mu, f; z) > 0\right\}. \tag{1.10}$$

A class defined by $J(\lambda, f; z)$ was studied by Dinggong [1], and also, for $f \in \mathcal{A}$, the general case of $\mathfrak{T}_\mu$ was studied by Özkan and Altıntaş [2]. Given two functions f and g, which are analytic in $\mathbb{U}$, the function f is said to be *subordinate* to g, written as

$$f \prec g, \qquad f(z) \prec g(z), \quad (z \in \mathbb{U}) \tag{1.11}$$

if there exists a Schwarz function ω analytic in $\mathbb{U}$, with

$$\omega(0) = 0, \qquad |\omega(z)| < 1, \quad (z \in \mathbb{U}) \tag{1.12}$$

and such that

$$f(z) = g(\omega(z)), \quad (z \in \mathbb{U}). \tag{1.13}$$

In particular, if g is univalent in $\mathbb{U}$, then

$$f \prec g \quad \text{iff } f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \tag{1.14}$$

2. The main results

In proving our main results, we need the following lemma due to Miller and Mocanu.

Lemma 2.1 (see [3, page 132]). *Let q be univalent in $\mathbb{U}$ and let θ and ϕ be analytic in a domain $\mathfrak{D}$ containing $q(\mathbb{U})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{U})$. Set*

$$Q(z) = zq'(z) \cdot \phi[q(z)], \qquad h(z) = \theta[q(z)] + Q(z), \tag{2.1}$$

and suppose that either

(i) *Q is starlike, or*

(ii) *h is convex.*

In addition, assume that

(iii) $\mathfrak{Re}(zh'(z)/Q(z)) = \mathfrak{Re}[\theta'[q(z)]/\phi[q(z)] + zQ'(z)/Q(z)] > 0.$

If P is analytic in $\mathbb{U}$, with $P(0) = q(0)$, $P(\mathbb{U}) \subset \mathfrak{D}$ and

$$\theta[P(z)] + zP'(z) \cdot \phi[P(z)] \prec \theta[q(z)] + zq'(z) \cdot \phi[q(z)] = h(z), \tag{2.2}$$

then $P \prec q$, and q is the best dominant.

Lemma 2.2. *Let $q \in \mathcal{H}_p$ be univalent, $q(z) \neq 0$ and satisfies the following conditions:*

$$\begin{aligned} &\text{(i)} \quad \frac{zq'(z)}{q(z)} \text{ is starlike;} \\ &\text{(ii)} \quad \mathfrak{Re}\left\{\frac{q(z)}{\lambda} + 1 + \frac{zq''(z)}{q'(z)} - \frac{z'q(z)}{q(z)}\right\} > 0 \end{aligned} \tag{2.3}$$

for $\lambda \neq 0$ and for all $z \in \mathbb{U}$. For $P \in \mathcal{H}_p$ with $P(z) \neq 0$ in $\mathbb{U}$ if

$$P(z) + \lambda \frac{zP'(z)}{P(z)} \prec q(z) + \lambda \frac{zq'(z)}{q(z)}, \tag{2.4}$$

then $P \prec q$, and q is the best dominant.

Proof. Define the functions θ and ϕ by

$$\theta(w) := w, \quad \phi(w) := \frac{\lambda}{w}, \quad \mathfrak{D} = \{w : w \neq 0\} \tag{2.5}$$

in Lemma 2.1. Then, the functions

$$\begin{aligned} Q(z) &= zq'(z)\cdot\phi[q(z)] = \lambda\frac{zq'(z)}{q(z)}, \\ h(z) &= \theta[q(z)] + Q(z) = q(z) + \lambda\frac{zq'(z)}{q(z)}. \end{aligned} \tag{2.6}$$

□

Using (2.3), we obtain that Q is starlike in $\mathbb{U}$ and $\mathfrak{Re}\{zh'(z)/Q(z)\} > 0$ for all $z \in \mathbb{U}$. Since it satisfies preconditions of Lemma 2.1 and using (2.4), it follows from Lemma 2.1 that $P \prec q$, and q is the best dominant.

Theorem 2.3. *Let $q \in \mathcal{H}_p$ be univalent, $q(z) \neq 0$ and satisfies the conditions* (2.3) *in Lemma 2.2. For $f \in \mathcal{A}_p$ if*

$$J(\lambda, f; z) \prec q(z) + \lambda\frac{zq'(z)}{q(z)}, \tag{2.7}$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z), \tag{2.8}$$

and q is the best dominant.

Proof. Let us put

$$P(z) := \frac{zf'(z)}{f(z)}, \quad (z \in \mathbb{U}), \tag{2.9}$$

where $P(0) = p$. Then, we obtain easily the following result:

$$P(z) + \lambda\frac{zP'(z)}{P(z)} = J(\lambda, f; z). \tag{2.10}$$

Thus, using Lemma 2.1 and (2.7), we can obtain the result (2.8). □

Lemma 2.4. *Let $q \in \mathcal{H}_1$ be univalent and satisfies the following conditions:*

$$\begin{aligned} &\text{(i)} \ \ q(z) \text{ is convex;} \\ &\text{(ii)} \ \ \mathfrak{Re}\left\{\left(\frac{1}{\mu} + p\right) + \frac{zq''(z)}{q'(z)}\right\} > 0, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \end{aligned} \tag{2.11}$$

for $\mu \neq 0$ and for all $z \in \mathbb{U}$. For $P \in \mathcal{H}_1$ in $\mathbb{U}$ if

$$(1 - \mu + \mu p)P(z) + \mu zP'(z) \prec (1 - \mu + \mu p)q(z) + \mu zq'(z), \tag{2.12}$$

then $P \prec q$, and q is the best dominant.

Proof. For $\mu \neq 0$ real number, we define the functions θ and ϕ by

$$\theta(w) := (1-\mu+\mu p)w, \qquad \phi(w) := \mu, \quad \mathfrak{D} = \{w : w \neq 0\} \tag{2.13}$$

in Lemma 2.1. Then, the functions

$$\begin{aligned} Q(z) &= zq'(z)\cdot\phi[q(z)] = \mu z q'(z), \\ h(z) &= \theta[q(z)] + Q(z) = (1-\mu+\mu p)q(z) + \mu z q'(z). \end{aligned} \tag{2.14}$$

□

Using (2.11), we obtain that Q is starlike in $\mathbb{U}$ and $\mathfrak{Re}\{zh'(z)/Q(z)\} > 0$ for all $z \in \mathbb{U}$. Since it satisfies preconditions of Lemma 2.1 and using (2.12), it follows from Lemma 2.1 that $P \prec q$, and q is the best dominant.

Theorem 2.5. *Let $q \in \mathcal{H}_1$ be univalent and satisfies the conditions* (2.11) *in Lemma 2.4. For $f \in \mathcal{A}_p$ if*

$$I_p(\mu, f; z) \prec (1-\mu+\mu p)q(z) + \mu z q'(z). \tag{2.15}$$

Then,

$$\frac{f(z)}{z^p} \prec q(z), \tag{2.16}$$

and q is the best dominant.

Proof. Let us put

$$P(z) := \frac{f(z)}{z^p}, \tag{2.17}$$

where $P(0) = 1$. Then, we have

$$(1-\mu+\mu p)P(z) + \mu z P'(z) = I_p(\mu, f; z). \tag{2.18}$$

Thus, using (2.15) and Lemma 2.4, we can obtain the result (2.16). □

Corollary 2.6. *Let $q \in \mathcal{H}_1$ be univalent and satisfies the following conditions:*

$$\begin{aligned} &\text{(i)}\ \ q(z)\ \textit{is convex}; \\ &\text{(ii)}\ \ \mathfrak{Re}\left\{\left(\frac{1}{\mu}+1\right) + \frac{zq''(z)}{q'(z)}\right\} > 0, \quad (p \in \mathbb{N} = \{1,2,3,\ldots\}) \end{aligned} \tag{2.19}$$

for $\mu \neq 0$ and for all $z \in \mathbb{U}$. For $P \in \mathcal{H}_1$ in $\mathbb{U}$ if

$$P(z) + \mu z P'(z) \prec q(z) + \mu z q'(z), \tag{2.20}$$

then $P \prec q$, and q is the best dominant.

Proof. By putting $p = 1$ in Lemma 2.4, we obtain Corollary 2.6. □

Corollary 2.7. *Suppose $q \in \mathcal{S}$ satisfies the conditions* (2.19) *in Corollary 2.6. For $f \in \mathcal{A}$ if*

$$I(\mu, f; z) \prec q(z) + \mu z q'(z). \tag{2.21}$$

Then,

$$\frac{f(z)}{z} \prec q(z), \tag{2.22}$$

and q is the best dominant.

Proof. By putting $p = 1$ in Theorem 2.5, we obtain Corollary 2.7. □

Corollary 2.8. *Let $q \in \mathcal{H}_1$ be univalent; $q(z)$ is convex for all $z \in \mathbb{U}$. For $P \in \mathcal{H}_1$ in $\mathbb{U}$ if*

$$P(z) + zP'(z) \prec q(z) + zq'(z), \tag{2.23}$$

then $P \prec q$, and q is the best dominant.

Proof. In Corollary 2.6, we take $\mu = 1$. □

Corollary 2.9. *Let $q \in \mathcal{S}$ be convex. For $f \in \mathcal{A}$ if*

$$f'(z) \prec q(z) + zq'(z). \tag{2.24}$$

Then,

$$\frac{f(z)}{z} \prec q(z), \tag{2.25}$$

and q is the best dominant.

Proof. In Corollary 2.7, we take $\mu = 1$. □

Corollary 2.10. *Let $q \in \mathcal{H}_1$ be univalent, $q(z)$ is convex for all $z \in \mathbb{U}$. For $P \in \mathcal{H}_1$ in $\mathbb{U}$ if*

$$pP(z) + zP'(z) \prec pq(z) + zq'(z), \tag{2.26}$$

then $P \prec q$, and q is the best dominant.

Proof. In Lemma 2.4, we take $\mu = 1$. □

Corollary 2.11. *Let $q \in \mathcal{H}_1$ be univalent, $q(z)$ is convex, for all $z \in \mathbb{U}$. If $f \in \mathcal{A}_p$, and*

$$\frac{f'(z)}{z^{p-1}} \prec pq(z) + zq'(z), \tag{2.27}$$

then

$$\frac{f(z)}{z} \prec q(z), \tag{2.28}$$

and q is the best dominant.

Proof. In Theorem 2.3, we take $\mu = 1$. □

Corollary 2.12. *Let $q \in \mathcal{S}$ satisfies*

$$I_p(\mu, f; z) \prec \frac{(1-\mu+\mu p) + 2[\mu - \alpha - \alpha\mu p]z - (1-2\alpha)(1-\mu+\mu p)z^2}{(1-z)^2}, \tag{2.29}$$

where $f \in \mathcal{A}_p$, then

$$\frac{f(z)}{z^p} \in \mathcal{CS}^*(\alpha), \tag{2.30}$$

and q is the best dominant.

Proof. In Theorem 2.5, we take

$$q(z) = \frac{1+(1-2\alpha)z}{1-z}. \tag{2.31}$$

□

Corollary 2.13. *Let $q \in \mathcal{S}$ satisfies*

$$\frac{f'(z)}{z^{p-1}} \prec \frac{p + 2[1-\alpha-\alpha p]z - (1-2\alpha)pz^2}{(1-z)^2}, \tag{2.32}$$

where $f \in \mathcal{A}_p$, then

$$\frac{f(z)}{z^p} \in \mathcal{CS}^*(\alpha), \tag{2.33}$$

and q is the best dominant.

Proof. In Corollary 2.12, we take $\mu = 1$. □

Corollary 2.14. *Let $q \in \mathcal{S}$ satisfies*

$$\frac{f'(z)}{z^{p-1}} \prec \frac{p+2z-pz^2}{(1-z)^2}, \tag{2.34}$$

where $f \in \mathcal{A}_p$, then

$$f \in \mathcal{CS}^*, \tag{2.35}$$

and q is the best dominant.

Proof. In Corollary 2.13, we take $\alpha = 0$. □

Corollary 2.15. *Let $q \in \mathcal{S}$ satisfies*

$$f'(z) \prec \frac{1+2z-z^2}{(1-z)^2}, \tag{2.36}$$

where $f \in \mathcal{A}_p$, then

$$f \in \mathcal{CS}^*, \tag{2.37}$$

and q is the best dominant.

Proof. In Corollary 2.14, we take $p = 1$. □

References

[1] D. Yang, "On a criterion for multivalently starlikeness," *Taiwanese Journal of Mathematics*, vol. 1, no. 2, pp. 143–148, 1997.

[2] Ö. Özkan and O. Altıntaş, "Applications of differential subordination," *Applied Mathematics Letters*, vol. 19, no. 8, pp. 728–734, 2006.

[3] S. S. Miller and P. T. Mocanu, "Differential Subordinations: Theory and Applications," vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.